FINITE GROUPS WITH A CERTAIN NUMBER OF ELEMENTS PAIRWISE GENERATING A NON-NILPOTENT SUBGROUP†

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ABSTRACT. Let \( n > 0 \) be an integer and \( \mathcal{X} \) be a class of groups. We say that a group \( G \) satisfies the condition \((\mathcal{X}, n)\) whenever in every subset with \( n + 1 \) elements of \( G \) there exist distinct elements \( x, y \) such that \( \langle x, y \rangle \) is in \( \mathcal{X} \). Let \( \mathcal{N} \) and \( \mathcal{A} \) be the classes of nilpotent groups and abelian groups, respectively. Here we prove that: (1) If \( G \) is a finite semi-simple group satisfying the condition \((\mathcal{N}, n)\), then \( |G| < c^{2[\log_{21} n]^2[\log_{21} n]!} \), for some constant \( c \). (2) A finite insoluble group \( G \) satisfies the condition \((\mathcal{N}, 21)\) if and only if \( G/Z^*(G) \cong A_5 \), the alternating group of degree 5, where \( Z^*(G) \) is the hypercentre of \( G \). (3) A finite non-nilpotent group \( G \) satisfies the condition \((\mathcal{N}, 4)\) if and only if \( G/Z^*(G) \cong S_3 \), the symmetric group of degree 3. (4) An insoluble group \( G \) satisfies the condition \((\mathcal{A}, 21)\) if and only if \( G \cong Z(G) \times A_5 \), where \( Z(G) \) is the centre of \( G \). (5) If \( d \) is the derived length of a soluble group satisfying the condition \((\mathcal{A}, n)\), then \( d = 1 \) if \( n \in \{1, 2\} \) and \( d \leq 2n - 3 \) if \( n \geq 2 \).

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1. Introduction and results

Let $n > 0$ be an integer and $\mathcal{X}$ be a class of groups. We say that a group $G$ satisfies the condition $(\mathcal{X}, n)$ whenever in every subset with $n+1$ elements of $G$ there exist distinct elements $x, y$ such that $\langle x, y \rangle$ is in $\mathcal{X}$. If $\mathcal{X}$ is subgroup-closed, then every group which is the union of $n$ $\mathcal{X}$-subgroups satisfies the condition $(\mathcal{X}, n)$. Let $\mathcal{N}$ be the class of nilpotent groups. Tomkinson in [23] proved that if $G$ is a finitely generated soluble group satisfying the condition $(\mathcal{N}, n)$, then $|G/Z^*(G)| < n^{3n}$, where $Z^*(G)$ is the hypercentre of $G$. This result gives a bound for the size of every finite soluble centerless group satisfying the condition $(\mathcal{N}, n)$; on the other hand, Endimioni in [10] proved that if $n \leq 20$, then every finite group satisfying the condition $(\mathcal{N}, n)$ is soluble, and $A_5$, the alternating group of degree 5, satisfies the condition $(\mathcal{N}, 21)$. Hence for $n \leq 20$ and all soluble groups, we have a positive answer to the following question:

Does there exist a bound (depending only on $n$) for the size of every centerless finite group satisfying the condition $(\mathcal{N}, n)$?

Here we find a bound for the size of finite semi-simple groups satisfying the condition $(\mathcal{N}, n)$ and also for all finite centerless groups satisfying the condition $(\mathcal{N}, 21)$. We also obtain a characterization for $A_5$ (see Corollary 2.11, below). The main results are

**Theorem 1.1.** Let $G$ be a finite semi-simple group satisfying the condition $(\mathcal{N}, n)$. Then $|G| < c^{2[\log_{21} n]^n [\log_{21} n]}$, for some constant $c$.

**Theorem 1.2.** Let $G$ be a finite insoluble group. Then $G$ satisfies the condition $(\mathcal{N}, 21)$ if and only if $G/Z^*(G) \cong A_5$.

In [10] Endimioni proved that if $n \leq 3$, then every finite group satisfying the condition $(\mathcal{N}, n)$ is nilpotent, and $S_3$, the symmetric group of degree 3, satisfies the condition $(\mathcal{N}, 4)$. In fact, the only non-trivial finite centerless group satisfying the condition $(\mathcal{N}, 4)$ is
$S_3$. In section 2, we investigate finite groups satisfying the condition $(\mathcal{N}, 4)$.

**Theorem 1.3.** Let $G$ be a non-nilpotent finite group. Then $G$ satisfies the condition $(\mathcal{N}, 4)$ if and only if $G/Z^*(G) \cong S_3$.

It follows from Corollaries 2.12 and 3.4 below that a finite group satisfies the condition $(\mathcal{N}, 4)$ (respectively, $(\mathcal{N}, 21)$) if and only if it is the union of 4 (respectively, 21) nilpotent subgroups. Another natural question is: “For which positive integers $n$ is every finite group satisfying the condition $(\mathcal{N}, n)$ the union of $n$ nilpotent subgroups?”

In section 3, we investigate (not necessarily finite) groups satisfying the condition $(\mathcal{A}, n)$, where $\mathcal{A}$ is the class of abelian groups. Indeed, in a group satisfying the condition $(\mathcal{A}, n)$, the largest set of non-commuting elements (or the largest set of elements in which no two generate an abelian subgroup) has size at most $n$. By a result of B.H. Neumann [19] a group satisfies the condition $(\mathcal{A}, n)$ for some $n \in \mathbb{N}$ if and only if it is centre-by-finite. In fact, Neumann answered affirmatively the following question of P. Erdős [19]: Let $G$ be an infinite group. If there is no infinite subset of $G$ whose elements do not mutually commute, is there then a finite bound on the cardinality of each such set of elements? Neumann [19] proved that a group has the condition of Erdős’s question if and only if it is centre-by-finite. This result has initiated a great deal of research towards the determination of the structure of groups having some similar properties (for example see [1],[2],[3],[4],[5],[8],[9],[11],[13],[16],[17],[18],[22]).

Pyber in [20] gave a bound for the index of the centre of a group satisfying the condition $(\mathcal{A}, n)$. Here we characterize insoluble groups satisfying the condition $(\mathcal{A}, 21)$. Note that every group satisfying the condition $(\mathcal{A}, n)$ also satisfies the condition $(\mathcal{N}, n)$. 
Theorem 1.4. Let $G$ be an insoluble group. Then $G$ satisfies the condition $(\mathcal{A}, 21)$ if and only if $G \cong Z(G) \times A_5$.

We also obtain a result which is of independent interest, namely, the derived length of soluble groups satisfying the condition $(\mathcal{A}, n)$ is bounded by a function depending only on $n$.

Theorem 1.5. Let $G$ be a soluble group satisfying the condition $(\mathcal{A}, n)$ and let $d$ be the derived length of $G$. Then $d = 1$ if $n \in \{1, 2\}$ and $d \leq 2n - 3$ if $n \geq 2$.

2. Semi-simple groups satisfying the condition $(\mathcal{N}, n)$ and insoluble groups satisfying the condition $(\mathcal{N}, 21)$

Recall that a group $G$ is semi-simple if $G$ has no non-trivial normal abelian subgroups. If $G$ is a finite group then we call the product of all minimal normal non-abelian subgroups of $G$ the centerless CR-radical of $G$; it is a direct product of non-abelian simple groups (see page 88 of [21]).

We first prove a result on the direct product of (not necessarily finite) groups not satisfying the condition $(\mathcal{X}, n)$, for a certain class $\mathcal{X}$ of groups. This result may also be useful in other investigations on groups satisfying the condition $(\mathcal{X}, n)$. For example, if one can find a bound depending only on $n$ for the size of finite non-abelian simple groups satisfying the condition $(\mathcal{X}, n)$, then by the aid of Lemma 2.1 below, it is easy to see that there exists a bound depending only on $n$ for the size of every semi-simple finite group satisfying the condition $(\mathcal{X}, n)$ (for instance see Theorem 1.1).

Lemma 2.1. Let $\mathcal{X}$ be a class of groups which is closed with respect to homomorphic images. Suppose for $i \in \{1, \ldots, t\}$ that $H_i$ is a group not satisfying the condition $(\mathcal{X}, n_i)$. Then $H_1 \times \cdots \times H_t$ does not satisfy the condition $(\mathcal{X}, m)$, where $m = n_1 + \cdots + n_t$. 
Proof. It suffices to show that if $H$ and $K$ are two groups which do not satisfy $(X, n)$ and $(X, m)$, respectively, then $H \times K$ does not satisfy the condition $(X, n + m)$. By the hypothesis, there exist $x_1, \ldots, x_{n+1}$ in $H$ and $y_1, \ldots, y_{m+1}$ in $K$ such that $\langle x_i, x_j \rangle \notin X$ for $1 \leq i < j \leq n+1$ and $\langle y_k, y_l \rangle \notin X$ for $1 \leq k < l \leq m+1$. Now it is easy to see that the subgroup generated by each pair of distinct elements of the set
\[
\{(x_2, 1), \ldots, (x_{n+1}, 1), (x_1, y_1), (x_1, y_2), \ldots, (x_1, y_{m+1})\},
\]
does not have the property $X$. □

Our next lemma is about the direct product of finite groups not satisfying $(N, n)$. For finite groups, this is a better result than Lemma 2.1.

Lemma 2.2. Suppose that $H_i$ is a finite group not satisfying the condition $(N, n_i)$ for $i \in \{1, \ldots, t\}$. Then $H_1 \times \cdots \times H_t$ does not satisfy the condition $(N, m)$, where $m = (n_1 + 1) \cdots (n_t + 1) - 1$.

Proof. By the hypothesis, for every $i \in \{1, \ldots, t\}$ there exists a subset $X_i$ in $H_i$ of size $n_i + 1$ such that no pair of its distinct elements generate a nilpotent subgroup. Now we show that the subgroup generated by each pair of distinct elements of the set $X = X_1 \times \cdots \times X_t$ is not nilpotent. Let $a = (a_1, \ldots, a_t), b = (b_1, \ldots, b_t)$ be two distinct elements of $X$. Then for some $i \in \{1, \ldots, t\}, a_i \neq b_i$. Since $a_i, b_i \in X_i$, we have that $K := \langle a_i, b_i \rangle$ is not nilpotent. Since $K$ is a finite non-nilpotent group, it is not an Engel group by a result of Zorn (see Theorem 12.3.4 of [21]). Therefore there exist elements $x, y \in K$ such that $[x, n y] \neq 1$ for all $n \in \mathbb{N}$. Suppose that
\[
x = a_i^{\delta_1} b_i^{\epsilon_2} \cdots a_i^{\delta_{r-1}} b_i^{\epsilon_r} \quad \text{and} \quad y = a_i^{\epsilon_1} b_i^{\epsilon_2} \cdots a_i^{\epsilon_{s-1}} b_i^{\epsilon_s},
\]
where $\delta_p, \epsilon_q \in \{0, 1, -1\}$ for all $p \in \{1, \ldots, r\}$ and $q \in \{1, \ldots, s\}$. Suppose, for a contradiction, that $\langle a, b \rangle$ is nilpotent. Then there exists a positive integer $m$ such that $[\bar{x}, n \bar{y}] = 1$ where
\[
\bar{x} = a_i^{\delta_1} b_i^{\delta_2} \cdots a_i^{\delta_{r-1}} b_i^{\delta_r} \quad \text{and} \quad \bar{y} = a_i^{\epsilon_1} b_i^{\epsilon_2} \cdots a_i^{\epsilon_{s-1}} b_i^{\epsilon_s}.
\]
But
\[[x_m y] = ([x_1 m y_1], \ldots, [x_t m y_t]),\]
where
\[x_j = a_j^\delta_1 b_j^{\delta_1} \cdots a_j^{\delta_{r-1}} b_j^{\delta_{r-1}} \text{ and } y_j = a_j^{\epsilon_1} b_j^{\epsilon_1} \cdots a_j^{\epsilon_{s-1}} b_j^{\epsilon_{s-1}}\]
for all \( j \in \{1, \ldots, t\} \). Hence \([x_m y] = [x_i m y_i] = 1\), a contradiction. This completes the proof. □

**Lemma 2.3.** Let \( M_1, \ldots, M_m \) be non-abelian finite simple groups. Then \( M_1 \times \cdots \times M_m \) does not satisfy the condition \((N, 21^m - 1)\).

**Proof.** Since by Proposition 2 of [10], \( M_i \) does not satisfy the condition \((N, 20)\) for all \( i \in \{1, \ldots, m\} \), the proof follows easily from Lemma 2.2. □

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( R \) be the centerless CR-radical of \( G \). Then \( R \) is a direct product of a finite number \( m \) of finite non-abelian simple groups and \( G \) is embedded in \( \text{Aut}(R) \). Then by Lemma 2.3, we have \( 21^m - 1 < n \) and so \( m \leq \lceil \log_{21} n \rceil \). On the other hand, since \( Z(G) = 1 \), by Lemma 3.3 of [23] every prime divisor of \( |G| \) is less than \( n \). Thus by Remark 5.5 of [6], there is a constant \( c \) such that the order of every non-abelian simple section of \( G \) is less than \( c \sqrt{n} \). Hence \( |G| < 2^{\sqrt{n} \lceil \log_{21} n \rceil} \). Now using the following well-known facts that: (a) for a finite simple group \( S \) we have \( |\text{Aut}(S)| < |S|^2 \) and (b) if \( R \) is the product of \( m \) simple groups \( S_i \), then \( G \) acts on these factors, the quotient group is embeddable into \( \text{Sym}(m) \) and the kernel \( K \) of the action is embeddable into the product of groups \( \text{Aut}(S_i) \); hence \( |K| < |R|^2 \). Thus \( |G| < 2^{2 \sqrt{n} \lceil \log_{21} n \rceil} \). □

Since in every finite group \( G \), the quotient \( G/\text{Sol}(G) \) is semisimple, where \( \text{Sol}(G) \) is the soluble radical (the largest soluble normal subgroup) of \( G \), we have
**Corollary 2.4.** Let $G$ be a finite group satisfying $(N, n)$. Then

$$|G/\text{Sol}(G)| < c^{2n^2[\log_{21} n][\log_{21} n]}$$

for some constant $c$.

Combining the result of Tomkinson quoted in the introduction and Corollary 2.4, we obtain as a further nice corollary that in fact:

**Corollary 2.5.** Let $G$ be a finite group satisfying $(N, n)$. Then

$$|G/F(G)| < n^{n^4}c^{2n^2[\log_{21} n][\log_{21} n]}$$

for some constant $c$, where $F(G)$ is the largest nilpotent normal subgroup of $G$.

We need the following proposition, which is of independent interest, in the proof of Proposition 2.7.

**Proposition 2.6.** Let $p$ be a prime number, $n$ a positive integer and $r$ and $q$ be two odd prime numbers dividing respectively $p^n + 1$ and $p^n - 1$. Then the number of Sylow $r$-subgroups (respectively, $q$-subgroups) of $G = \text{PSL}(2, p^n)$ is $\frac{p^n(p^n-1)}{2}$ (respectively, $\frac{p^n(p^n+1)}{2}$). Also the intersection of every two distinct Sylow $r$-subgroups or $q$-subgroups is trivial.

**Proof.** Our proof uses Theorems 8.3 and 8.4 in chapter II of [14]. Let $q$ be an odd prime dividing $p^n - 1$ and let $k = \gcd(p^n - 1, 2)$. By Theorem 8.3 in Chapter II of [14], $\text{PSL}(2, p^n)$ possesses a cyclic subgroup $U$ of order $u = \frac{p^n-1}{k}$ such that

1. The intersection of every two distinct conjugates of $U$ is trivial.
2. For every non-trivial element $w$ of $U$, the normalizer $N_G(\langle w \rangle)$ of $\langle w \rangle$ is a dihedral group of order $2u$.

Since $q$ is an odd prime number, $q$ divides $u$, and since $|G| = \frac{p^n(p^n+1)(p^n-1)}{k}$, we have $\gcd(p^n(p^n+1), q) = 1$. It follows that any Sylow $q$-subgroup of $U$ is also a Sylow $q$-subgroup of $G$ and each of them is cyclic. Therefore it follows from (2) that the number of Sylow $q$-subgroups of $G$ is $\frac{p^n(p^n+1)}{2}$. Now (1) implies that the
intersection of every two distinct Sylow $q$-subgroups of $G$ is trivial. By a similar argument the second statement of the proposition follows from the corresponding parts of Theorem 8.4 in Chapter II of [14], namely that the group $G$ contains a cyclic subgroup $K$ of order $s = \frac{p^n+1}{k}$ such that

(1) The intersection of every two distinct conjugates of $K$ is trivial.

(2) For every non-trivial element $t$ of $K$, the normalizer $N_G(\langle t \rangle)$ of $\langle t \rangle$ is a dihedral group of order $2s$. □

**Proposition 2.7.** The only non-abelian finite simple group satisfying the condition $(N, 21)$ is $A_5$.

**Proof.** Suppose, for a contradiction, that there exists a non-abelian finite simple group satisfying the condition $(N, 21)$ which is not isomorphic to $A_5$. Let $G$ be such a group of least order. Thus every proper non-abelian simple section of $G$ is isomorphic to $A_5$. Therefore by Proposition 3 of [7], $G$ is isomorphic to one of the following:

PSL(2, $2^p$), $p = 4$ or a prime;  
PSL(2, $3^p$), PSL(2, $5^p$), $p$ a prime;  
PSL(2, $p$), $p$ a prime $\geq 7$;  
PSL(3, 3), PSL(3, 5);  
PSU(3, 4) (the projective special unitary group of degree 3 over the finite field of order $4^2$) or  
Sz$(2^p)$, $p$ an odd prime.

For each prime divisor $p$ of $|G|$, let $\nu_p(G)$ be the number of all Sylow $p$-subgroups of $G$. If $p$ is a prime number dividing $|G|$ such that the intersection of any two distinct Sylow $p$-Subgroups is trivial, then by Lemma 3 of [10], $\nu_p(G) \leq 21$ (*). Now, for every prime number $p$ and every integer $n > 0$, we have $\nu_p(\text{PSL}(2, p^n)) = p^n + 1$ and the intersection of any two distinct Sylow $p$-subgroups is trivial (see chapter II Theorem 8.2 (b),(c) of [14]). Thus among the projective special linear groups, we only need to investigate the following groups:

PSL(2, $3^2$), PSL(2, 8), PSL(2, $2^4$), PSL(3, 3), PSL(3, 5), PSL(2, $p$)
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for \( p \in \{7, 11, 13, 17, 19\} \). Now if in Proposition 2.6, we take \( q = 7 \) for \( \text{PSL}(2,8) \); \( q = 5 \) for \( \text{PSL}(2,16) \); \( r = 5 \) for \( \text{PSL}(2,9) \); \( q = 3 \) for \( \text{PSL}(2,7) \), \( \text{PSL}(2,13) \) and \( \text{PSL}(2,19) \); and \( r = 3 \) for \( \text{PSL}(2,11) \) and \( \text{PSL}(2,17) \); we see, by (\(*)\), that \( G \) cannot be isomorphic with any of these groups. Therefore we must consider the groups \( \text{PSL}(3,3) \), \( \text{PSL}(3,5) \), \( \text{PSU}(3,4) \) or \( \text{Sz}(2^p) \), \( p \) an odd prime.

\( H := \text{PSL}(3,3) \) has order \( 2^4 \times 3^3 \times 13 \), so \( \nu_{13}(H) = 1 + 13k \), for some \( k > 0 \) and since 14 does not divide \( |H| \), \( \nu_{13}(H) > 26 \).

\( K := \text{PSL}(3,5) \) has order \( 5^3 \times 2^5 \times 3 \times 31 \), so \( \nu_{31}(K) = 1 + 31k > 21 \) for some \( k > 0 \).

\( L := \text{PSU}(3,4) \) has order \( 2^6 \times 5^2 \times 13 \) (see Theorem 10.12(d) of chapter II in [14] and note that \( L \) is the projective special unitary group of degree 3 over the finite field of order \( 4^2 \)). Therefore \( \nu_{13}(L) = 1 + 13k > 21 \) for some \( k > 0 \) and since 14 does not divide \( |L| \), \( \nu_{13}(L) > 26 \).

\( M := \text{Sz}(2^p) \) (\( p \) an odd prime) has order \( 2^{2p}(2^p - 1)(2^p + 1) \) and \( \nu_2(M) = 2^p + 1 \geq 65 \) (see Theorem 3.10 (and its proof) of chapter XI in [15]). This completes the proof by (\(*)\). □

**Lemma 2.8.** \( S_5 \), the symmetric group of degree 5, does not satisfy the condition \( (N,21) \).

**Proof.** Every subgroup generated by a pair of distinct elements of 22-element subset
\[
\{(3,4,5), (2,3,4), (2,3,4,5), (1,4,5), (2,3,5,4), (2,3,5), (2,4,5), (1,2,3), (1,2,3,4), (1,2,4,5,3), (1,2,4,3,5), (1,2,5), (1,3,4,5), (1,3,5), (1,3,2,4,5), (1,4,2), (1,5,4,3,2), (1,5,3,2), (1,5,4,2), (1,5,2,4,3), (1,5,3,2,4)\}
\] is not nilpotent. □

**Remark 2.9.** Here we state two properties of \( A_5 \) which we use in the sequel. Suppose that \( P_1, \ldots , P_{21} \) are all the Sylow subgroups of \( A_5 \). Then

(i) For all \( x_i \in P_i \setminus \{1\} \ (i = 1, \ldots , 21) \), the set \( \{x_1, \ldots , x_{21}\} \) is a subset of \( A_5 \) such that no pair of its distinct elements generate a
nilpotent subgroup. (See the proof of Proposition 2 of [10]).

(ii) $A_5 = \bigcup_{i=1}^{21} P_i$.

We use the following fact in the sequel without any specific reference. If $G$ is any group such that $G/Z_m(G)$ is nilpotent for some integer $m \geq 0$, then $G$ is nilpotent. Because $Z_n(G/Z_m(G)) = G/Z_m(G)$ for some integer $n \geq 0$ and so by Theorem 5.1.11 (iv) of [21], we have $Z_{m+n}(G) = G$, which implies that $G$ is nilpotent.

**Lemma 2.10.** Let $G$ be a finite insoluble group satisfying the condition $(\mathcal{N}, 21)$ and let $S = \text{Sol}(G)$ be the soluble radical of $G$. Then $G/S \cong A_5$, and for all $a \in S$ and for all $x \in G\setminus S$, the subgroup $\langle a, x \rangle$ is nilpotent. In particular, $Z^*(G) = Z^*(S)$.

**Proof.** Let $S$ be the soluble radical of $G$ and consider the semisimple group $\overline{G} = G/S$. Let $\overline{R}$ be the centerless CR-radical of $\overline{G}$. Then $\overline{R}$ is a direct product of non-abelian simple groups. Since $G$ is insoluble, $\overline{R}$ is non-trivial. Now, by Lemma 2.3 and Proposition 2.7, $\overline{R} \cong A_5$. Since $C_{\overline{G}}(\overline{R}) = 1$, we have $\overline{G} \cong A_5$ or $S_5$. By Lemma 2.8, $\overline{G} \cong A_5$. Now, let $Q_1, \ldots, Q_{21}$ be the Sylow subgroups of $G/S$. For each $i \in \{1, \ldots, 21\}$, let $x_iS$ be a non-trivial element of $Q_i$. Then, by Remark 2.9(i), $\langle x_i, x_j \rangle_S \not\in \mathcal{N}$ and so $\langle x_i, x_j \rangle \notin \mathcal{N}$ for all distinct $i, j \in \{1, \ldots, 21\}$. Now, fix $k \in \{1, \ldots, 21\}$ and for an arbitrary element $a \in S$ consider the elements

$$x_k, x_1, \ldots, x_{k-1}, ax_k, x_{k+1}, \ldots, x_{21}.$$ 

For $k, j \in \{1, \ldots, 21\}$ and $j \neq k$, $\langle ax_k, x_j \rangle$ is not nilpotent, since $\langle ax_k, x_j \rangle S = \langle x_k, x_j \rangle S$. Since $G$ satisfies the condition $(\mathcal{N}, 21)$, the subgroup $\langle x_k, ax_k \rangle$ is nilpotent and hence so is $\langle a, x_k \rangle$ for all $k \in \{1, \ldots, 21\}$. On the other hand, the union of the subgroups $Q_1, \ldots, Q_{21}$ is $G/S$, by Remark 2.9(ii), and so $\langle a, x \rangle$ is nilpotent for all $x \in G\setminus S$ and for all $a \in S$.

Since $S$ is finite, $Z^*(S) = Z_m(S)$ for some $m \in \mathbb{N}$. Now for all $a \in Z_m(S)$ and for all $b \in S$, the subgroup $T := \langle a, b \rangle$ is nilpotent, since $TZ_m(S)/Z_m(S) \cong T/(T \cap Z_m(S))$ is cyclic and $T \cap Z_m(S) \leq Z_m(T)$.  

Thus $\langle a, x \rangle$ is nilpotent for all $a \in Z^*(S)$ and for all $x \in G$. Since $G$ is finite, $a$ is a right Engel element for all $a \in Z^*(S)$ (see Theorem 12.3.7 of [21]) and so $Z^*(S) \leq Z^*(G)$. Hence $Z^*(S) = Z^*(G)$. This completes the proof. □

Proof of Theorem 1.2. Suppose that $G$ satisfies the condition $(\mathcal{N}, 21)$ and suppose, for a contradiction, that $G$ is a counterexample of least order subject to $G/Z^*(S) \not\leq A_5$. Let $S = \text{Sol}(G)$ be the soluble radical of $G$. We claim that $Z(S) = 1$. For if $Z(S) \neq 1$ then $G/Z(S)$ is a finite insoluble group satisfying the condition $(\mathcal{N}, 21)$ and since $|G/Z(S)| < |G|$ and the soluble radical of $G/Z(S)$ is $S/Z(S)$, we have that the assertion of Theorem 1.2 is true for the group $G/Z(S)$, i.e.

$$\frac{G/Z(S)}{Z^*(G/Z(S))} \cong A_5.$$  \hfill (*)

Now Lemma 2.10 implies that $Z^*(S/Z(S)) = Z^*(G/Z(S))$. On the other hand

$$Z^*(S/Z(S)) = Z^*(S)/Z(S) = Z^*(G)/Z(S),$$

by Lemma 2.10 (note that for a finite group $K$ we have $Z^*(K) = Z_m(K)$ for some integer $m > 0$). Thus it follows from (*) that $G/Z^*(G) \cong A_5$ which is a contradiction. Hence $Z(S) = 1$, which implies that $Z^*(S) = 1$.

Now, let $x \in G \setminus S$ be such that $x^2 \in S$. Thus for all $b \in S$, we have $bx \in G \setminus S$ and $(bx)^2 \in S$. By Lemma 2.10, $\langle bx, a \rangle$ is nilpotent for all $a \in S$, and so also is $\langle (bx)^2, a \rangle$. Therefore $(bx)^2$ is a right Engel element of $S$ and so $(bx)^2 \in Z^*(S) = 1$. Thus for all $b \in S$, $(bx)^2 = 1$. Now, again by Lemma 2.10, $\langle bx, x \rangle = \langle b, x \rangle$ is nilpotent and so is $\langle b, x^2 \rangle$. Thus as before $x^2 = 1$. Therefore $D := \langle b, x \rangle$ is a finite dihedral group which is nilpotent and so $|D|$ is a power of 2 and $b$ is a 2-element. Hence $S$ is a 2-group, and since $Z(S) = 1$, we conclude that $S$ must be trivial. Therefore, by Lemma 2.10, $Z^*(G) = 1$ and $G/Z^*(G) = G/S \cong A_5$, a contradiction.
Conversely, suppose that $G/Z^*(G) \cong A_5$. By Remark 2.9(ii),
\[
\frac{G}{Z^*(G)} = \bigcup_{i=1}^{21} \frac{P_i}{Z^*(G)},
\]
where $P_1/Z^*(G), \ldots, P_{21}/Z^*(G)$ are the Sylow subgroups of $G/Z^*(G)$. But $G$ is finite, so $Z^*(G) = Z_m(G)$ for some $m \in \mathbb{N}$. Since $Z_m(G) \leq Z_m(P_i)$ for all $i \in \{1, \ldots, 21\}$ and $P_i/Z_m(G)$ is nilpotent, we conclude that each $P_i$ is nilpotent. Now the proof is complete, since $G = \bigcup_{i=1}^{21} P_i$. \qed

From Theorem 1.2 we have a nice characterization for $A_5$.

**Corollary 2.11.** The only finite centerless insoluble group satisfying the condition $(N, 21)$ is $A_5$.

Theorem 1.2 also gives us the following consequences.

**Corollary 2.12.** A finite insoluble group satisfies the condition $(N, 21)$ if and only if it is covered by 21 nilpotent subgroups.

**Corollary 2.13.** Let $G$ be a finite group satisfying the condition $(N, 21)$. If the centerless CR-radical of $G$ is non-trivial, then $G \cong A_5 \times Z^*(G)$.

**Proof.** Let $R$ be the centerless CR-radical of $G$. Then $R$ is a non-trivial direct product of some non-abelian simple groups and so by Lemma 2.3 and Proposition 2.7, $R \cong A_5$. Since $R$ is simple, $R \cap Z^*(G) = 1$. But, by Theorem 1.2, $|G| = |Z^*(G)||A_5|$, and so $G \cong A_5 \times Z^*(G)$. \qed

**Remark 2.14.** We note that not every finite insoluble group satisfying the condition $(N, 21)$ is necessarily isomorphic to a direct product as in Corollary 2.13. For example if $K := SL(2, 5)$ then $K/Z(K) \cong A_5$ and so $K$ satisfies the condition $(N, 21)$, by Theorem 1.2. However we conjecture that every finite insoluble group
satisfying the condition \((N,21)\) is a direct product of a nilpotent group and a group isomorphic to either \(A_5\) or \(\text{SL}(2,5)\).

3. Finite groups satisfying the condition \((N,4)\)

In this section, we investigate finite groups satisfying the condition \((N,4)\), and give the proof of Theorem 1.3.

**Lemma 3.1.** Let \(G\) be a finite \(\{2,3\}\)-group. If \(G\) satisfies the condition \((N,4)\), then \(G\) is \(2\)-nilpotent.

**Proof.** Suppose that \(G\) is a counterexample of least order. Thus by a result of Itô (see Theorem 5.4 on page 434 of [14]), \(G\) is a minimal non-nilpotent group and \(G\) has a unique Sylow 2-subgroup \(P\) and a cyclic Sylow 3-subgroup \(Q\) such that \(\Phi(Q) \leq Z(G)\) and \(\Phi(P) \leq Z(G)\) (see Theorem 5.2 on page 281 of [14]). If \(Z(G) \neq 1\) then \(G/Z(G)\) is nilpotent and so \(G\) is nilpotent, a contradiction. Thus \(Z(G) = 1\) and so \(|Q| = 3\) and \(P\) is an elementary abelian 2-group.

Let \(Q = \langle a \rangle\). Then \(C_P(a) \leq Z(G)\), and so \(C_P(a) = 1\). On the other hand by Lemma 3.4 of [23], \(|P : C_P(a)| \leq 4\) and so \(|P| \leq 4\). If \(|P| = 4\) then \(G \cong A_4\), the alternating group of degree 4. But \(A_4\) does not satisfy the condition \((N,4)\); thus \(|P| = 2\). Therefore \(G \cong S_3\), a contradiction, since \(S_3\) is \(2\)-nilpotent. This completes the proof. \(\square\)

**Lemma 3.2.** Let \(G = RX\) be an extension of an elementary abelian 3-group \(R\) by an abelian 2-group \(X\) such that \(X\) acts faithfully on \(R\) and \(R = [R, X]\). If \(G\) satisfies the condition \((N,4)\), then \(|X| \leq 2\) and \(|R| \leq 3\).

**Proof.** The proof follows from the argument of Lemma 3.7 of [23]. \(\square\)
We are now ready to give a proof for Theorem 1.3, the outline of which is in fact a refinement of that of Theorem C in [23] for \( n = 4 \).

**Proof of Theorem 1.3.** Suppose that \( G \) satisfies the condition \((N,4)\). By factoring out \( Z^*(G) \), we may assume that \( G \) is a finite non-trivial group with trivial centre satisfying the condition \((N,4)\). We note that \( G \) is a \( \{2,3\} \)-group by Lemma 3.3 of [23].

Let \( H_p/O_p(G) \) be the hypercentre of \( G/O_p(G) \), for \( p = 2, 3 \). Then, since \( G \) is finite, there is a positive integer \( m \) such that
\[
[H_2 \cap H_3, m G] \leq O_p(G) \cap O_3(G) = 1,
\]
and so \( H_2 \cap H_3 \leq Z^*(G) = 1 \). But \( O_2(G) = O_3(G) \) and by Lemma 3.1, is the unique Sylow 3-subgroup of \( G \). Thus \( G/O_2(G) \) is a 2-group and so \( G = H_2 \). Therefore \( H_3 = 1 \) and so \( O_2(G) = 1 \). Hence \( P = \text{Fitt}(G) = O_3(G) \). Let \( \overline{G} = G/\Phi(P) \) and \( \overline{P} = P/\Phi(P) \), thus \( \overline{G}/\overline{P} \) acts faithfully on the GF(3)-vector space \( \overline{P} \) (see [12], Theorem 6.3.4). We note that \( \overline{P} \) is an elementary abelian normal 3-subgroup of \( \overline{G} \), that \( \overline{P} = O_3(\overline{G}) \), and that \( C_{\overline{G}}(\overline{P}) = \overline{P} \). Let \( Q/\overline{P} \) be the socle of \( \overline{G}/\overline{P} \), so that \( Q/\overline{P} \) is an abelian 2-subgroup. We may write \( Q = \overline{P}X \), where \( X \) is an abelian 2-subgroup of \( Q \). Let \( R = [\overline{P}, Q] \), so that \( \overline{P} = R \times C_{\overline{P}}(Q) \). If \( C = C_{\overline{G}}(R) \) then \( C \cap Q \) centralizes \( R \times C_{\overline{P}}(Q) = \overline{P} \) and so \( C \cap Q = \overline{P} \). It follows that \( C_{\overline{G}}(R) = \overline{P} \) and so \( \overline{G}/\overline{P} \) acts faithfully on \( R \). Now \( R \) and \( X \) satisfy the conditions of Lemma 3.2 and so \( |R| \leq 3 \). Since \( \overline{G}/\overline{P} \) acts faithfully on \( R \), the order of \( G/P \) is no more than 2. Let \( T \) be a Sylow 2-subgroup of \( G \); then \( |T| \leq 2 \) and hence \( T \) is cyclic and by Lemma 3.4 of [23], \( |P : C_P(T)| \leq 3 \). Now, we have \( |C_{P}(T)_{m}P| = |C_{P}(T)_{m}P| = 1 \) for some \( m \in \mathbb{N} \). Thus \( C_{P}(T) \leq Z^*(G) = 1 \) and so \( |G| = |T||P| \leq 2 \times 3 = 6 \). Therefore \( G \cong S_3 \).

Conversely, suppose that \( G/Z^*(G) \cong S_3 \). Since \( S_3 \) is covered by 4 abelian subgroups, \( G \) is also covered by 4 nilpotent subgroups. This completes the proof. □
Corollary 3.3. Every finite group satisfying the condition \((N,4)\) is supersoluble. The alternating group \(A_4\) satisfies the condition \((N,5)\).

Proof. Let \(G\) be a finite group satisfying the condition \((N,4)\). By Proposition 1 of [10], \(G = H \times K\), where \(H\) is a nilpotent \(\{2,3\}'\)-group and \(K\) is a \(\{2,3\}\)-group. If \(K\) is nilpotent, then there is nothing to prove. Assume that \(K\) is not nilpotent. By Theorem 1.3, \(K/Z^*(K) \cong S_3\) and so \(K\) is supersoluble. Thus \(G\) is also a supersoluble group.

The group \(A_4\) is the union of its five Sylow subgroups, so \(A_4\) satisfies the condition \((N,5)\). □

Corollary 3.4. A finite group satisfies the condition \((N,4)\) if and only if it is the union of four nilpotent subgroups.

Proof. Let \(G\) be a finite group satisfying the condition \((N,4)\). Then by Theorem 1.3, \(G/Z^*(G)\) is the union of 4 nilpotent subgroups and hence so is \(G\). The converse is clear. □

4. Finite groups satisfying the condition \((A,n)\)

Now suppose that \(A\) is the class of abelian groups. Then every group satisfying the condition \((A,n)\) also satisfies the condition \((N,n)\). The converse is not true, since as we have observed already, \(SL(2,5)\) satisfies the condition \((N,21)\). However \(SL(2,5)\) does not satisfy the condition \((A,21)\).

Lemma 4.1. \(SL(2,5)\) does not satisfy the condition \((A,21)\).

Proof. Let \(P_1, \ldots, P_5\) be the Sylow 2-subgroups of \(SL(2,5)\), \(Q_1, \ldots, Q_{10}\) the Sylow 3-subgroups of \(SL(2,5)\), and \(R_1, \ldots, R_6\) the Sylow 5-subgroups of \(SL(2,5)\). We note for each \(i = 1, \ldots, 5\) that \(P_i\) is a quaternion group of order 8 and \(Z(P_i) = Z(SL(2,5))\) (see, for example, Theorem 8.10 in chapter II of [14]). Let \(x_i \in P_i \setminus Z(P_i)\)
\[ \{x_1, \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\} \]

commute. Now since \( P_1 \) is a quaternion group of order 8 and \( x \in P_1 \setminus \text{Z}(P_1) \), there exists an element \( x \in P_1 \setminus \text{Z}(P_1) \) such that \( x_1 x \neq xx_1 \). On the other hand, as above, no two distinct elements in

\[ \{x, x_1, \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\} \]

commute. Therefore no two distinct elements in the set

\[ \{x, x_1, \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\} \]

commute, which completes the proof. \( \square \)

**Lemma 4.2.** Let \( G \) be a finite group satisfying the condition \((A, 21)\). If there exists a central subgroup \( B \) of \( G \) of order no more than 2 such that \( G/B \cong A_5 \), then \( G \cong B \times A_5 \).

**Proof.** Since \( G/B \cong A_5 \) it follows that \( G = G'B \) and \( G'/(B \cap G') \cong A_5 \). Therefore if \( G' \cap B = 1 \) then the proof is complete. So suppose, for a contradiction, that \( G' \cap B \neq 1 \). Thus \( |B| = 2 \). According to the Universal Coefficients Theorem (see Theorem 11.4.18 of [21]) the central extension \( B \to G \to G/B \) determines a homomorphism \( \delta : M(G/B) \to B \) so that \( \text{Im} \delta = G' \cap B \), where \( M(G/B) \) is the Shur multiplicator of \( G/B \) (see for example Exercise 10 on page 354 of [21]). But we know that the Shur multiplicator of the alternating group \( A_5 \) is \( \mathbb{Z}_2 \). Hence \( G' \cap B = B \) and so \( B \leq G' \). It follows that \( G \) is a perfect group of order 120. But it is well-known that the only perfect group of order 120 is \( \text{SL}(2, 5) \). Now Lemma 4.1 gives a contradiction and the proof is complete. \( \square \)

We need the following lemma in the proof of Theorem 1.4.

**Lemma 4.3.** Let \( G \) be a group satisfying the condition \((A, n)\) \((n > 1)\). Then for any normal non-abelian subgroup \( N \) of \( G \), the quotient \( G/N \) satisfies the condition \((A, n-1)\).
Proof. Suppose, for a contradiction, that $G/N \notin (A, n - 1)$. Then there exist elements $x_1, \ldots, x_n$ in $G$ such that $[x_i, x_j] \notin N$ for all distinct $i, j \in \{1, \ldots, n\}$ (*). Let $a, b$ be two distinct arbitrary elements of $N$ and consider the subset $X = \{ax_1, \ldots, ax_n, bx_1\}$. By the hypothesis, there exist two distinct commuting elements in $X$. But, by (*), the only commuting pair of elements of $X$ are $bx_1$ and $ax_1$. Therefore for all $a, b \in N$, we have $ax_1b = bx_1a$ (**)) and in particular for $b = 1$, we have $ax_1 = x_1a$ for all $a \in N$. Thus for all $x, y \in N$ we have

$$xyx_1 = xx_1y = yx_1x = yxx_1$$

(the middle equality follows from (**)) and so $xy = yx$. Hence $N$ is abelian, a contradiction. □

Proof of Theorem 1.4. Suppose that $G \cong Z(G) \times A_5$. Then $G$ is covered by 21 abelian subgroups as $A_5$ has this property, by Remark 2.9(ii).

Now, suppose that $G$ satisfies the condition $(A,21)$. Then by a famous Theorem of B. H. Neumann [19], $G/Z(G)$ is finite. Thus, by Theorem 1.2,

$$\frac{G/Z(G)}{Z^*(G/Z(G))} \cong G/Z^*(G) \cong A_5.$$ 

If $H := Z^*(G)$ is not abelian, then Lemma 4.3 shows that $A_5$ satisfies the condition $(A,20)$, which contradicts Proposition 2 of [10]. Thus $H$ is abelian; we show that in fact $H = Z(G)$. To prove this let $Q_1, \ldots, Q_{21}$ be the Sylow subgroups of $G := G/H$. For each $i \in \{1, \ldots, 21\}$, let $x_iH$ be a non-trivial element of $Q_i$. Then $[x_i, x_j] \notin H$ and so $[x_i, x_j] \neq 1$ for all distinct $i, j \in \{1, \ldots, 21\}$, by Remark 2.9(i). Now fix $k \in \{1, \ldots, 21\}$ and consider the elements

$$x_k, x_1, \ldots, x_{k-1}, ax_k, x_{k+1}, \ldots, x_{21},$$

for an arbitrary element $a \in H$. Then for $j \in \{1, \ldots, 21\}$ and $j \neq k$, we have $[ax_k, x_j] \neq 1$, since $[ax_k, x_j]H = [x_k, x_j]H$. Since $G$ satisfies the condition $(A,21)$, $[x_k, ax_k] = 1$ and so $[a, x_k] = 1$ for all $k \in \{1, \ldots, 21\}$. Since the union of $Q_1, \ldots, Q_{21}$ is $G$, by Remark
2.9(ii), we have $[a, x] = 1$ for all $x \in G \setminus H$ and for all $a \in H$. Therefore $H = Z(G)$.

Now by the same argument as in Lemma 4.2, by considering the central extension $Z(G) = H \rightarrow G \rightarrow \overline{G}$, we have that $K = G' \cap Z(G)$ is of order no more than 2, $G = G'Z(G)$ and $G'/K \cong A_5$. Thus Lemma 4.2 implies that there is a subgroup $L$ of $G'$ such that $G'' = K \times L$ and $L \cong A_5$. Therefore $G = G'Z(G) = LKZ(G) = LZ(G)$ and it is clear that $L \cap Z(G) = 1$. Therefore $G = L \times Z(G) \cong A_5 \times Z(G)$. □

We end this paper by proving Theorem 1.5.

**Proof of Theorem 1.5.** We first prove that if $n = 2$, then $G$ is abelian. Consider two distinct elements $x, y \in G$. Then $X = \{x, y, xy\}$ is a subset of size 3. Thus by the hypothesis two distinct elements of $X$ commute. But commutativity of each pair of distinct elements of $X$ implies the commutativity of $x$ and $y$. Hence $G$ is abelian.

Now suppose that $n \geq 2$ and use induction on $n$. If $n = 2$ then $G$ is abelian and $d = 1$. So let $n > 2$. Then $2 < 2n - 3$. Thus we may assume that $d > 2$. Therefore $G'^{d-2}$ is not abelian and so $G/G'^{d-2}$ satisfies the condition $(A, n-1)$ by Lemma 4.3. Thus by induction the derived length of $G/G'^{d-2}$ is at most $2(n-1) - 3$ and so $d - 2 \leq 2(n-1) - 3$. Hence $d \leq 2n - 3$. □

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