On Conditions for an Endomorphism to be an Automorphism

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Abstract. If $K$ is a set of automorphisms of a group $G$, an endomorphism $\theta : G \to G$ is said to be $K$-pointwise if for each element $t \in G$, there exists an element $\varphi \in K$ such that $\theta(t) = \varphi(t)$. This generalizes the notion of pointwise inner automorphism. We show that in some special cases, a $K$-pointwise endomorphism is necessarily an automorphism (it is not true in general).

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1 Introduction

By definition, a pointwise inner automorphism of a group $G$ is an automorphism $\theta : G \to G$ such that $t$ and $\theta(t)$ are conjugate for any $t \in G$. This notion appears already in the famous book of Burnside [1, Note B, p. 463]. There are many examples of groups (finite or infinite) admitting a pointwise inner automorphism which is not inner (see, for instance, [2], [9], [7] or [8], where these groups are besides nilpotent). On the other hand, each pointwise inner automorphism is inner for example in a free group [4] or in a free nilpotent group [3].

In the same way, we define a pointwise inner endomorphism of $G$ as an endomorphism $\theta : G \to G$ such that $t$ and $\theta(t)$ are conjugate for any $t \in G$. Equivalently, an endomorphism $\theta$ of $G$ is pointwise inner if and only if $\theta(t^G) \subseteq t^G$ for each $t \in G$, where $t^G$ is the conjugacy class of $t$ in $G$.

Clearly, a pointwise inner endomorphism is always injective. But it is not necessarily surjective. Indeed, let $S(\mathbb{N})$ be the finitary symmetric group on the set of
natural integers. In other words, $S(N)$ is the set of permutations of $N$ having finite support, with the operation of composition (the support of a permutation $f$ is the set $\{ n \in N \mid f(n) \neq n \}$). Consider the mapping $\theta : S(N) \to S(N)$, where for each element $f \in S(N)$, the permutation $\theta(f)$ is defined by

$$
\theta(f)(0) = 0, \quad \text{and} \quad \theta(f)(n) = 1 + f(n - 1) \quad \text{when} \quad n > 0
$$

($\theta$ could be called the $1$-right-shift). It is easy to verify that $\theta$ is an endomorphism of $S(N)$. Moreover, for any $f \in S(N)$, we have

$$
\theta(f) = (0 \ 1 \ 2 \ \ldots \ k \ k+1) \circ f \circ (0 \ 1 \ 2 \ \ldots \ k \ k+1)^{-1},
$$

where the integer $k$ is chosen such that the set $\{0, 1, 2, \ldots, k\}$ contains the support of $f$. Therefore, $\theta$ is a pointwise inner endomorphism. But $\theta$ is not surjective since $\theta(S(N)) = \{ f \in S(N) \mid f(0) = 0 \} \neq S(N)$.

However, we shall see that in a soluble group, each pointwise inner endomorphism is surjective (and so is an automorphism). In fact, we shall prove a slightly stronger result. For that, we shall consider the notion of pointwise inner endomorphism as a particular case of a more general concept.

2 Results

We denote by $A(G)$ the set of automorphisms of a group $G$ and by $I(G)$ the set of inner automorphisms. Let $K$ be a subset of $A(G)$. An endomorphism $\theta : G \to G$ is said to be $K$-pointwise if for each element $t \in G$, there exists an element $\varphi \in K$ such that $\theta(t) = \varphi(t)$. When $K$ is a subgroup, that means that for each $t \in G$, we have $\theta(t^K) \subseteq t^K$, where $t^K$ denotes the orbit of $t$ under the action of $K$. In particular, when $K = I(G)$, the notions of “pointwise inner endomorphism” and “$K$-pointwise endomorphism” coincide.

Obviously, any $K$-pointwise endomorphism is injective. We aim in this paper to show that under a suitable hypothesis on $G$ and $K$, each $K$-pointwise endomorphism is an automorphism. For example, suppose that $K$ is a finite subgroup of $A(G)$. A $K$-pointwise endomorphism $\theta$ induces an injection $\theta_t : t^K \to t^K$ for any $t \in G$. But since $t^K$ is finite, $\theta_t$ is bijective and so $\theta$ is an automorphism. It turns out that more generally, the result remains true when $K$ is a finite subset of $A(G)$.

Proposition 2.1. Let $G$ be a group and $K$ a finite subset of $A(G)$. Then each $K$-pointwise endomorphism $\theta : G \to G$ is an automorphism.

In addition, the proof of this result will show that there exist a subgroup $H \leq G$ of finite index and an automorphism $\varphi \in K$ such that $\theta(x) = \varphi(x)$ for all $x \in H$. If $|K| = 2$, $\theta$ is actually in $K$ but this is not always the case when $|K| \geq 3$ (see the remark following the proof of Proposition 2.1 in the next section).

Now recall that a normal automorphism $\varphi$ of a group $G$ is an automorphism such that $\varphi(H) = H$ for each normal subgroup $H$ of $G$. We shall denote by $A_n(G)$ the set of normal automorphisms of $G$. 
Proposition 2.2. In a soluble group $G$, each $A_n(G)$-pointwise endomorphism is an automorphism.

Notice that the automorphism obtained in the conclusion of this proposition is obviously a normal automorphism.

Since $A_n(G)$ contains $I(G)$, each pointwise inner endomorphism is also an $A_n(G)$-pointwise endomorphism. Thus, as an immediate consequence of Proposition 2.2, we can state:

Corollary 2.3. In a soluble group, each pointwise inner endomorphism is an automorphism.

As mentioned in the introduction, this automorphism is not necessarily inner. Proposition 2.2 cannot be extended to $A(G)$-pointwise endomorphisms. Indeed, consider for example the free abelian group of countably infinite rank. If \{\(e_0, e_1, e_2, \ldots\)\} is a free basis, the endomorphism $\theta$ defined by $\theta(e_i) = e_{i+1}$ \((i = 0, 1, 2, \ldots)\) is $A(G)$-pointwise but is not an automorphism. However, we have:

Proposition 2.4. Let $G$ be a soluble group of finite rank. Then each $A(G)$-pointwise endomorphism is an automorphism.

Recall that a group is said to be of finite rank (in the sense of Prüfer) if there is a positive integer $n$ such that every finitely generated subgroup can be generated by $n$ elements.

We do not know whether in a finitely generated soluble group $G$, each $A(G)$-pointwise endomorphism is an automorphism. Nevertheless, we can positively answer in a particular case:

Proposition 2.5. In a 2-generated nilpotent-by-abelian group $G$, each $A(G)$-pointwise endomorphism is an automorphism.

We end this section with a question:

Question. Suppose that $K = I(G)$ or $K = A(G)$. Is it possible to find a finitely generated group $G$ and a $K$-pointwise endomorphism $\theta : G \to G$ which is not an automorphism?

Obviously, a positive answer when $K = I(G)$ would imply a positive answer when $K = A(G)$.

3 Proofs

In the following, we shall use the evident fact that any $K$-pointwise endomorphism is injective.

The proof of the first lemma is straightforward and is omitted.

Lemma 3.1. Let $H$ be a normal subgroup of a group $G$. Let $\theta$ be an endomorphism of $G$ such that $\theta(H) \leq H$. Denote by $\overline{\theta}$ (respectively, $\theta_0$) the endomorphism induced by $\theta$ in $G/H$ (respectively, in $H$). If $\overline{\theta}$ and $\theta_0$ are surjective, then so is $\theta$. 

Lemma 3.2. In a group $G$, consider an injective endomorphism $\tau$. Suppose that the subgroup $H = \{ x \in G \mid \tau(x) = x \}$ is of finite index in $G$. Then $\tau$ is an automorphism.

Proof. We must prove that $\tau$ is surjective. Recall that the core $H_G$ of $H$ in $G$ is the normal subgroup of $G$ defined by $H_G = \bigcap_{x \in G} x^{-1} H x$. Since $H$ is of finite index in $G$, so is $H_G$. Clearly, $\tau$ induces in the subgroup $H_G$ the identity automorphism, and in the quotient $G/H_G$ an injective endomorphism. But this quotient is finite, so $\tau$ induces an automorphism in $G/H_G$. Hence, it follows from Lemma 3.1 that $\tau$ is surjective. \qed

Proof of Proposition 2.1. Suppose that $K = \{ \varphi_1, \ldots, \varphi_n \}$. For each integer $i = 1, \ldots, n$, consider the subgroup $\{ x \in G \mid \tau_i(x) = x \}$, where $\tau_i = \varphi_i^- \circ \theta$. Notice that $\tau_i$ is injective. Since $\theta$ is $K$-pointwise, we have $G = H_1 \cup \cdots \cup H_n$. It follows then from a well-known result of B.H. Neumann [5] that among the subgroups $H_1, \ldots, H_n$, at least one, say $H_j$, is of finite index in $G$. By Lemma 3.2, $\tau_j$ is an automorphism; hence so is $\theta$. \qed

Remark. We keep here the notation used in the proof of Proposition 2.1. If $n = 2$, then $G = H_1 \cup H_2$. It is well known and easy to see that such an equality implies $G = H_1$ or $G = H_2$. In other words, $\theta = \varphi_1$ or $\theta = \varphi_2$. On the other hand, if $K$ contains three elements, a $K$-pointwise endomorphism is not always in $K$. For example, let $G$ be the Klein 4-group. This group can be considered as a vector space of dimension 2 over the finite field of order 2. Let $\varphi_1, \varphi_2, \varphi_3$ be the automorphisms of $G$ corresponding to the matrices

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

Then, if we take $K = \{ \varphi_1, \varphi_2, \varphi_3 \}$, the identity automorphism $\theta = \text{Id}_G$ is a $K$-pointwise endomorphism but it is not in $K$.

Proof of Proposition 2.2. Let $\theta$ be an $A_n(G)$-pointwise endomorphism of $G$. We argue by induction on the derived length $r$ of $G$. First suppose that $r = 1$. It suffices to prove that $\theta$ is surjective. For that, consider an arbitrary element $y \in G$. There exists a normal automorphism $\varphi \in A_n(G)$ such that $\theta(y) = \varphi(y)$. Since $G$ is abelian, the subgroup $\langle y \rangle$ is normal and so $\varphi(\langle y \rangle) = \langle y \rangle$. It follows that $y = \varphi(y^m)$ for some integer $m$ and thus

$$y = \varphi(y)^m = \theta(y)^m = \theta(y^m).$$

Therefore, $\theta$ is surjective (and so is an automorphism).

Now consider a soluble group $G$ of derived length $r > 1$ and denote by $G^{(r-1)}$ its $(r-1)$th derived subgroup. Then $\theta$ induces in the quotient $G/G^{(r-1)}$ an endomorphism $\overline{\theta}$ which is clearly an $A_n(G/G^{(r-1)})$-pointwise endomorphism. Thus, by induction, we can say that $\overline{\theta}$ is an automorphism. By Lemma 3.1, the proof will be complete if we show that the endomorphism $\theta_0 : G^{(r-1)} \to G^{(r-1)}$ induced by $\theta$ in $G^{(r-1)}$ is surjective. Hence, consider an element $y \in G^{(r-1)}$. According to the
Clearly, since \( p \) is prime, \( G \) is a derived subgroup of \( \text{A}(G) \), which is in fact an automorphism. Consequently, we have \( \frac{G}{G'} \) is a torsion-free abelian group of finite rank. Hence, there is an element \( \theta \) such that \( \theta(v_i) = w_i \theta(v_i) \). Notice that \( w_i \) and \( \theta(v_i) \) are elements of \( G \). Thus, we may write
\[
y = \theta(v_1)^{-1} w_1^{-1} \theta(y) w_1 \theta(v_1) \cdots \theta(v_k)^{-1} w_k^{-1} \theta(y) w_k \theta(v_k).
\]
Hence, \( \theta_0 \) is surjective as required. 

Recall that the class of groups of finite rank is closed under taking subgroups and quotients. This property, easy to verify, will be freely used in the next proof.

**Proof of Proposition 2.4.** Let \( G \) be a soluble group of finite rank and let \( \theta \) be an \( \text{A}(G) \)-pointwise endomorphism. We argue by induction on the derived length of \( G \). First suppose that \( G \) is abelian. We must show that \( \theta \) is surjective. For each prime \( p \), denote by \( T_p \) the \( p \)-primary component of \( G \) and by \( T = \prod_p T_p \) its torsion-subgroup. Since \( T_p \) is of finite rank, it satisfies the minimal condition on subgroups [6, Theorem 4.3.13] and so is co-hopfian (namely, each injective endomorphism is an automorphism). It follows that \( \theta(T_p) = T_p \), therefore we have \( \theta(T) = T \). Clearly, since \( \theta \) is injective, so is the endomorphism \( \overline{\theta} \) of \( \overline{G} = G/T \) induced by \( \theta \). Moreover, \( G \) is a torsion-free abelian group of finite rank. Hence, there is a positive integer \( c \) such that \( \overline{G}^c \leq \overline{\theta(G)} \) and in addition, the quotient \( \overline{G}/\overline{G}^c \) is finite (see the proof of Theorem 15.2.3 in [6]). It follows that the quotient \( G/T \overline{G}^c \) (isomorphic to \( \overline{G}/\overline{G}^c \)) is finite. Therefore, \( \theta \) induces on \( G/T \overline{G}^c \) an \( \text{A}(G/T \overline{G}^c) \)-pointwise endomorphism which is in fact an automorphism. Consequently, we have the equality \( T \overline{G}^c = T \theta(G) \). But \( T \overline{G}^c \leq T \theta(G) \) for \( \overline{G}/\overline{G}^c \leq \overline{\theta(G)} \), so \( G = T \theta(G) \). Since \( T = \theta(T) \), we obtain \( G = \theta(G) \), as required.

Now suppose that \( G \) is soluble of derived length \( r > 1 \). Then \( \theta \) induces in \( G/G^{(r-1)} \) (respectively, in \( G^{(r-1)} \)) an \( \text{A}(G/G^{(r-1)}) \)-pointwise endomorphism (respectively, an \( \text{A}(G^{(r-1)}) \)-pointwise endomorphism). By induction, these endomorphisms are automorphisms. Then the result follows from Lemma 3.1.

**Proof of Proposition 2.5.** Let \( G \) be a 2-generated nilpotent-by-abelian group. Hence, the derived subgroup \( G' \) is nilpotent, of class say \( c \). If \( c = 0 \) (namely \( G' = \{1\} \),
the result follows from Proposition 2.4. Now suppose that \( c > 0 \) and argue by induction on \( c \). Let \( \theta \) be an \( \Lambda(G) \)-pointwise endomorphism. If \( G \) is generated by \( a \) and \( b \), consider the commutator \( d = [a, b] = a^{-1}b^{-1}ab \). By assumption, there exists an automorphism \( \varphi \in \Lambda(G) \) such that \( \theta(d) = \varphi(d) \). Therefore, if \( \tau = \varphi^{-1} \circ \theta \), we have \( \tau(d) = d \). Obviously, \( \tau \) is an \( \Lambda(G) \)-pointwise endomorphism. Moreover, if \( \gamma_c(G') \) denotes the \( c \)th term of the lower central series of \( G' \), \( \tau \) induces in \( G/\gamma_c(G') \) an \( \Lambda(G/\gamma_c(G')) \)-pointwise endomorphism \( \tau' \). By the inductive hypothesis, \( \tau' \) is an automorphism and so \( G = \gamma_c(G').\tau'(G) \). Since the derived subgroup \( G' \) is clearly the normal closure of \( d \) in \( G \), it is generated by the set of elements of the form

\[
\tau(v)^{-1}w^{-1}dw\tau(v) = \tau(v)^{-1}d\tau(v) = \tau(v)^{-1}dv \quad (v \in G, \ w \in \gamma_c(G')).
\]

It follows that \( \tau(G) \) contains \( G' \). Since \( G = \gamma_c(G').\tau(G) \), we obtain \( G = \tau(G) \). Hence, \( \tau \) is an automorphism and so is \( \theta = \varphi \circ \tau \).

References


