

A Property Equivalent to \(n\)-Permutability for Infinite Groups

Alireza Abdollahi* and Aliakbar Mohammadi Hassanabadi †

Department of Mathematics, University of Isfahan, Isfahan, Iran

and

Bijan Taeri‡

Department of Mathematics, University of Technology of Isfahan, Isfahan, Iran

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Let \(n\) be an integer greater than 1. A group \(G\) is said to be \(n\)-permutable whenever for every \(n\)-tuple \((x_1, \ldots, x_n)\) of elements of \(G\) there exists a non-identity permutation \(\sigma\) of \(\{1, \ldots, n\}\) such that \(x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}\). In this paper we prove that an infinite group \(G\) is \(n\)-permutable if and only if for every \(n\) infinite subsets \(X_1, \ldots, X_n\) of \(G\) there exists a non-identity permutation \(\sigma\) on \(\{1, \ldots, n\}\) such that \(X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset\). © 1999 Academic Press

1. INTRODUCTION

Permutable groups have been studied by various people (for example, see [1–3, 5, 6]). Let \(n\) be an integer greater than 1. Recall that a group \(G\) is called \(n\)-permutable whenever for every \(n\)-tuple \((x_1, \ldots, x_n)\) of elements of \(G\) there exists a non-identity permutation \(\sigma\) of \(\{1, \ldots, n\}\) such that \(x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}\). Also a group is said to be permutable if it is \(n\)-permutable for some integer \(n > 1\). The main result for groups in this class was obtained by Curzio et al. in [3], where it was shown that such

* E-mail: abdolahi@math.ui.ac.ir.
† E-mail: aamohaha@math.ui.ac.ir.
‡ E-mail: b.taeri@cc.iut.ac.ir.

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groups are finite-by-abelian-by-finite. Let \( n > 1 \) and \( m \) be positive integers. Let \( S_n \) denote the group of all permutations on the set \( \{1, \ldots, n\} \).

A natural extension of permutable groups, namely \((m, n)\)-permutable groups, groups in which \( X_1 \cdots X_n \subseteq \bigcup_{\sigma \in S_n \setminus \{1\}} X_{\sigma(1)} \cdots X_{\sigma(n)} \) for all subsets \( X_i \) of \( G \) where \( |X_i| = m \) for all \( i = 1, \ldots, n \), was introduced by Mohammadi Hassanabadi and Rhemtulla in [9]. It was proved there that such a group either is \( n \)-permutable or is finite of order bounded by a function of \( m \) and \( n \). In [8] Mohammadi Hassanabadi investigated another extension of \((m, n)\)-permutable groups as follows. For positive integers \( n > 1 \) and \( m \) a group \( G \) is called restricted \((m, n)\)-permutable if \( X_1 \cdots X_n \cap \bigcup_{\sigma \in S_n \setminus \{1\}} X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset \) for all subsets \( X_i \) of \( G \) where \( |X_i| = m \) for all \( i = 1, \ldots, n \). It was proved there that such a group is finite-by-abelian-by-finite. In [4] Longobardi et al. called a group \( G \) a \( P_n^* \)-group \((n \) an integer greater than 1) if for every sequence \( X_1, \ldots, X_n \) of infinite subsets of \( G \) there exist \( x_i \) in \( X_i \) such that \( x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)} \) for some non-trivial permutation \( \sigma \) in \( S_n \). They proved that every infinite \( P_n^* \)-group is an \( n \)-permutable group. Here we deal with another extension of infinite restricted \((m, n)\)-permutable and \( P_n^* \)-groups.

Let \( n \) be an integer greater than 1. We call a group \( G \) a restricted \((1, n)\)-permutable group if \( X_1 \cdots X_n \cap \bigcup_{\sigma \in S_n \setminus \{1\}} X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset \) for all infinite subsets \( X_1, \ldots, X_n \) of \( G \).

Our main result is the following, which sharpens and generalizes that of [8] and also generalizes the result of [4] concerning \( P_n^* \).

**Theorem.** Every infinite restricted \((\infty, n)\)-permutable group is \( n \)-permutable.

2. PROOFS

To prove the theorem, we need the following results.

**Lemma 2.1.** Let \( G \) be an infinite residually finite group which is a restricted \((\infty, n)\)-permutable group. Then \( G \) is an \( n \)-permutable group.

**Proof.** Let \( x_1, \ldots, x_n \) be arbitrary elements of \( G \) and

\[
S = \{ x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \mid \sigma \in S_n \setminus \{1\} \}.
\]

Suppose, for a contradiction, that \( 1 \notin S \). Since \( G \) is residually finite and \( S \) is finite, there exists a normal subgroup \( N \) of finite index in \( G \) such that \( S \cap N = \emptyset \). Now considering infinite subsets \( N x_1, \ldots, N x_n \), there exists \( \sigma \in S_n \setminus \{1\} \) such that \( N x_1 \cdots N x_n \cap N x_{\sigma(1)} \cdots N x_{\sigma(n)} \neq \emptyset \) and so \( x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \in N \), which is a contradiction. \( \blacksquare \)
Lemma 2.2. Let $G = \prod_{i \in I} G_i$ be an infinite direct product of non-abelian subgroups. Then $G$ is not a restricted $(\infty, n)$-permutable group for all integers $n > 1$.

Proof. Suppose, for a contradiction, that $G$ is a restricted $(\infty, n)$-permutable group for some integer $n > 1$. We show that $G$ is an $n$-permutable group, which contradicts Corollary 2.9 of [1]. Let $x_1, \ldots, x_n \in G$ and put $S = \{x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \mid \sigma \in S_n \setminus \{1\}\}$. Let $k$ be any integer greater than $|S|$. Since $G$ is an infinite direct product of normal subgroups, there exist $k$ infinite normal subgroups $N_1, \ldots, N_k$ of $G$ such that $N_i \cap N_j = 1$ for all distinct $i, j \in \{1, \ldots, k\}$. Let $l \in \{1, \ldots, k\}$ and consider infinite subsets $N_i x_1, \ldots, N_i x_n$. By the hypothesis, there exist two distinct $i, j \in \{1, \ldots, k\}$ and an element $s \in S$ such that $s \in N_i \cap N_j = 1$ and so $G$ is an $n$-permutable group. \qed

We denote by $A^{-1}$ the set $\{a^{-1} \mid a \in A\}$ for any non-empty subset $A$ of a group. Let $a$ and $g$ be arbitrary elements of a group $G$. We define $S(a, g) := \{x \in G \mid a^x = g\}$ which is either an empty set or a right coset of the centralizer of $a$ in $G$.

A key result required in the proof of the theorem is the following:

Lemma 2.3. Let $G$ be an infinite restricted $(\infty, n)$-permutable group. Then the FC-centre of $G$ is non-trivial.

Proof. Suppose, for a contradiction, that the FC-centre of $G$ is trivial. We construct $n$ infinite subsets $X_1, \ldots, X_n$ of $G$ such that
\[X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \emptyset\]
for all non-identity permutations $\sigma$ in $S_n$. For this, for each $m \in \mathbb{N}$ we construct $n$ subsets $X_{i,m} = \{a_{i,1}, \ldots, a_{i,m}\}$ of $G$ ($i = 1, \ldots, n$), such that
\[X_{1,m} \cdots X_{n,m} \cap X_{\sigma(1),m} \cdots X_{\sigma(n),m} = \emptyset\] (#)
for all non-identity permutations $\sigma$ in $S_n$. We argue by induction on $m$. Let $m = 1$. By Lemma 2.1 in [3], $G$ is not an $n$-permutable group and so there exist $a_{1,1}, \ldots, a_{n,1} \in G$ such that $a_{1,1} \cdots a_{n,1} \neq a_{\sigma(1),1} \cdots a_{\sigma(n),1}$ for all $\sigma \in S_n \setminus \{1\}$. Now suppose that we have already defined subsets $X_{i,m} = \{a_{i,1}, \ldots, a_{i,m}\}$ of $G$ ($i = 1, \ldots, n$) satisfying (#) for all $\sigma \in S_n \setminus \{1\}$.

Suppose that we have already defined $a_{i,m+1}$ and so $X_{i,m+1}$ for $i = 1, \ldots, r$ such that for all $\sigma \in S_n \setminus \{1\}$
\[X_{1,m+1} X_{2,m+1} \cdots X_{r,m+1} X_{r+1,m+1} \cdots X_{n,m} \cap X_{\sigma(1),m+1} \cdots X_{\sigma(n),m+1} = \emptyset\]
where \( j_i = m + 1 \) whenever \( \sigma(t) \in \{1, \ldots, r\} \) and otherwise \( j_i = m \). Let \( T_{r+1} \) be the union of all the following sets where \( \sigma \) varies over \( S_n \setminus 1 \),

\[
X_{\sigma(1), j_1} \cdots X_{\sigma(n), s_n}
\]

where \( s_j = m + 1 \) whenever \( \sigma(l) \in \{1, \ldots, r\} \) and otherwise \( s_j = m \); also \( i \) varies over \( \{1, \ldots, n\} \) and if \( i = 1 \) or \( i = n \) then we define respectively \( X_{\sigma(l), s_l} \) and \( \sigma(l) \) for all \( \sigma \in S_n \setminus 1 \).

Let

\[
U_{r+1} = X_{r, m+1}^{-1} \cdots X_{1, m+1}^{-1} \left( \bigcup_{1 \neq \sigma \in S_n} X_{\sigma(1), j_1} \cdots X_{\sigma(n), j_n} \right) X_{n, m}^{-1} \cdots X_{r+2, m}^{-1}
\]

where \( j_i = m + 1 \) whenever \( \sigma(l) \in \{1, \ldots, r\} \) and otherwise \( j_i = m \). Now we prove that there exists an element \( a_{r+1, m+1} \in G \setminus U_{r+1} \) such that if \( X_{r+1, m+1} = \{a_{r+1, 1}, \ldots, a_{r+1, m+1}\} \) then for all \( \sigma \in S_n \setminus 1 \)

\[
X_{1, m+1} \cdots X_{r+1, m+1} X_{r+2, m+1} \cdots X_{n, m} \cap X_{\sigma(1), j_1} \cdots X_{\sigma(n), j_n} = \emptyset
\]

where \( j_i = m + 1 \) whenever \( \sigma(l) \in \{1, \ldots, r + 1\} \) and otherwise \( j_i = m \). Suppose not. Therefore \( a_1, a_2, \ldots, a_{i_1, i_2, \ldots, i_n} = a_{\sigma(1), j_1} a_{\sigma(2), j_2} \cdots a_{\sigma(n), j_n} \) for some \( 1 \leq i_1, \ldots, i_{r+1} \leq m + 1, 1 \leq i_{r+2}, \ldots, i_n \leq m, 1 \leq j_1, \ldots, j_n \leq n \), and \( 1 \leq j_1 \leq m + 1 \) whenever \( \sigma(s) \in \{1, \ldots, r + 1\} \). Suppose that \( \sigma(t) = r + 1 \). If \( i_{r+1} \neq m + 1 \) or \( j_i \neq m + 1 \) then we get contradiction with the induction hypothesis or the choice of \( a_{r+1, m+1} \). Therefore we must always have \( i_{r+1} = j_i = m + 1 \) and so

\[
\left( a_{r+2, i_2, \ldots, i_n} a_{\sigma(1), j_1} \cdots a_{\sigma(n), j_n} \right)^{-1}
\]

\[
= a_{r+1, i_2, \ldots, i_n}^{-1} a_{\sigma(1), j_1} \cdots a_{\sigma(n), j_n} a_{r+1, j_i}
\]

Now we define \( g_{\sigma} \) and \( f_{\sigma} \) for all \( \sigma \in S_n \setminus 1 \) as

\[
f_{\sigma} = \begin{cases} 
  a_{r+2, i_2, \ldots, i_n} (a_{\sigma(1), j_1} \cdots a_{\sigma(n), j_n})^{-1} & \text{if } 1 \leq t \leq n - 1 \\
  a_{r+2, i_2, \ldots, i_n} a_{\sigma(n), j_n} & \text{if } t = n
\end{cases}
\]

and

\[
g_{\sigma} = \begin{cases} 
  (a_{1, i_1} \cdots a_{r, i_r})^{-1} a_{\sigma(1), j_1} \cdots a_{\sigma(t-1), j_{t-1}} & \text{if } 2 \leq t \leq n \\
  (a_{1, i_1} \cdots a_{r, i_r})^{-1} & \text{if } t = 1
\end{cases}
\]

where \( t = \sigma^{-1}(r + 1) \). Hence \( a_{r+1, m+1} \in S(g_{\sigma}, f_{\sigma}) \) and so

\[
G = U_{r+1} \cup \left( \bigcup_{\sigma \in S_n \setminus 1} S(g_{\sigma}, f_{\sigma}) \right)
\]

(*)
where $\sigma$ in (\*) varies over the set of all non-identity permutations in $S_n$ such that $S(g_{\sigma}, f_{\sigma}) \neq \emptyset$. Obviously the set of pairs $(g_{\sigma}, f_{\sigma})$ is finite. Therefore (\*) shows that $G$ is a finite union of right cosets of the centralizers of $g_{\sigma}$’s. Thus by the famous theorem of Neumann [10] there exists $g_{\sigma}$ in the FC-centre of $G$ such that $S(g_{\sigma}, f_{\sigma}) \neq \emptyset$. But by the hypothesis $g_{\sigma} = f_{\sigma} = 1$. Thus there exist $(n-1)$-tuples $(i_1, \ldots, i_r, i_{r+2}, \ldots, i_n)$ and $(j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_n)$ where $1 \leq i_1, \ldots, i_r \leq m + 1$, $1 \leq i_{r+2}, \ldots, i_n \leq m$, $t = \sigma^{-1}(r + 1)$, and $j_i = m + 1$ whenever $1 \leq \sigma(i) \leq r$ and otherwise $j_i = m$ such that

$$d_{r+2, i_{r+2}} \cdots d_{n, i_n} \left( d_{\sigma(t+1), j_{t+1}} \cdots d_{\sigma(n), j_n} \right)^{-1} = \left( d_{1, i_1} \cdots d_{r, i_r} \right)^{-1} d_{\sigma(1), j_1} \cdots d_{\sigma(t-1), j_{t-1}} = 1.$$ 

So for any $a \in X_{r+1, m}$ we have the following, which contradicts the induction hypothesis:

$$a_{1, i_1} \cdots a_{r, i_r} a_{r+2, i_{r+2}} \cdots a_{n, i_n} = a_{\sigma(1), j_1} a_{\sigma(2), j_2} \cdots a_{\sigma(t-1), j_{t-1}} a_{\sigma(t+1), j_{t+1}} \cdots a_{\sigma(n), j_n}.$$

Therefore we have defined $X_{r+1, m+1}$. Thus we have inductively defined $X_{i, m} = \{a_{i, 1}, \ldots, a_{i, m}\}$ for all $m \in \mathbb{N}$ such that for all $\sigma \in S_n \backslash 1$

$$X_{1, m} \cdots X_{n, m} \cap X_{\sigma(1), m} \cdots X_{\sigma(n), m} = \emptyset.$$ 

Now set $X_i = \bigcup_{m=1}^{\infty} X_{i, m}$ $(i = 1, \ldots, n)$, then $X_i$ is infinite and

$$X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \emptyset$$

for all $\sigma \in S_n \backslash 1$. Otherwise there exist $n$-tuples $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ on $\mathbb{N}$ and $\pi \in S_n \backslash 1$ such that $a_{i_1, i_1} \cdots a_{i_n, i_n} = a_{\pi(1), j_1} \cdots a_{\pi(n), j_n}$. Let $s = \text{Max}\{i_1, \ldots, i_n, j_1, \ldots, j_n\}$. Then $X_{1, s} \cdots X_{n, s} \cap X_{\pi(1), s} \cdots X_{\pi(n), s} \neq \emptyset$, which is a contradiction with the construction of $X_{i, s}$ $(i = 1, \ldots, n)$. \]

By Lemma 2.3, every non-trivial restricted $(\infty, n)$-permutable group has a non-trivial FC-element and since the class of restricted $(\infty, n)$-permutable groups is closed under homomorphic images we have:

**Corollary 2.4.** Every restricted $(\infty, n)$-permutable group is FC-hypercentral.

**Lemma 2.5.** Let $G$ be an infinite restricted $(\infty, n)$-permutable group. If $G$ is finitely generated or non-periodic then $G$ is an $n$-permutable group.
Proof. Suppose that $G$ is finitely generated. By Corollary 2.4, $G$ is FC-hypercentral. Now by a result of McLain [7] (or see p. 133 of [11]) a finitely generated FC-hypercentral group is nilpotent-by-finite. Therefore $G$ is a finitely generated nilpotent-by-finite group and so $G$ is residually finite. Thus $G$ is $n$-permutable by Lemma 2.1. Now assume that $G$ is non-periodic. Then there is an element $x$ of infinite order in $G$. Let $x_1, \ldots, x_n$ be arbitrary elements of $G$. By the previous part \langle x, x_1, \ldots, x_n \rangle is an $n$-permutable group and so $G$ is $n$-permutable.

Lemma 2.6. Let $G$ be a restricted $(\infty, n)$-permutable group. Then $G$ is hyperabelian-by-finite.

Proof. We may assume that $G$ is infinite, and it suffices to show that $G$ contains a non-trivial normal abelian subgroup. Suppose no such normal abelian subgroup exists, and let $x$ be a non-identity element in the FC-centre of $G$ which exists by Lemma 2.3. Let $N_1 := \langle x \rangle^G$ be the normal closure of $\langle x \rangle$ in $G$, and let $C := C_G(N_1)$. Then $|G : C|$ is finite and $N_1 \cap C = Z(N_1)$ is a normal abelian subgroup of $G$. Hence $N_1 \cap C = 1$. Therefore $N_1$ is finite and, having a trivial centre, it is certainly non-abelian. Now suppose, inductively, that we have already defined normal non-abelian finite subgroups $N_1, \ldots, N_t$ of $G$ such that $N_1, \ldots, N_t$ generate their direct product in $G$. Write $D := C_G(N_1 \cdots N_t)$; thus $|G : D|$ is finite. Now using Lemma 2.3 we can choose a non-trivial element $y$ in the FC-centre of $D$. Then $y$ is an element of the FC-centre of $G$. Let $N_{t+1} := \langle y \rangle^G$. It is easily seen that $N_{t+1}$ is a finite non-abelian group. Moreover, $N_{t+1} \subset D$, so that $N_1, \ldots, N_t, N_{t+1}$ generate their direct product in $G$. Thus we have found in $G$ an infinite direct product $N_1 \times N_2 \times \cdots \times N_t \times \cdots$ of finite non-abelian groups, which together with Lemma 2.2 gives a contradiction.

Lemma 2.7. Let $G$ be an infinite restricted $(\infty, n)$-permutable group which is not Černikov. Then $G$ is an $n$-permutable group.

Proof. By Lemma 2.5, we may assume that $G$ is periodic. By Lemma 2.6, there exists a normal hyperabelian subgroup $H$ of finite index in $G$. Therefore $H$ is a periodic locally soluble group and $G$ is locally finite. Let $x_1, \ldots, x_n$ be arbitrary elements in $G$ and let $A$ be the finite subgroup generated by $x_1, \ldots, x_n$. We note that $H$ is not a Černikov group and $A$ can be regarded as a finite group of automorphisms of $H$. Now by a result of Zaicev [13], there exists an abelian subgroup $B$ of $H$ which is not Černikov and $B$ is a normal subgroup of $AB$. Since $B$ is periodic it is a direct product of the Sylow $p$-subgroups $B_p$ of $B$. If infinitely many $B_p$ are non-trivial, then since $|A|$ has only finitely many prime divisors, there exists an infinite subgroup $D$ of $B$ which is normal in $AB$ such that $A \cap D = 1$. Consider the $n$ infinite subsets $Dx_1, \ldots, Dx_n$. By the hypothesis there exists
and so \( x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \in A \cap D = 1 \). Therefore \( x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)} \) as required. So assume that there exist only finitely many \( B_p \) which are non-trivial. Since \( B \) is not a Černikov group and since the product of two normal Černikov subgroups of a group is a Černikov group, then there exists a prime number \( p \) such that \( B_p \) is not Černikov. Thus by Theorem 4.3.13 of [12], \( C = \{ b \in B \mid b^p = 1 \} \) is an infinite elementary abelian \( p \)-group. Clearly \( C \) is normal in \( AB \). Now the infinite group \( AC \) is a residually finite-by-finite group and so \( AC \) is residually finite. Therefore by Lemma 2.1, \( AC \) is an \( n \)-permutable group and the proof is complete.

\begin{proof}
We need the following remark in the final step of the proof of the theorem. Here \( |x| \) denotes the order of an element \( x \) of a group.

\textbf{Remark 2.8.} We note that if \( x_1, \ldots, x_n \) \((n > 1)\) are \( p \)-elements \((p \) a prime\) of distinct orders in an abelian group then \( r < |x_1 \cdots x_n| \leq t \) where \( r = \min\{|x_1|, \ldots, |x_n|\} \) and \( t = \max\{|x_1|, \ldots, |x_n|\} \).

\textbf{Proof of the Theorem.} Let \( G \) be an infinite restricted \((\infty, n)\)-permutable group. By Lemma 2.7, we may assume that \( G \) is a Černikov group. Thus there exists an infinite normal subgroup \( A \) of \( G \) which is a direct product of finitely many groups isomorphic to \( C_{p^\infty} \), the quasicyclic \( p \)-group, for some prime number \( p \). Let \( x_1, \ldots, x_n \in G \) and let \( X \) be the finite subgroup generated by \( x_1, \ldots, x_n \) (we note that \( G \) is locally finite). Let \( Y \) be the group of automorphisms of \( A \) induced by the elements of \( X \) under conjugation. Then \( Y \) is finite. Let \( a_0 \) be an integer such that \( |a| \leq p^{a_0} \) for any \( a \in X \cap A \). By Lemma 3.5 of [4] there are infinite sequences \( a_0 < a_1 < \cdots \) of integers and \( a_1, a_2, \ldots \) of elements of \( A \) such that for any \( i \), \( |a_i| = p^{a_i} \), and \([a_i, y]| > p^{a_i-1}, \) for any \( y \in Y \setminus C_Y(a_i) \). Now partition the set \( \{ a_i \mid i \geq 1 \} \) into \( n \) infinite disjoint subsets \( J_i, i = 1, \ldots, n \). Consider the set \( J_i x_i, i = 1, \ldots, n \), and let \( \sigma \in S_n \setminus 1 \) be such that

\begin{equation}
(a_{i_1} x_{i_1}) \cdots (a_{i_n} x_{i_n}) = (a_{j_1} x_{\sigma(1)}) \cdots (a_{j_n} x_{\sigma(n)})
\end{equation}

for suitable \( a_i \in J_1, \ldots, a_n \in J_n \) and \( a_{j_1} \in J_{\sigma(1)}, \ldots, a_{j_n} \in J_{\sigma(n)} \). Therefore

\[ x = (x_1 \cdots x_n)^{-1} x_{\sigma(1)} \cdots x_{\sigma(n)} = a_{i_1}^{-1} \cdots a_{i_n}^{-1} a_{j_1}^{x_{\sigma(1)}} \cdots a_{j_n}^{x_{\sigma(n)}}. \]

We note that \( i_1, \ldots, i_n \) are pairwise distinct as are \( j_1, \ldots, j_n \). If

\[ \{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\} = \emptyset \]
then by Remark 2.8, $p^{\alpha t} < |x|$ where $r = M \ln \{i_1, \ldots, i_n, j_1, \ldots, j_n\}$, which is a contradiction, since $\alpha > \alpha_t$ and $x \in X \cap A$. Thus $F = \{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\} \neq \emptyset$. Let $|F| = s$. We may assume, without loss of generality, that $i_1 = j_1, \ldots, i_s = j_s$. Then we may write

$$x = [a_{i_1}, y_1]^{\alpha_1} \cdots [a_{i_s}, y_s]^{\alpha_s} a_{i_{s+1}}^{\alpha_{s+1}} \cdots a_{i_n}^{\alpha_n} (a_{j_{s+1}}^{\alpha_{j_{s+1}}} \cdots a_{j_n}^{\alpha_{j_n}})^{-1},$$

for some $y_1, \ldots, y_n, z_1, \ldots, z_n \in X$. Now suppose, for a contradiction, that $x \neq 1$. If $[a_{i_1}, y_1] = \cdots = [a_{i_s}, y_s] = 1$ then $s < n$, since $x \neq 1$. Then since $i_{s+1}, \ldots, i_n, j_{s+1}, \ldots, j_n$ are pairwise distinct, by Remark 2.8, $|x| > p^{\alpha_t}$ where $k = M \ln \{i_{s+1}, \ldots, i_n, j_{s+1}, \ldots, j_n\}$, which is a contradiction. Thus we may assume, without loss of generality, that $y_l \in Y \backslash C_Y(a_l)$, for $l = 1, \ldots, s$ and $i_1 < \cdots < i_s$. Now we claim that the elements

$$[a_{i_1}, y_1], [a_{i_2}, y_2], [a_{i_{s+1}}, y_{s+1}], \ldots, [a_{i_n}, y_n], [a_{j_{s+1}}, y_{s+1}], \ldots, [a_{j_n}, y_n]$$

have distinct orders. For, since $p^{\alpha_{l-1}} < |[a_{i_l}, y_l]| \leq p^{\alpha_l}$ for $l = 1, \ldots, s$ and $\alpha_0 < \cdots < \alpha_s$, then the elements $[a_{i_1}, y_1], \ldots, [a_{i_s}, y_s]$ have distinct orders. Clearly $a_{i_{s+1}}^{\alpha_{s+1}}, \ldots, a_{i_n}^{\alpha_n}$ have distinct orders. If there exist $l \in \{1, \ldots, s\}$ and $k \in \{s+1, \ldots, n\}$ such that $|[a_{i_l}, y_l]| = |a_{i_k}|$ or $|[a_{i_l}, y_l]| = |a_{i_k}|$, then since $p^{\alpha_{l-1}} < |[a_{i_l}, y_l]| \leq p^{\alpha_l}, \alpha_l = \alpha_{l_k}$ or $\alpha_l = \alpha_{l_k}$ and so $i_l = i_k$ or $i_l = j_k$, a contradiction. Now by Remark 2.8, $p^l < |x|$, where

$$p' = M \ln \{|[a_{i_1}, y_1]|, \ldots, |[a_{i_s}, y_s]|, |a_{i_{s+1}}^{\alpha_{s+1}}|, \ldots, |a_{i_n}^{\alpha_n}|, |a_{j_{s+1}}^{\alpha_{s+1}}|, \ldots, |a_{j_n}^{\alpha_n}|\}.$$