A Condition on a Certain Variety of Groups.

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ABSTRACT - Let $a_1, \ldots, a_n$ be nonzero integers whose greatest common divisor is $d$. We prove that an infinite group $G$ is of finite exponent dividing $d$ if and only if for every $n$ infinite subsets $X_1, \ldots, X_n$ of $G$ there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $x_1^{a_1} \cdots x_n^{a_n} = 1$.

1. Introduction.

Let $\mathcal{V}$ be a variety of groups defined by the law $v(x_1, \ldots, x_n) = 1$. We say that $G$ is a $\mathcal{V}^\#$-group if for every $n$ infinite subsets $X_1, \ldots, X_n$ of $G$ there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $v(x_1, \ldots, x_n) = 1$.

In [4] P.S. Kim, A.H. Rhemtulla and H. Smith posed the following question: for which variety $\mathcal{V}$, is every infinite $\mathcal{V}^\#$-group a $\mathcal{V}$-group?

There exist positive answers for the variety $\mathcal{C}_1$ of abelian groups defined by the law $[x, y] = 1$ (see [7]), the variety $\mathcal{C}_{2,2}$ defined by the law $[x, [y, y]] = 1$ (see [5]), the varieties $\mathcal{B}_2$ and $\mathcal{B}_3$ of 2-Engel and 3-Engel groups defined by the laws $[x, y, y] = 1$ and $[x, y, y, y] = 1$, respectively (see [10] and [11]), the variety $\mathcal{B}_n$ of groups of exponent dividing $n$, defined by the law $x^n = 1$ (see [6]), the variety $\mathcal{N}_k$ of nilpotent groups of nilpotency class at most $k$, defined by the law $[x_1, \ldots, x_{k+1}] = 1$ (see [6]), the variety $\mathcal{C}_3$ of 3-abelian groups defined by the law $(xy)^3 y^{-3} x^{-3} = 1$.

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(see [1]) and, the variety generated by the law \( x_1 \ldots x_n = 1 \) (see [3]).

Now let \( \mathcal{V} \) and \( \mathcal{W} \) be varieties of groups defined by different laws
\( v(x_1, \ldots, x_n) = 1 \) and \( w(y_1, \ldots, y_m) = 1 \), respectively. C. Delizia in [2]
posed the question of whether \( \mathcal{V}^e = \mathcal{W}^e \) when \( \mathcal{V} = \mathcal{W} \), and for \( k > 1 \), he
investigated the question for the varieties \( \mathcal{V} = \mathcal{V}_k \) (the variety defined
by the law \([x_1, \ldots, x_k, x_1] = 1 \) and \( \mathcal{W} = N_k \).

For nonzero integers \( \alpha_1, \ldots, \alpha_n \) with the greatest common divisor \( d \),
we have considered the question for the varieties \( \mathcal{V} \) defined by the law
\( x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 1 \) and \( \mathcal{W} = \mathcal{B}_d \); proving

**Theorem.** \( \mathcal{V}^e = \mathcal{B}_d^e \).

2. Proofs.

**Lemma 1.** Let \( \mathcal{V} \) be the variety defined by the law \( w(x_1, \ldots, x_n) = 1 \). If \( G \) is an infinite FC-group in \( \mathcal{V}^e \), then \( G \in \mathcal{V} \).

**Proof.** Let \( g_1, \ldots, g_n \in G \), then \( |G : \bigcap_{i=1}^n C_G(g_i)| \leq \prod_{i=1}^n |G : C_G(g_i)| < \infty \). Thus \( \bigcap_{i=1}^n C_G(g_i) \) is infinite. Let \( A \) be an infinite abelian subgroup of \( \bigcap_{i=1}^n C_G(g_i) \). Then \( A \in \mathcal{V} \) by Lemma 3 of [3]. Now consider the infinite sets \( g_1 A, \ldots, g_n A \). By the property \( \mathcal{V}^e \) there exist \( a_1, \ldots, a_n \in A \), such that

\[
1 = w(g_1, \ldots, g_n)w(a_1, \ldots, a_n) = w(g_1, \ldots, g_n) = 1,
\]

and \( G \in \mathcal{V} \). \( \blacksquare \)

From now on, for nonzero constant integers \( \alpha_1, \ldots, \alpha_n \) and any nonzero integer \( \alpha \) we denote the varieties defined by the laws \( w = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \) and \( w = x^n \), respectively, by \( \mathcal{V} \) and \( \mathcal{B}_n \). Also we denote the greatest common divisor of \( \alpha_1, \ldots, \alpha_n \) by \( d \).

**Lemma 2.** Let \( G \in \mathcal{V}^e \) be an infinite group. If the set \( T = \{ g \in G \mid g^d = 1 \} \) is infinite, then \( G \in \mathcal{V} \). In particular an infinite group in \( \mathcal{V}^e \setminus \mathcal{V} \) can not have an infinite subgroup in \( \mathcal{V} \).

**Proof.** Let \( X \) be an infinite subset of \( G \). Let \( 1 \leq j \leq n \) be fixed. By considering the infinite subsets \( X_i = T, \ i = 1, \ldots, j - 1, j + 1, \ldots, n \) and \( X_j = X \), and using the property \( \mathcal{V}^e \), we find the elements \( g_i \in T, \ i = 1, \ldots, j - 1, j + 1, \ldots, n \) and \( x \in X \) such that
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$g_1^{a_1} \cdots g_n^{a_n} = 1$. But $g_1^{a_1} = 1$, so we have $x_1^{a_1} = 1$. Thus $G \in \mathcal{B}_{a_1}$, for all $j = 1, \ldots, n$. Since $\mathcal{B}_{a_j} = \mathcal{B}_{a_j} \cup \mathcal{F}$, we have $G \in \mathcal{B}_{a_j}$, for all $j = 1, \ldots, n$. So $G \in \mathcal{V}$, as required.

We define the class $\mathcal{V}(\infty)$ including all groups $G$ in which any infinite subset $X$ contains distinct elements $x_1, \ldots, x_n$ such that $x_1^{a_1} \cdots x_n^{a_n} = 1$. Also, if $k \geq n$ is an integer, we define the class $\mathcal{V}(k)$ including all groups $G$ in which any subset $X$ with $|X| = k$, contains distinct elements $x_1, \ldots, x_n \in X$ such that $x_1^{a_1} \cdots x_n^{a_n} = 1$. It is clear that, $\mathcal{V}(k) \subseteq \mathcal{V}(\infty)$ and $\mathcal{V}(\infty) \subseteq \mathcal{V}(\infty)$.

**Lemma 3.** Let $G \in \mathcal{V}(\infty)$ be an infinite group. Then $|H : C_H(g^d)|$ is finite, for any $g \in G$, and any infinite subgroup $H$ of $G$.

**Proof.** Let $H$ be an infinite subgroup of $G$ and suppose, for a contradiction, that there is an element $g \in G$, such that $|H : C_H(g^d)|$ is infinite. Let $T$ be a right transversal to $C_H(g^d)$ in $H$. List the elements of $T$ as $t_1, t_2, \ldots$ under some well ordering $\leq$, so that $t_i < t_j$ if $i < j$.

Consider the set $T(n)$ of all $n$-element subsets of $T$. For each $t \in T(n)$, list the elements $t_i, \ldots, t_n$ of $t$ in ascending order given by $\leq$, and write $t = (t_1, \ldots, t_n)$. Create $n! + 1$ sets, one $U_{\sigma}$ for each permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$ and $V$: For each $t \in T(n)$, $t = (t_1, \ldots, t_n)$ put $t \in U_{\sigma}$ if

$$(g^{t_{\sigma(1)}})^{a_1} \cdots (g^{t_{\sigma(n)}})^{a_n} = 1,$$

and put $t \in V$ if $t \notin U_{\sigma}$, for any $\sigma$.

By Ramsey's Theorem, there exists an infinite subset $T_0 \subseteq T$, such that $T_0^{(n)} \subseteq U_{\sigma}$, for some $\sigma$, or $T_0^{(n)} \subseteq V$. Suppose, for a contradiction, that $T_0^{(n)} \subseteq V$, then the set $\{g^x \mid x \in T_0\}$, is infinite. By the property $\mathcal{V}(\infty)$, there exist $n$ distinct elements $t_1, \ldots, t_n \in T_0$, such that $(g^{t_1})^{a_1} \cdots (g^{t_n})^{a_n} = 1$, but then $t = \{t_1, \ldots, t_n\}$ lies in some $U_{\sigma}$, a contradiction.

Thus $T_0^{(n)} \subseteq U_{\sigma}$, for some $\sigma$. Moreover, by restricting the order $\leq$ to $T_0$ we may assume that $T_0 = \{t_1, t_2, \ldots\}$ and $t_i < t_j$ if $i < j$. Hence for any $i_1 < i_2 < \cdots < i_n$, we have

$$(g^{t_{i_1}})^{a_1} \cdots (g^{t_{i_n}})^{a_n} = 1.$$

Now consider a sequence

$$i_1 < i_2 \ldots < i_{n-1} < i_n < i_{n+1} < i_{n+2} < \cdots < i_{2^{n+1}}.$$
of $2n + 1$ indices and suppose that $k \in \{1, 2, \ldots, n\}$ is fixed and let $\sigma(s) = k$. Define a sequence $j_1 < \ldots < j_n$ as follows: the first $k - 1$ elements are $j_1 = \hat{i}_1, \ldots, j_{k-1} = \hat{i}_{k-1}$ (if $k = 1$ this is empty), $j_k = \hat{i}_{n+1}$, and the last $n - k$ elements are $j_{k+1} = \hat{i}_{n+3}, \ldots, j_n = \hat{i}_{2n-k+2}$ (if $k = n$ this is empty). Thus

$$(g^{t_{j_1}})^{x_s} = ((g^{t_{j_1}})^{x_1}(g^{t_{j_2}})^{x_2} \ldots (g^{t_{j_k}})^{x_k})^{x_1}((g^{t_{j_{k+1}}})^{x_{k+1}} \ldots (g^{t_{j_n}})^{x_n})^{-1}.$$  

Now if we put $j_k = \hat{i}_{n+2}$ in the above sequence, we have

$$(g^{t_{j_1}})^{x_s} = ((g^{t_{j_1}})^{x_1}(g^{t_{j_2}})^{x_2} \ldots (g^{t_{j_k}})^{x_k})^{x_1}((g^{t_{j_{k+1}}})^{x_{k+1}} \ldots (g^{t_{j_n}})^{x_n})^{-1},$$  

and thus $(g^{t_{j_1}})^{x_s} = (g^{t_{j_k}})^{x_s}$. Therefore $t_{j_{k+1}} t_{j_{k+2}}^{-1} \in C_H(g^{x_s})$. Now if $k$ runs through the set $\{1, 2, \ldots, n\}$, so does $s$. Thus

$$t_{j_{k+1}} t_{j_{k+2}}^{-1} \in \bigcap_{s=1}^{n} C_H(g^{x_s}) = C_H(g^{x}),$$

a contradiction.  

It is natural to ask whether every infinite $\forall(\infty)$-group belongs to $\forall$. The answer, in general, is negative. In fact, there is an infinite group satisfying the condition $\forall(k)$ for some positive integer $k$ which does not belong to $\forall$.

For example, we can take a group $K$ (with $m = 2$ and $n$ to be a sufficiently large prime, $n > 10^{10}$) in Theorem 31.5 in [8]. This group $K$ is given by all relations of the form $A^{n^2} = 1$ and $A^n = B^n$, where $A$ and $B$ run through a special set of words. The group $K$ is 2-generated and of exponent $n^2$, and it is a central extension of $B(2, n)$ (this is the 2-generated free Burnside group of exponent $n$) by a cyclic central subgroup $(z)$ of order $n$.

Thus $K$ is infinite, since $B(2, n)$ is infinite, and $K^n = \langle z \rangle$ is a finite (central) cyclic subgroup of $K$. Now we show that the set $X$ of all elements of order $n$ in $K$ coincides with $\langle z \rangle$ and thus is finite. For this purpose we need to show that all non-central elements of $K$ have order $n^2$.

Take an element $x$ outside $Z(K) = \langle z \rangle$. It follows from the construction of $K$ that $x$ is conjugate in $K$ to a product $A^k z^t$, where $A^n = z$. Then $(A^k z^t)^n = (A^n)^k (z^n)^t = z^k = 1$, since $k$ is not divided by $n$. Thus $x$ is of order $n^2$ as required.

Let, now $O$ be the variety defined by the law $x_1^n \ldots x_n^n = 1$. Then $K \in O(n^2)$. For, let $x_1, \ldots, x_n$ be distinct elements of $K$. Thus $x_1^n, \ldots, x_n^n$
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are in $K^n$ and so there exists $\{y_1, \ldots, y_n\} \subseteq \{x_1, \ldots, x_{n^2}\}$ such that $y_1^n = \cdots = y_n^n = e \in Z(K) = \langle z \rangle$. Therefore $y_1^n \cdots y_n^n = c^n = 1$, since $|K| = |Z(K)| = n$.

**Corollary 4.** Let $G \in \forall^\#$ be an infinite group, then $G^d = \langle g^d \mid g \in G \rangle$ is finite.

**Proof.** By Lemma 3, $|G : C_G(g^d)|$ is finite, for all $g \in G$. Thus $G^d \leq F$, the FC-center of $G$. If $F$ is infinite then $F \in \forall$, by Lemma 1, and so $G \in \forall = \beta_d \cup \mathcal{F}$ by Lemma 2, and thus $G^d = 1$. If $F$ is finite, then $G^d$ is also finite, as required. ■

**Proof of the Theorem.** Let $G$ be an infinite group in $\forall^\#$ and assume $G \notin \forall$. Then $G^d$ is finite, by Corollary 4. Write $H := C_G(G^d)$. Then $G/H$ is finite and $a^d \in Z(H)$, for any $a \in H$. We show that $H/FC(H)$ has exponent 2 where $FC(H)$ is the FC-centre of $H$. Let $a \in H$, $a^2 \not\in FC(H)$. Then the set $X = \{a^g \mid g \in H\}$ is infinite. Let $X_1 = X_2 = \cdots = X_n = X$. Then, since $G \in \forall^\#$, there exist $g_1, \ldots, g_n \in H$ such that

$$(a^{g_1})^{a_1} (a^{g_2})^{a_2} \cdots (a^{g_n})^{a_n} = 1 = a^{a_1} a^{a_2} \cdots a^{a_n}$$

since $a^d \in Z(H)$. Write $\alpha_i = d \beta_i$ for all $i \in \{1, \ldots, n\}$. Then

$$(*) \quad (a^d)^{\beta_1 + \beta_2 + \cdots + \beta_n} = 1.$$ 

Now let $Y = \{(a^g)^d \mid g \in H\}$, then $Y$ is infinite, since $a^2 \notin FC(H)$. For $X_1 = Y$, $X_2 = X_3 = \cdots = X_n = X$, there exist $h_1, \ldots, h_n \in H$ such that

$$(a^{2h_1})^{a_1} (a^{2h_2})^{a_2} \cdots (a^{2h_n})^{a_n} = 1.$$ 

Then $a^{2 \alpha_1 + \alpha_2 + \cdots + \alpha_n} = 1$ and

$$(**) \quad (a^d)^{2 \beta_1 + \beta_2 + \cdots + \beta_n} = 1.$$ 

From $(*)$ and $(**)$, we get $(a^d)^{\beta_1} = 1$, and arguing similarly with $X = X_1 = \cdots = X_{i-1} = X_{i+1} = \cdots = X_n$ and $X_i = Y$, we obtain $(a^d)^{\beta_i} = 1$ for any $i$ in the set $\{1, \ldots, n\}$. Then $a^d = 1$, since gcd$(\beta_1, \beta_2, \ldots, \beta_n) = 1$. But from $a^d = 1$, we have $(a^g)^d = 1$, for any $g$ in $H$ and, by Lemma 2, the set $\{a^g \mid g \in H\}$ is finite, that is $a \in FC(H)$, a contradiction.

Therefore $H/FC(H)$ is a locally finite group, $FC(H)$ is finite by Lemmas 1 and 2, and so $H$ is locally finite. Then $H$ can not be infinite by Lemma 2. Thus $H$ is finite which forces $G$ to be finite. This is a contradiction, which completes the proof. ■
REFERENCES


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