Finitely Generated Soluble Groups
with an Engel Condition on Infinite Subsets.

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ABSTRACT - In this note, we prove that, in every finitely generated soluble group $G$, $G/Z_2(G)$ is finite if and only if in every infinite subset $X$ of $G$ there exist different $x, y$ such that $[x, y, y] = 1$.

B. H. Neumann proved in [9] that a group $G$ is centre-by-finite if and only if every infinite subset $X$ of $G$ contains two different commuting elements. This answered a question posed by Paul Erdös. Extensions of problems of this type are studied in [1], [4], [5], [8] and [11].

We denote by $E(\infty)$ (respectively, $N(\infty)$) the class of groups $G$ such that, every infinite subset $X$ of $G$, contains different elements $x$ and $y \in X$ such that $[x, k, y] = 1$ (respectively, $(x, y)$ is nilpotent of class at most $k$) for some $k = k(x, y) \geq 1$. If the integer $k$ is the same for all infinite subsets of $G$, we say that $G$ is in the class $E_k(\infty)$ (respectively, $N_k(\infty)$).

It is easy to see that the above classes are closed with respect to forming subgroups and homomorphic images.

In [6] J. C. Lennox and J. Wiegold studied the class $N(\infty)$ and proved that a finitely generated soluble group is in $N(\infty)$ if and only if it is finite-by-nilpotent.

Also, in [7] P. Longobardi and M. Maj studied the class $E(\infty)$ and proved that a finitely generated soluble group is in $E(\infty)$ if and only if it is finite-by-nilpotent. Moreover, they proved that a finitely generated soluble group $G$ is in $E_2(\infty)$ if and only if $G/R(G)$ is finite, where $R(G)$ is

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the characteristic subgroup of $G$ consisting of all right 2-Engel elements of $G$.

In [2] and [3] C. Delizia proved that, a finitely generated soluble (or residually finite) group $G$ is in $N_2$, if and only if $G/Z_2(G)$ is finite.

Here we prove the following:

**Theorem.** Let $G$ be a finitely generated soluble group. Then $G \in E_2(\infty)$ if and only if $G/Z_2(G)$ is finite.

**Proof.** Let $G$ be a finitely generated soluble $E_2(\infty)$-group. By Theorem 1 of [7], $G$ contains a finite normal subgroup $N$ such that $G/N$ is torsion-free nilpotent. Now by Theorem 2 of [7], $R(G)$ has finite index in $G$, where $R(G) = \{ a \in G | [a, x, x] = 1 \text{ for all } x \in G \}$, thus $R(G) N/N$ has finite index in $G/N$. So $R(G) N/N$ is a torsion-free 2-Engel group, therefore by Theorem 7.14 in [10], $R(G) N/N$ is nilpotent group of class at most 2. Since $G/N$ is torsion-free nilpotent and $R(G) N/N$ is of finite index in $G/N$, thus $G/N$ is nilpotent group of class at most 2. We note that $G$ is residually finite since it is a finitely generated nilpotent-by-finite group. Thus it contains a normal subgroup $L$ of finite index such that $L \cap N = 1$. Now $[L, G] \leq N \cap L = 1$. Then $L \leq Z_2(G)$ as required to be shown.

Conversely, if $G/Z_2(G)$ is finite and $\{ x_i : i \in I \}$ is an infinite set of elements of $G$, there exist $i, j \in I$ with $i \neq j$ such that $x_i \equiv x_j \mod Z_2(G)$. Therefore $x_i x_j^{-1} = z \in Z_2(G)$, so $\langle x_i, x_j \rangle = \langle z, x_j \rangle$ is nilpotent of class at most 2. Hence $G \in N_2(\infty) \subset E_2(\infty)$.

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**REFERENCES**

Finitely generated soluble groups etc.


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