A CHARACTERIZATION OF INFINITE ABELIAN GROUPS

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In this note we prove that an infinite group $G$ is abelian if and only if for every two infinite subsets $X$ and $Y$ of $G$, $XY \cap YX \neq \emptyset$, where $AB = \{ab | a \in A, b \in B\}$ for $A, B \subseteq G$.

1. Introduction

Many properties equivalent to commutativity have been considered by various people, for example in [6] Lennox, Mohammadi Hassanabadi and Wiegold introduced the class $P_n^*$-groups, where $n$ is a positive integer, as follows: $G \in P_n^*$ if and only if every infinite set of $n$-sets in $G$ contains a pair $X, Y$ of different members such that $XY = YX$, where $AB = \{ab | a \in A, b \in B\}$ for $A, B \subseteq G$. In [6] it is shown that infinite groups

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in $P_n^*$ are abelian if $n = 2$ or $n = 3$. This result was later extended in [5], where it was shown that if $n > 1$ then every infinite $P_n^*$-group is abelian.

Here we consider another characterization of infinite abelian groups. We say that a group $G$ is an $\mathcal{A}$-group or satisfies the property $\mathcal{A}$ if and only if for every two infinite subsets $X$ and $Y$ of $G$, $XY \cap YX \neq \emptyset$. Our main result is the following.

**Theorem.** Every infinite $\mathcal{A}$-group is abelian.

Now we present a reformulation of the class $\mathcal{A}$. Let $X$ and $Y$ be two infinite subsets of a group $G$. Then it is clear that $XY \cap YX \neq \emptyset$ if and only if $1_G \in X^{-1}Y^{-1}XY$. Therefore $G \in \mathcal{A}$ if and only if for every two infinite subsets $X$ and $Y$ of $G$, $1_G \in X^{-1}Y^{-1}XY$. Since the variety of abelian groups is generated by the law $x^{-1}y^{-1}xy = 1$, the above observation leads us to the following definitions.

Let $w = x_{i_1}^{\epsilon_1} \ldots x_{i_t}^{\epsilon_t}$ be a word in the free group of rank $n > 0$, where $\epsilon_1, \ldots, \epsilon_t \in \{-1, 1\}$. Suppose that $G$ is a group and $X_1, \ldots, X_n$ are subsets of $G$, we define

$$w(X_1, \ldots, X_n) := \{x_{i_1}^{\epsilon_1} \ldots x_{i_t}^{\epsilon_t} \mid x_{i_j} \in X_{i_j}, 1 \leq j \leq t, 1 \leq i_j \leq n\}.$$

Let $V(w)$ be the variety generated by the law $w(x_1, \ldots, x_n) = 1$. We denote by $V(w^*)$ and $V(\bar{w})$ the class of groups $G$ satisfying the following conditions, respectively:

$G \in V(w^*)$ whenever $X_1, \ldots, X_n$ are infinite subsets of $G$, there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $w(x_1, \ldots, x_n) = 1$.

$G \in V(\bar{w})$ whenever $1_G \in w(X_1, \ldots, X_n)$ for all infinite subsets $X_1, \ldots, X_n$ of $G$.

Clearly, we have $\mathcal{F} \cup V(w) \subseteq V(w^*) \subseteq V(\bar{w})$, $\mathcal{F}$ being the class of finite groups. Apparently, there is no example of an infinite group in
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\( V(w^*) \) which does not belong to \( V(w) \). Note that for certain words \( w \), the equality \( \mathcal{F} \cup \mathcal{V}(w) = \mathcal{V}(w^*) \) is known: it is the case with \( w = x^n \), \( w = [x, y] = x^{-1}y^{-1}xy \) and more generally \( w = [x_1, \ldots, x_n] \) [11]. This equality is also proved for \( w = [x, y]^2 \) [8], \( w = [x, y, y] \) [17], \( w = [x, y, y, y] \) [18], \( w = x_1 \cdots x_n \) [3] and \( w = (x_1x_2)^3(x_1^2x_2^3)^{-1} \) [2]. The origin of these results is a question of P. Erdös [12], since this first paper, problems of similar nature have been the object of several articles, (for example [1],[4],[7],[8],[10],[13],[14]).

The class \( V(w^*) \) have been firstly introduced in [4] and [11]. As far as we know the class \( V(w^*) \) has not been considered in the literature so far.

2. Proofs

In order to establish our main result, we need two lemmas, the first of which is the key to the proof of the Theorem.

**Lemma 2.1.** Let \( G \) be an infinite \( \mathcal{A} \)-group. Then \( C_G(x) \) is infinite for all \( x \) in \( G \).

**Proof.** Assume that \( K = \{ x \in G \mid C_G(x) \) is infinite } \( \neq G \) and let \( T = G \setminus K \). Then \( T \) is infinite. For otherwise, for \( x \in T \), \( \{ x^g \mid g \in G \} \subseteq T \). Therefore \( |G : C_G(x)| \) is finite and so \( x \in K \), a contradiction. Thus \( T \) is infinite.

Now, for \( g, a \in G \) let \( S(a, g) = \{ x \in G \mid a^x = g \} \). Then either \( S(a, g) = \varnothing \) or it is a right coset of \( C_G(a) \) and so is finite if \( a \in T \). Suppose, for a contradiction, that \( T \) is infinite. We construct two infinite subsets \( A = \{ a_1, a_2, \ldots \} \) and \( B = \{ b_1, b_2, \ldots \} \) of \( T \) such that \( AB \cap BA = \varnothing \). We define by induction \( A_n = \{ a_1, \ldots, a_n \} \) and \( B_n = \{ b_1, \ldots, b_n \} \) for all \( n \in \mathbb{N} \) such that \( A_nB_n \cap B_nA_n = \varnothing \). Let \( n = 1 \). There exist \( a_1, b_1 \in T \) such that \( a_1b_1 \neq b_1a_1 \), since otherwise the infinite set \( T \) is a subset of the centralizer of any element in \( T \), which is a contradiction. Thus we have \( A_1B_1 \cap B_1A_1 = \varnothing \). Now, suppose inductively that \( n > 1 \).
and we have already defined the subsets $A_{n-1}$ and $B_{n-1}$ of $T$ such that $A_{n-1}B_{n-1} \cap B_{n-1}A_{n-1} = \emptyset$. To obtain $A_n$ and $B_n$, we first define $b_n$ as follows:

Let

$$M = \left( \bigcup_{1 \leq i,j \leq n-1} S(a_i, a_j) \right) \bigcup \{a_i^{-1}b_ja_k, a_i b_j a_k^{-1} \mid 1 \leq i,j,k \leq n-1\}.$$ 

$M$ is a finite set and we may choose $b_n$ to be any element of the infinite set $T \setminus M$. Then we define $a_n$ as follows:

Let

$$N = \left( \bigcup_{1 \leq i,j \leq n} S(b_i, b_j) \right) \bigcup \{b_i^{-1}a_j b_k, b_i a_j b_k^{-1} \mid 1 \leq i,j,k \leq n, j \neq n\}.$$ 

Now $N$ is finite and $T \setminus N$ is infinite. We choose $a_n$ to be any element in the infinite set $T \setminus N$. Set $A_n = A_{n-1} \cup \{a_n\}$ and $B_n = B_{n-1} \cup \{b_n\}$. By considering the following equalities

$$A_n B_n = A_{n-1} B_{n-1} \cup A_{n-1}\{b_n\} \cup \{a_n\} B_{n-1} \cup \{a_n b_n\}$$

$$B_n A_n = B_{n-1} A_{n-1} \cup B_{n-1}\{a_n\} \cup \{b_n\} A_{n-1} \cup \{b_n a_n\}$$ 

and the method of choosing $a_n$ and $b_n$, one can easily check that $A_n B_n \cap B_n A_n = \emptyset$. Therefore $AB \cap BA = \emptyset$, since if $a_i b_j = b_k a_l$ for some $i,j,k,l \in N$, then $a_i b_j = b_k a_l \in A_i B_i \cap B_i A_i$ where $i = \max\{i,j,k,l\}$, which is a contradiction. Hence $G$ is not a $\tilde{A}$-group, a contradiction.

Lemma 2.2. Let $G$ be an infinite $\tilde{A}$-group. Then every infinite subgroup $H$ of $G$ contains the derived subgroup $G'$ of $G$. In particular, $G$ is nilpotent of class at most 2.

Proof. Let $H$ be an infinite subgroup of $G$ and $g \in G$. Let $X = \{h \in H \mid g^{-1}hg \notin H\}$. If $X$ is infinite, then $(gX)X \cap X(gX) \neq \emptyset$,
which is easily seen to lead to a contradiction. Thus $X$ is finite and $g^{-1}hg \in H$ for all but finitely many elements $h$ of $H$ and hence $H$ is normal in $G$. Now let $x, y \in G$ and consider the infinite subsets $xH$ and $yH$. Then $(xH)(yH) \cap (yH)(xH) \neq \emptyset$, from which it follows that $[x, y] \in H$. Therefore $G'$ lies in $H$ and by Lemma 2.1, $G'$ lies in the intersection of the centralizers of all the elements of $G$ and so is in the center of $G$; thus $G$ is nilpotent of class at most 2.

Now we see how the property $A$ will induce the commutativity of an infinite group.

**Proof of the Theorem.** Let $G$ be an infinite $A$-group. Suppose that $G$ is non-periodic and $a$ is an element of infinite order in $G$. Let $x, y \in G$. Consider $H = \langle a, x, y \rangle$ which is an infinite finitely generated $A$-group, and thus $H$ is residually finite, by Lemma 2.2. Now $H$ is abelian, since by Lemma 2.2, the derived subgroup of $H$ lies in the intersection of all normal subgroups of finite index in $H$. Thus $xy = yx$ which means that $G$ is abelian.

Therefore we may assume that $G$ is a torsion group. Suppose, for a contradiction, that $G$ is non-abelian. Now observe that by Lemma 2.2 there cannot be a subgroup of $G$ of the form $U \times V$ where $U$ and $V$ are both infinite. Thus each abelian subgroup of $G$ satisfies the minimal condition, by Theorem 4.3.11 of [16]. Hence $G$ satisfies the minimal condition (see Theorem 15.2.2 of [16]). Therefore by Theorem 3.14 of [16], $G$ is a finite extension of a central Prüfer group $D$. Thus $D$ is a quasicyclic $p$-group, for some prime $p$. Now, let $a_1, a_2, \ldots$ be a set of generators of the quasicyclic $p$-group $D$ which satisfy the following relations:

$$a_ia_j = a_ja_i, \quad a_i^p = 1 \quad \text{and} \quad a_i = a_{i+1}^p \quad \text{for all} \quad i, j \in \mathbb{N}.$$  

Now suppose that $[x, y] \neq 1$ for some $x, y \in G$. By Lemma 2.2, $[x, y] \in D$ and so $[x, y]$ is of order $p^n$ for some $n \in \mathbb{N}$. Let $X = \{a_{n+2}, a_{n+3}, \ldots \}$. By
the property $\bar{A}$, there exist $a_i, a_j, a_k, a_l \in X$ such that $ya_i xa_j = xa_k ya_l$ and so $[x, y] = a_k a_i a_j^{-1} a_l^{-1}$. Thus $a_k a_i a_j^{-1} a_l^{-1} \neq 1$ and so is of order at least $p^{n+1}$, a contradiction. Therefore $xy = yx$ and the proof is complete.

References


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