GROUPS WITH SPECIFIC NUMBER OF CENTRALIZERS

A. ABDOLLAHI, S.M. JAFARIAN AMIRI AND A. MOHAMMADI HASSANABADI

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Abstract. Let $G$ be a group and let $\text{cent}(G)$ denote the set of centralizers of single elements of $G$. A group $G$ is called $n$-centralizer if $|\text{cent}(G)| = n$. In this paper, for a finite group $G$, we give some interesting relations between $|\text{cent}(G)|$ and the maximum number of the pairwise non-commuting elements in $G$. Also we characterize all $n$-centralizer finite groups for $n = 7$ and 8. Using these results we prove that there is no finite group $G$ with the property that $|\text{cent}(G)| = |\text{cent}(G/Z(G))| = 8$, where $Z(G)$ denotes the centre of $G$. This latter result answers positively a conjecture posed by A. R. Ashrafi.

1. Introduction and results

Throughout we will use the usual notation, for example $C_n$ denotes the cyclic group of order $n$; $S_n$ and $A_n$ denote respectively symmetric and alternating groups on $n$ letters and $D_{2n}$ stands for dihedral group of order $2n$.

Let $G$ be a group. We denote by $\text{cent}(G) = \{C_G(g) | g \in G\}$, where $C_G(g)$ is the centralizer of the element $g$ in $G$. Let $n > 0$ be an integer. A group $G$ is called $n$-centralizer if $|\text{cent}(G)| = n$ and $G$ is called primitive $n$-centralizer if $|\text{cent}(G/Z(G))| = |\text{cent}(G)| = n$, where $Z(G)$ denotes the centre of $G$. A subgroup $H$ of $G$ is called a proper centralizer of $G$ if $H = C_G(x)$ for some $x \in G \setminus Z(G)$.

It is clear that a group is 1-centralizer if and only if it is abelian. Belcastro and Sherman have the following results in [5]:

(i) There is no 2-centralizer and no 3-centralizer group.
(ii) A finite group $G$ is 4-centralizer if and only if $G/Z(G) \cong C_2 \times C_2$.

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A finite group $G$ is 5-centralizer if and only if $\frac{G}{Z(G)} \cong C_3 \times C_3$ or $S_3$.

Using these results, it is easy to see that there is no finite primitive 4-centralizer group and a finite group $G$ is primitive 5-centralizer if and only if $\frac{G}{Z(G)} \cong S_3$.

Ashrafi in [2, 3] has shown that if $G$ is a finite 6-centralizer group, then $\frac{G}{Z(G)} \cong D_8, A_4, C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$.

Our main results are:

**Theorem A.** Let $G$ be a finite group. Then $G$ is a 7-centralizer group if and only if $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}$ or $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$.

**Theorem B.** Let $G$ be a finite 8-centralizer group. Then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2, A_4$ or $D_{12}$.

Using Theorem B, we have confirmed Conjecture 2.3 of [2], namely:

**Theorem C.** There is no finite primitive 8-centralizer group.

### 2. 7-centralizer groups

In this section, we give the proof of Theorem A.

A cover $\Gamma$ for a group $G$ is a collection of proper subgroups whose union is the whole $G$. We use the term $n$-cover for a cover with $n$ members. A cover is called irredundant if no proper subcollection is also a cover. A cover is called a partition with kernel $K$ if the intersection of pairwise members of the cover is $K$. B. H. Neumann in [11] obtained a uniform bound for the index of the intersection of an irredundant $n$-cover in terms of $n$ and Tomkinson [13] improved this bound. For a natural number $n$, let $f(n)$ denote the largest index $|G : D|$, where $G$ is a group with an irredundant $n$-cover whose intersection is $D$. We know that $f(3) = 4$, $f(4) = 9$, $f(5) = 16$ and $f(6) = 36$ (see [12], [9], [6] and [1], respectively). We will use these results in the sequel.

We shall use the following lemma of Tomkinson repeatedly.

**Lemma 2.1.** (Lemma 3.3 of [13]) Let $M$ be a proper subgroup of the (finite) group $G$ and let $H_1, \ldots, H_k$ be subgroups of $G$ with $|G : H_i| = \beta_i$ and $\beta_1 \leq \cdots \leq \beta_k$. If $G = M \cup H_1 \cup \cdots \cup H_k$, then $\beta_1 \leq k$.

Furthermore, if $\beta_1 = k$, then $\beta_1 = \beta_2 = \cdots = \beta_k = k$ and $H_i \cap H_j \leq M$ for any two distinct $i$ and $j$.

Now we have
Proposition 2.2. Let \( n \geq 2 \) be an integer and let \( G \) be a finite group such that \( \frac{G}{Z(G)} \cong D_{2n} \). Then \( |\text{cent}(G)| = n + 2 \).

Proof. For an element \( x \) in \( G \), we write \( \overline{x} \) for \( xZ(G) \). By the hypothesis there exist elements \( r, s \in G \) such that

\[
\overline{G} = \frac{G}{Z(G)} = \langle \overline{r}, \overline{s} | \overline{r}^n = \overline{s}^2 = 1, \ \overline{r} \overline{s} = \overline{s} \overline{r}^{-1} \rangle.
\]

Since \( r \in C_G(r) \setminus Z(G) \) and \( \langle \overline{r} \rangle \) is a maximal subgroup of \( \overline{G} \), \( \frac{C_G(r)}{Z(G)} = \frac{C_G(\overline{r})}{Z(G)} = \langle \overline{r} \rangle \).

If \( n \) is even, then \( C_{\overline{G}}(\overline{r} \overline{s}) = \langle \overline{r} \overline{s} \rangle \times \langle \overline{r} \overline{s} \rangle \). If \( \frac{C_G(r^s)}{Z(G)} = \frac{C_G(r^s)}{Z(G)} \), then \( r^s \in C_G(r \overline{s}) \) and so \( r^s \in C_G(r \overline{s}) = C_G(r) \), a contradiction. Therefore \( \frac{C_G(r^s)}{Z(G)} = \langle \overline{r} \overline{s} \rangle = \frac{\langle r^s \rangle Z(G)}{Z(G)} \). Thus \( \text{cent}(G) = \{ G, C_G(r), C_G(r^i) : 1 \leq i \leq n \} \) and so \( |\text{cent}(G)| = n + 2 \). If \( n \) is odd, then \( C_{\overline{G}}(\overline{r} \overline{s}) = \langle \overline{r} \overline{s} \rangle = \frac{\langle r^s \rangle Z(G)}{Z(G)} \). Hence \( \text{cent}(G) = \{ G, C_G(r), C_G(r^i) : 1 \leq i \leq n \} \) and the proof is complete. \( \square \)

Corollary 2.3. Let \( G \) be a finite group and \( \frac{G}{Z(G)} \cong D_{2n} \). Then \( G \) is primitive \( n \)-centralizer if and only if \( n > 1 \) is an odd integer.

Proof. Note that if \( n > 1 \) is an integer, then \( |\text{cent}(D_{2n})| = \begin{cases} \frac{n}{2} + 2 & \text{if } n \text{ is even} \\ n + 2 & \text{if } n \text{ is odd} \end{cases} \),

(see for example; [2, Lemma 2.4] or [3, Lemma 2.4]). Now the proof follows from Proposition 2.2. \( \square \)

Definition 2.1. A non-empty subset \( X = \{ x_1, \ldots, x_r \} \) of a finite group \( G \) is called a set of pairwise non-commuting elements if \( x_i x_j \neq x_j x_i \) for all distinct \( i, j \in \{ 1, \ldots, r \} \). A set of pairwise non-commuting elements of \( G \) is said to have maximal size if its cardinality is the largest one among all such sets.

Remark 2.1. Let \( G \) be a finite group and \( \{ x_1, \ldots, x_r \} \) be a set of pairwise non-commuting elements of \( G \) having maximal size. Then

1. \( \{ C_G(x_i) \mid i = 1, \ldots, r \} \) is an irredundant \( r \)-cover with the intersection \( Z(G) = \bigcap_{i=1}^r C_G(x_i) \) (see Theorem 5.1 of [13]).
2. \( |\frac{G}{Z(G)}| \leq f(r) \) (see Corollary 5.2 of [13]).
3. \( f(3) = 4, f(4) = 9, f(5) = 16 \) and \( f(6) = 36 \) (see [12], [9], [6] and [1], respectively).
Lemma 2.4. Let $G$ be a finite non-abelian group and \{x_1, \ldots, x_r\} be a set of pairwise non-commuting elements of $G$ with maximal size. Then

1. $r \geq 3$.
2. $r + 1 \leq |\text{cent}(G)|$.
3. $r = 3$ if and only if $|\text{cent}(G)| = 4$.
4. $r = 4$ if and only if $|\text{cent}(G)| = 5$.

Proof. (1) Since $G$ is not abelian, there exist elements $x, y \in G$ such that $xy \neq yx$. Thus \{x, y, xy\} is a set of pairwise non-commuting elements of $G$, and so $r$ is at least 3, as required.

(2) It is clear, since $C_G(x_i)$s are distinct for $1 \leq i \leq r$.

(3) Suppose that $r = 3$. Then $|\frac{G}{Z(G)}| \leq f(3) = 4$ by Remark 2.1(1), (2) and (3). Since $G$ is not abelian, $\frac{G}{Z(G)} \cong C_2 \times C_2$. It now follows from Theorem 2 of [5] that $|\text{cent}(G)| = 4$.

Conversely, suppose that $|\text{cent}(G)| = 4$. Then by part (2), $r \leq 3$ and part (1) forces $r$ to be 3.

(4) Suppose that $r = 4$. Then $|\frac{G}{Z(G)}| \leq f(4) = 9$, by Remark 2.1(2) and (3). Assume that $|G : C_G(x_i)| = \alpha_i$ for $1 \leq i \leq r$ and $\alpha_1 \leq \cdots \leq \alpha_r$. We claim that $\frac{G}{Z(G)}$ is not a 2-group. Assume, on the contrary, that $\frac{G}{Z(G)}$ is a 2-group. Then $|\frac{G}{Z(G)}| = 4$ or 8 and so $C_G(x)$ is abelian for each $x \in G \setminus Z(G)$, this is because $\frac{C_G(x)}{Z(G)} = 2$ or 4 and $\frac{C_G(x)}{Z(C_G(x))}$ is cyclic.

By Remark 2.1(4), we have $Z(G) = C_G(x) \cap C_G(y)$ for distinct proper centralizers $C_G(x)$ and $C_G(y)$. We have $\alpha_2 \leq 3$ by Lemma 2.1 and so $\alpha_2 = 2$. It follows that $\frac{G}{Z(G)} \cong C_2 \times C_2$. Thus $|\text{cent}(G)| = 4$ by Theorem 2 of [5], which is a contradiction. Hence $|\frac{G}{Z(G)}| = 6$ or 9 from which it follows that $\frac{G}{Z(G)} \cong S_3$ or $C_3 \times C_3$. Now Theorem 5 of [5] implies that $|\text{cent}(G)| = 5$. 

(4) Let $G$ be a group such that every proper centralizer in $G$ is abelian. Then for all $a, b \in G \setminus Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$. For, if $z \in (C_G(a) \cap C_G(b)) \setminus Z(G)$, then $C_G(z)$ contains both $C_G(a)$ and $C_G(b)$, since $C_G(a)$ and $C_G(b)$ are abelian. Since $z$ is not in $Z(G)$, $C_G(z) \leq C_G(a)$ and $C_G(z) \leq C_G(b)$. Thus $C_G(z) = C_G(a) = C_G(b)$.

Hence, in such a group $G$, \{\frac{C_G(x)}{Z(G)} \mid x \in G \setminus Z(G)\} forms a partition with kernel $Z(G)$. It follows that \{\frac{C_G(x)}{Z(G)} \mid x \in G \setminus Z(G)\} forms a partition whose kernel is the trivial subgroup (see also Proposition 1.2 of [10]).
Conversely, if $|\text{cent}(G)| = 5$, then $r \leq 4$ by part (1). On the other hand $r \neq 3$ by part (3). Thus $r = 4$, and the proof is complete.

\[ \square \]

Proposition 2.5. Let $G$ be a finite group and let $X = \{x_1, \ldots, x_r\}$ be a set of pairwise non-commuting elements of $G$ having maximal size.

(a) If $|\text{cent}(G)| < r + 4$, then

1. For each element $x \in G$, $C_G(x)$ is abelian if and only if $C_G(x) = C_G(x_i)$ for some $i \in \{1, \ldots, r\}$.
2. If $C_G(x_i)$ is a maximal subgroup of $G$ for some $i \in \{1, \ldots, r\}$, then $Z(G) = C_G(x_i) \cap C_G(x_j)$ for all $j \in \{1, \ldots, r\} \setminus \{i\}$. In particular, if $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, then $|\text{cent}(G)| = 4$, and if $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $|\text{cent}(G)| = 5$.

(b) If $|\text{cent}(G)| = r + 2$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1}), C_G(x_{i_2})$ and $C_G(x_{i_3})$ for three distinct $i_1, i_2, i_3 \in \{1, \ldots, r\}$.

(c) If $|\text{cent}(G)| = r + 3$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$ and $C_G(x_{i_2})$ for two distinct $i_1, i_2 \in \{1, \ldots, r\}$.

Proof. (a)(1) Suppose, for a contradiction, that there exists an index $i$ such that $K := C_G(x_i)$ is non-abelian. Then $|\text{cent}(C_G(x_i))| \geq 4$ and so $C_G(x_i)$ contains at least three proper centralizers, say $C_K(y_i), i = 1, 2, 3$. By the hypothesis $C_G(x_i) \neq C_G(y_j)$ for every $t \in \{1, \ldots, r\}$ and $j \in \{1, 2, 3\}$. Therefore $|\text{cent}(G)| \geq r + 4$, a contradiction.

Conversely, suppose that $C_G(x)$ is abelian. Since $\{x_1, \ldots, x_r\}$ is a set of pairwise non-commuting elements of $G$ of maximal size, there exists an index $j$ such that $x \in C_G(x_j)$. By the previous part, $C_G(x_j)$ is abelian which implies that $C_G(x_j) \subseteq C_G(x)$. Since $x_j$ is in $C_G(x)$, we have $C_G(x) \subseteq C_G(x_j)$, which completes the proof.

(a)(2) Let $x \in C_G(x_i) \cap C_G(x_j)$. By part (a)(1), $C_G(x_i)$ is abelian and so $C_G(x_i) \leq C_G(x)$. Since $x_j \in C_G(x)$ and $x_i \neq x_j$, $C_G(x_i) \neq C_G(x)$. Now since $C_G(x_i)$ is a maximal subgroup of $G$, $C_G(x) = G$, hence $x \in Z(G)$.

If $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, then by the hypothesis $|G : C_G(x_1)| = |G : C_G(x_2)| = 2$. Now by the previous part we have $\frac{G}{Z(G)} \cong C_2 \times C_2$, and so by Theorem 2 of [5], $|\text{cent}(G)| = 4$. If $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $C_G(x_2)$ is a maximal subgroup of $G$ and $Z(G) = C_G(x_1) \cap C_G(x_2)$, by the first part of (a)(2). Thus $|G/Z(G)| \leq 9$. Now since 3 divides $|G/Z(G)|$, we have $\frac{G}{Z(G)} \cong C_3 \times C_3$ or $S_3$ and the result holds by Theorem 4 of [5].

(b) By the hypothesis and part (a)(1), there exists a proper centralizer $K := C_G(x)$ which is not abelian. Therefore $|\text{cent}(K)| \geq 4$. Assume that $\{C_K(y_i) \mid i =
Lemma 2.7. Let \( \{C_G(y_i) \mid i = 1, 2, 3\} \) be a set of three proper centralizers of \( K \). Then \( \{C_G(y_i) \mid i = 1, 2, 3\} \) is a set of three proper centralizers of \( G \). Since \( \text{cent}(G) = r + 2 \), \( C_G(y_i) = C_G(x_j) \) for distinct \( j_1, j_2, j_3 \in \{1, \ldots, r\} \) and so for \( i = 1, 2, 3 \); \( C_G(x_j) \) is contained in \( C_G(x) \) by part (a)(1).

(c) By the hypothesis and part (a)(1), there exists a proper centralizer \( H := C_G(x) \) which is not abelian. Therefore \( |\text{cent}(H)| \geq 4 \). Assume that \( \{C_H(w_i) \mid i = 1, 2, 3\} \) is a set of three proper centralizers of \( H \). Then \( \{C_G(w_i) \mid i = 1, 2, 3\} \) is a set of three proper centralizers of \( G \). Since \( |\text{cent}(G)| = r + 3 \), we may assume that \( C_G(w_i) = C_G(x_j) \) for distinct \( j_1, j_2 \in \{1, \ldots, r\} \). Then \( C_G(w_i) \) is abelian for \( i = 1, 2 \) by part (a)(1). But \( x \in C_G(w_i) \) for \( i = 1, 2 \) and so \( C_G(x_j) = C_G(w_i) \) is contained in \( C_G(x) \) for \( i = 1, 2 \). This completes the proof. \( \square \)

Lemma 2.6. Let \( G \) be a finite non-abelian group. Then every proper centralizer of \( G \) is abelian if and only if \( |\text{cent}(G)| = r + 1 \), where \( r \) is the maximal size of a set of pairwise non-commuting elements of \( G \).

Proof. Suppose that every proper centralizer of \( G \) is abelian and let \( X = \{x_1, \ldots, x_r\} \) be a set of pairwise non-commuting elements having maximal size. Consider a proper centralizer \( C_G(x) \) of \( G \). Then there exists an \( i \in \{1, \ldots, r\} \) such that \( x \in C_G(x_i) \) by the maximality of \( X \). Since \( C_G(x_i) \) is abelian, \( C_G(x_i) \leq C_G(x) \). On the other hand \( x_i \in C_G(x) \) gives \( C_G(x) \leq C_G(x_i) \). Therefore \( C_G(x) = C_G(x_i) \) and so \( \text{cent}(G) = \{G, C_G(x_i) \mid 1 \leq i \leq r\} \). This completes the proof in one direction, since for any two distinct \( i \) and \( j \), \( x_ix_j \neq x_jx_i \).

Conversely suppose that \( |\text{cent}(G)| = r + 1 \), where \( r \) is the maximal size of a set of pairwise non-commuting elements of \( G \). Assume that \( X = \{x_1, \ldots, x_r\} \) is a set of pairwise non-commuting elements of \( G \). Then Proposition 2.5(a)(1) implies that for each \( i \in \{1, \ldots, r\} \), \( C_G(x_i) \) is abelian, which completes the proof. \( \square \)

Lemma 2.7. Let \( G \) be a finite 7-centralizer group. Then \( \frac{G}{Z(G)} \) is not a 2-group.

Proof. Suppose, on the contrary, that \( \frac{G}{Z(G)} \) is a 2-group. Suppose that \( \{x_1, \ldots, x_r\} \) is a set of pairwise non-commuting elements of \( G \) having maximal size such that \( |G : C_G(x_i)| = \alpha_i \) with \( \alpha_1 \leq \cdots \leq \alpha_r \). Then \( \{C_G(x_i) \mid i = 1, \ldots, r\} \) is an irredundant cover for \( G \) with intersection \( Z(G) \). It follows from Lemma 2.4 that \( r = 5 \) or 6.

Suppose that \( r = 5 \). Then \( \alpha_2 \leq 4 \) by Lemma 2.1. It follows from Proposition 2.5(a)(2) that \( \alpha_2 = 4 \) and Lemma 2.1 yields that \( \alpha_i = 4 \) for \( i \geq 2 \). Now Proposition 2.5(b) implies that there exists a proper centralizer \( C_G(x) \) which is not abelian and contains at least three \( C_G(x_i) \)s, say for \( i = 2, 3, 4 \). Therefore
Lastly suppose, on the contrary, that there exists a group $G$ and

$L \in Z(H)$.  

Lemma 2.8. The groups $\langle x, y \mid x^5 = y^4 = 1, y^{-1} xy = x^{-1} \rangle$, $C_3 \times D_{10}$, $S_3 \times C_5$ and $\langle x, y \mid x^6 = 1, y^3 = y^{-1} xy = x^{-1} \rangle$ are not capable.

Proof. Suppose, for a contradiction, that there exists a group $H$ such that

\[
\frac{H}{Z(H)} \cong L := \langle x, y \mid x^5 = y^4 = 1, y^{-1} xy = x^{-1} \rangle.
\]

Then there exist elements $h_1, h_2 \in H$ such that $|h_1 Z(H)| = 4$ and $|h_2 Z(H)| = 10$ with $h_1^2 Z(H) = h_2^2 Z(H)$. Therefore $C_H(h_1)$ and $C_H(h_2)$ are subgroups of $C_H(h_1^2)$ and so the lcm(4, 10) divides $|\frac{C_H(h_1^2)}{Z(H)}|$. This forces $h_1^2$ to be in $Z(H)$, which is a contradiction.

Now assume, on the contrary, that there exists a group $G$ such that $\frac{G}{Z(G)} \cong D_{10} \times C_3$. Then there exist two elements $x$ and $y$ in $G$ such that $|xZ(G)| = 15$ and $|yZ(G)| = 6$ with $x^5 Z(G) = y^2 Z(G)$. It follows that $C_G(x^5)$ contains both of $C_G(x)$ and $C_G(y)$. Thus $C_G(x^5) = G$, so that $x^5 \in Z(G)$, which is not possible.

Next suppose, for a contradiction, that there exists a group $K$ such that $\frac{K}{Z(K)} \cong S_3 \times C_5$. Thus there exist elements $x, y \in K$ such that $|xZ(K)| = 15$ and $|yZ(K)| = 10$ with $x^3 Z(K) = y^2 Z(K)$. Therefore $C_G(x^3)$ contains both $C_G(x)$ and $C_G(y)$. It follows that $C_G(x^3) = G$, a contradiction.

Lastly suppose, on the contrary, that there exists a group $L$ such that

\[
\frac{L}{Z(L)} \cong \langle x, y \mid x^6 = 1, y^2 = x^3, y^{-1} xy = x^{-1} \rangle.
\]

Then there exist elements $l_1, l_2 \in L$ such that $|l_1 Z(L)| = 4$ and $|l_2 Z(L)| = 6$ with $l_1^2 Z(L) = l_2^3 Z(L)$. Therefore $C_L(l_1)$ and $C_L(l_2)$ are subgroups of $C_L(l_1^2)$ and so lcm(4, 6) divides $|\frac{C_L(l_1^2)}{Z(L)}|$. This implies that $l_1^2 \in Z(L)$, which cannot happen. \qed

Remark 2.2. Let $A$ be a finite abelian group. Let $p$ be a prime number and $i > 0$ be an integer. Suppose that $r(A, p^i)$ is the number of cyclic direct summands of order $p^i$ in the decomposition of $A$ into cyclic groups of prime power orders.
Baer in [4] showed that $A$ is capable if and only if for every prime number $p$, $r(A, p') = 1$ implies that $A$ contains elements of order $p^{i+1}$.

**Lemma 2.9.**

1. The only capable groups of order 12 are $D_{12}$ and $A_4$.
2. The only capable groups of order 20 are $D_{20}$ and $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$.
3. The only capable group of order 30 is $D_{30}$.

**Proof.** Firstly we note that, if $n \geq 2$ is an integer, then $D_{2n}$ is a capable group (since $\frac{D_{2n}}{Z(D_{2n})} \cong D_{2n}$). Also clearly every centreless group is capable.

1. There are exactly four non-cyclic groups of order 12, namely 

   $C_2 \times C_2 \times C_3$, $D_{12}$, $A_4$, and $\langle x, y | x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1} \rangle$.

   It follows from Remark 2.2 and Lemma 2.8 that $D_{12}$ and $A_4$ are the only non-cyclic capable groups of order 12.

2. There are exactly four non-cyclic groups of order 20, namely $C_2 \times C_2 \times C_5$, $D_{20}$, $T := \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$ and $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$. Since $Z(T) = 1$, it follows from Remark 2.2 and Lemma 2.8 that $D_{20}$ and $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$ are the only non-cyclic capable groups of order 20.

3. There are exactly three non-cyclic groups of order 30, namely $D_{30}$, $C_3 \times D_{10}$ and $S_3 \times C_5$. It follows from Lemma 2.8 that $D_{30}$ is the only capable group of order 30.

□

**Proof of Theorem A.** Suppose that $G$ is a 7-centralizer group and \{\(x_1, \ldots, x_r\)\} is a set of pairwise non-commuting elements of $G$ having maximal size. Let 

\[
|G : C_G(x_i)| = \alpha_i \text{ such that } \alpha_1 \leq \cdots \leq \alpha_r.
\]

Then \(\{C_G(x_i) \mid i = 1, \ldots, r\}\) is an irredundant $r$-cover for $G$ with intersection $Z(G)$ by Remark 2.1(1). It follows from Lemma 2.4 that $r = 5$ or 6.

If $r = 5$, then $\alpha_2 \leq 4$ by Lemma 2.1. If $\alpha_2 \leq 3$, then $|\text{cent}(G)| = 4$ or 5 by Proposition 2.5(a)(2). Therefore $\alpha_2 = 4$ and so $\alpha_2 = 4$ for $i \geq 2$ by Lemma 2.1. Now Proposition 2.5(b) implies that there exists a proper centralizer $C_G(x)$ which is not abelian and contains $\cup_{i \in S} C_G(x_i)$ for some $S \subset \{1, \ldots, 5\}$ with $|S| = 3$. Thus $G = (\cup_{i \in T} C_G(x_i)) \cup C_G(x)$, where $T = \{1, \ldots, 5\} \setminus S$. Now Lemma 2.1 implies that $\alpha_i = 2$ for some $2 \leq i \leq 5$, which is a contradiction.

Therefore $r = 6$. Then since $|\text{cent}(G)| = 7$, by Proposition 2.5(a)(1) all proper centralizers of $G$ are abelian. It follows from Remark 2.1(4) that $\{C_G(x_i) : 1 \leq i \leq 6\}$ is a partition with kernel $Z(G)$ for $G$. Now it follows from Lemma 2.1 that $\alpha_2 \leq 5$. Also Proposition 2.5(a)(2) implies that $\alpha_2 \neq 2, 3$. If $\alpha_2 = 4$, then
4 divides $|\frac{G}{Z(G)}|$ and $|\frac{G}{Z(G)}| \leq 16$. Now Lemma 2.7 implies that $|\frac{G}{Z(G)}| = 12$ and so $\frac{G}{Z(G)} \cong D_{12}$ or $A_4$ by Lemma 2.9. But this is impossible, since Proposition 2.2 and Theorem 3.6 of [2] imply that $|\text{cent}(G)| = 6$ or $8$. Therefore $\alpha_2 = 5$. If $\alpha_1 = 2$, then
$$\frac{G}{Z(G)} \cong D_{10},$$
as required.

If $\alpha_1 = 3$, then $\frac{G}{Z(G)}$ is of order 15 and so it is cyclic. Thus, in this case $|\text{cent}(G)| = 1$, which is not possible. If $\alpha_1 = 4$, then $|\frac{G}{Z(G)}| = 20$ and Lemma 2.9 implies that
$$\frac{G}{Z(G)} \cong \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle \text{ or } D_{20}.$$If $\frac{G}{Z(G)} \cong D_{20}$, then $|\text{cent}(G)| = 12$, a contradiction. Hence
$$\frac{G}{Z(G)} \cong \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle,$$as required.

Now assume that $\alpha_1 = 5$. Since $\alpha_2 = 5$, by Lemma 2.1 we have $\alpha_1 = \alpha_2 = \cdots = \alpha_6 = 5$. It follows that $|\frac{G}{Z(G)}| = 25$ and so
$$\frac{G}{Z(G)} \cong C_5 \times C_5,$$as required.

This completes the ‘only if’ part of the theorem.

If $\frac{G}{Z(G)} \cong D_{10}$ or $C_5 \times C_5$, then $|\text{cent}(G)| = 7$ by Theorem 5 of [5]. Suppose that $\frac{\frac{G}{Z(G)}}{Z(G)} \cong \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$. Then $|gZ(G)| \in \{1, 2, 4, 5\}$ for each $g \in G$.

If $|gZ(G)| = 4$ or 5, then $C_G(g) = \langle g \rangle Z(G)$ and if $|gZ(G)| = 2$, then there exists $g_1 \in G$ such that $|g_1Z(G)| = 4$ and $g = g_1^2z$ for some $z \in Z(G)$. So $C_G(g) = C_G(g_1^2) = C_G(g_1)$. On the other hand, $G$ has five Sylow 2-subgroups and one Sylow 5-subgroup which are all cyclic. Thus $|\text{cent}(G)| = 7$ and the proof is complete.

\[\Box\]

3. 8-CENTRALIZER GROUPS

In this section we prove Theorems B and C.

**Lemma 3.1.** Let $G$ be a finite 8-centralizer group. Then $\frac{G}{Z(G)}$ is a $\{2, 3\}$-group.

**Proof.** Let $\{x_1, \ldots, x_r\}$ be a set of pairwise non-commuting elements of $G$ having maximal size. Then $\{C_G(x_i) | i = 1, \ldots, r\}$ is an irredundant $r$-cover with intersection $Z(G)$ by Remark 2.1(1). It follows from Lemma 2.4 that $r = 5, 6$ or...
7. Assume that $|G : C_G(x_i)| = \alpha_i$, where $\alpha_1 \leq \cdots \leq \alpha_r$ and consider a $p$-element $x$ such that $p$ is a prime number greater than 5. Then $x \in Z(G)$ by Lemma 2.1 of [6] and so $\frac{G}{Z(G)}$ is a $\{2, 3, 5\}$-group. By Lemma 2.1, we have $\alpha_2 \leq 6$. If $r \leq 6$, then $\frac{G}{Z(G)} \leq 36$ by Remark 2.1(3). If $r = 7$ then, by Remark 2.1(4), $C_G(x_1) \cap C_G(x_2) = Z(G)$ which implies that $\frac{G}{Z(G)} \leq 36$.

If 5 is a divisor of $|\frac{G}{Z(G)}|$, then $\frac{G}{Z(G)} \in \{10, 15, 20, 25, 30, 35\}$. Since every group of order 15 or 35 is cyclic and $\frac{G}{Z(G)}$ is not cyclic, $|\frac{G}{Z(G)}| \in \{10, 20, 25, 30\}$. If $|\frac{G}{Z(G)}| = 10$ or 25, then $\frac{G}{Z(G)} \cong D_{10}$ or $C_5 \times C_5$ and so $|\text{cent}(G)| = 7$ by Theorem A, a contradiction. If $|\frac{G}{Z(G)}| = 20$ or 30, then Lemma 2.9 implies that $\frac{G}{Z(G)} \cong \langle x, y|x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle, D_{20}$ or $D_{30}$. Therefore $|\text{cent}(G)| = 7, 12$ or 17 by Theorem A and Proposition 2.2, which is impossible. This completes the proof.

\[\square\]

**Lemma 3.2.** Let $G$ be a finite 8-centralizer group. If $\frac{G}{Z(G)}$ is a 2-group, then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$.

**Proof.** Suppose that $X = \{x_1, \ldots, x_r\}$ is a set of pairwise non-commuting elements of $G$ having maximal size. Assume that $|G : C_G(x_i)| = \alpha_i$ such that $\alpha_1 \leq \cdots \leq \alpha_r$. Then $\{C_G(x_i) | i = 1, \ldots, r\}$ is an irredundant cover for $G$ with intersection $Z(G)$ by Remark 2.1(1). It follows from Lemma 2.4 that $r \neq 3$ or 4.

If $r = 5$, then $\alpha_2 \leq 4$. But $\alpha_2 \neq 2$ or 3 by Proposition 2.5(a)(2) and so $\alpha_i = 4$ for $i \geq 2$ by Lemma 2.1. Therefore by Proposition 2.5(c), there exists a proper centralizer $K := C_G(x)$ which is not abelian and contains at least two $C_G(x_i)$s, say for $i = 1, 2$. Thus $G = C_G(x_3) \cup C_G(x_4) \cup C_G(x_5) \cup C_G(x)$ which implies that $\alpha_3 \leq 3$ for some $i \in \{3, 4, 5\}$ by Lemma 2.1, a contradiction.

If $r = 6$, then $\alpha_2 = 4$ by Proposition 2.5(a)(2) and Lemma 3.1. Thus there exists a proper centralizer $C_G(x)$ which is not abelian and by Proposition 2.5(b) it contains at least three $C_G(x_i)$s, say for $i = 1, 2, 3$. Therefore $G = C_G(x_4) \cup C_G(x_5) \cup C_G(x_6) \cup C_G(x)$ which implies that $\alpha_i \leq 3$ for at least one $i \in \{4, 5, 6\}$. If $\alpha_1 = 2$, then Proposition 2.5(a)(2) implies that $\frac{G}{Z(G)} \leq 8$. This implies that $\frac{|C_G(x)|}{Z(G)} \leq 4$ and so $C_G(x)$ is abelian, a contradiction. Thus $\alpha_i \geq 4$ for every $i \in \{1, \ldots, 6\}$, which is impossible.

Finally suppose that $r = 7$. Then $C_G(x_i)$ is abelian for each $i \in \{1, \ldots, 7\}$ by Proposition 2.5(a)(1) and so $C_G(x_i) \cap C_G(x_j) = Z(G)$ by Remark 2.1(4) for any two distinct $i, j \in \{1, \ldots, 7\}$. It follows from Lemma 2.1 that $\alpha_2 = 4$ and so $\frac{G}{Z(G)} \leq \alpha_1 \alpha_2 \leq 16$. If $\frac{G}{Z(G)} = 8$, then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$, since other groups...
Lemma 3.3. Let $Z$ be a finite group of order 4 and so they cannot have an irredundant 7-cover. If $|\frac{G}{Z(G)}| = 16$, then $\alpha_1 = 4$ by Proposition 2.5(a)(2).

If $C_G(x_i)$ is not a normal subgroup of $G$ for $i = 1$ or 2, then $Z(G) = \text{core}_G(C_G(x_i))$ and so $|\frac{G}{Z(G)}|$ divides 24, which is wrong. Hence $C_G(x_i)$ is a normal subgroup of $G$ for $i = 1, 2$. It follows that $\frac{G}{Z(G)}$ is abelian and so $\frac{G}{Z(G)} \cong C_4 \times C_4$, $C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2$. However the group $C_2 \times C_2 \times C_2$ is not capable by Remark 2.2. The group $C_4 \times C_4$ cannot have such a partition. For, if $|\frac{G}{Z(G)}| = n_i$, then $\sum_{i=3}^{7} n_i = 22$ and so $\sum_{i=3}^{7} n_i = 14$. It yields that $n_3 = n_4 = 4$ and $n_5 = n_6 = n_7 = 2$ (if $r = 7$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 4$ and $\alpha_5 = \alpha_6 = \alpha_7 = 8$). This is impossible, since $C_4 \times C_4$ has only three elements of order 2. Finally we show that $\frac{G}{Z(G)} \not\cong C_2 \times C_2 \times C_2$.

Consider $P = \{C_G(x_i) \mid i = 1, \ldots, 7\}$ and $A_i = \frac{C_G(x_i)}{Z(G)}$ for $1 \leq i \leq 7$. For each $x \in G$, let $\pi = xZ(G)$, and let $A_1 = \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}$ and $A_2 = \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}$ for some $a, b, c, d \in G \setminus Z(G)$. Now using the fact that $P$ is a partition of $\frac{G}{Z(G)}$ with trivial kernel, having exactly three members of size 2 and four members of size 4; one can show that $P$ is equal to one of the following:

1. $\{A_1, A_2, \{\bar{1}, \bar{a}\}, \{\bar{1}, \bar{b}, \bar{c}, \bar{d}\}, \{\bar{1}, \bar{b}, \bar{c}, \bar{d}\}, \{\bar{1}, \bar{a}, \bar{b}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$
2. $\{A_1, A_2, \{\bar{1}, \bar{a}\}, \{\bar{1}, \bar{b}, \bar{c}, \bar{d}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$
3. $\{A_1, A_2, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$
4. $\{A_1, A_2, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$
5. $\{A_1, A_2, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$
6. $\{A_1, A_2, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}, \bar{c}\}, \{\bar{1}, \bar{a}, \bar{b}\}\}$

For any elements $x, y \in G$, we denote $x^{-1}y^{-1}xy$ by $[x, y]$. Note that if $\pi, \pi \in A_1$, then $[x, y] = 1$, and if $1 \neq \pi \in A_i$, $1 \neq \pi \in A_j$ and $i \neq j$, then $[x, y] \neq 1$. Now we reach to a contradiction in each of the above cases. For example in the case (1), we have $1 = [bc, ad] = [b, d][c, a]$ and $1 = [bd, ac] = [b, c][d, a][d, b]$ which yield that $[abcd] = 1$, a contradiction (note that in writing the latter equalities we are using the facts that $[x, y]^2 = 1$ and $[x, y] \in Z(G)$ for all $x, y \in G$).

Lemma 3.3. Let $G$ be a finite 8-centralizer group. Then $|\frac{G}{Z(G)}| \neq 24, 36$.

Proof. Let $\{x_1, \ldots, x_r\}$ be a set of pairwise non-commuting elements of $G$ having maximal size. Then $\{C_G(x_i) \mid i = 1, \ldots, r\}$ is an irredundant $r$-cover with intersection $Z(G)$. Assume that $|G : C_G(x_i)| = \alpha_i$, where $\alpha_1 \leq \cdots \leq \alpha_r$.

Suppose, for a contradiction, that $|\frac{G}{Z(G)}| = 36$. Then $r = 6$ or 7 by Remark 2.1(2,3). If $r = 6$, then $\alpha_2 \leq 4$ by Lemma 2.1 and Lemma 3.1. Now Proposition 2.5(a)(2) implies that $\alpha_2 \neq 2$ or 3 and so $\alpha_2 = 4$. If $\alpha_1 \leq 3$, then $|\frac{G}{Z(G)}| \leq 12$ by
Proposition 2.5(a)(2), which is not so. Thus $\alpha_1 = 4$ and so $\alpha_i \geq 4$ for each $i \geq 1$. On the other hand by Proposition 2.5(b), there exists a proper centralizer $C_G(x)$ which is not abelian and contains three subgroups $C_G(x_i)$ for $i \in \{1, \ldots, 6\}$. Thus $G = \bigcup_{i\in T} C_G(x_i) \cup C_G(x)$ for some $T \subset \{1, \ldots, 6\}$ such that $|T| = 3$. Therefore by Lemma 2.1, there exists at least one of $C_G(x_i)$s whose index is smaller than 4, which is a contradiction. Hence $r = 7$ and $\{C_G(x_i) \mid i = 1, \ldots, 7\}$ forms a partition with kernel $Z(G)$ by Remark 2.1(4). Since $\frac{|G|}{|Z(G)|} = 36$, by Lemma 2.1 and Proposition 2.5(a)(2), we have $\alpha_1 = \cdots = \alpha_7 = 6$. This gives that $\frac{C_G(x_i)}{Z(G)}$ is a cyclic subgroup of order 6 for every $i \in \{1, \ldots, 7\}$ and so $\frac{G}{Z(G)}$ has 14 elements of order 3, contradicting the fact that every group of order 36 has a unique Sylow 3-subgroup.

Now suppose that $\frac{|G|}{|Z(G)|} = 24$. If $r = 6$, then $\alpha_2 = 4$ by Lemma 2.1 and Proposition 2.5(a)(2). If $\alpha_1 \leq 3$, then $\frac{|G|}{|Z(G)|} \neq 24$ by Proposition 2.5(a)(2). Therefore $\alpha_i \geq 4$ for each $i \in \{1, \ldots, 6\}$. On the other hand by Proposition 2.5(b), $G$ contains a proper centralizer $C_G(x)$ which is not abelian and contains $C_G(x_i)$, $C_G(x_{i_2})$ and $C_G(x_{i_3})$ for some distinct $i, i_2, i_3 \in \{1, \ldots, r\}$. Again it follows that there exists an index $i$ such that $\alpha_i$ is smaller than 4 for some $i \in \{1, \ldots, 6\}$, which is not possible. If $r = 7$, then $\alpha_2 \leq 6$ by Lemma 2.1. Now Proposition 2.5(a)(2) implies that $\alpha_2 \neq 2$ or 3. If $\alpha_2 = 4$, then $\frac{|G|}{|Z(G)|} \leq \alpha_1 \alpha_2 \leq 16$, which is wrong. Therefore $\alpha_1 = 6$ for $2 \leq i \leq 7$ by Lemma 2.1. If $\alpha_1 = 6$, then $\frac{|G|}{|Z(G)|} = 36$ by inclusion-exclusion principal. Thus $\alpha_1 = 4$ and so $\frac{C_G(x_1)}{Z(G)}$ is the unique Sylow 3-subgroup of order 6. Therefore $\overline{G} = \frac{G}{Z(G)}$ has the unique Sylow 3-subgroup $\overline{U} = (\overline{w})$, where $\overline{w} = wZ(G)$ for some $w \in G$. Since $\overline{G} = \bigcup_{i=1}^7 \frac{C_G(x_i)}{Z(G)}$, $\frac{|C_G(x_i) \cap C_G(x_j)|}{Z(G)} = 1$ for distinct $i$ and $j$, $\alpha_1 = 4$ and $\alpha_i = 6$ for $2 \leq i \leq 7$, $\overline{G}$ has 20 elements of order 2 or 4. It follows that the number of Sylow 2-subgroups of $\overline{G}$ is three, namely $\overline{P}$, $\overline{P}_w$ and $\overline{P}_w'$. Also $|\overline{P} \cup \overline{P}_w \cup \overline{P}_w'| = 20$, so that $|\overline{P} \cap \overline{P}_w \cap \overline{P}_w'| = 2$. Thus $|\text{core}_{\overline{G}}(\overline{P})| = 2$. Next since $\overline{G} = \overline{U} \overline{P}$, we have $C_{\overline{G}}(\overline{U}) = \overline{U} \text{core}_{\overline{G}}(\overline{P})$. So that $|C_{\overline{G}}(\overline{U})| = 6$. But $\frac{\overline{U}}{\text{core}_{\overline{G}}(\overline{U})}$ embeds into $\text{Aut}(\overline{G}) \cong C_2$ and so $|\overline{G}| \leq 12$, giving the final contradiction.

**Proof of Theorem B.** Let $\{x_1, \ldots, x_r\}$ be a set of pairwise non-commuting elements of $G$ having maximal size. Then $\{C_G(x_i) \mid i = 1, \ldots, r\}$ is an irredundant $r$-cover with intersection $Z(G)$. Assume that $|G : C_G(x_i)| = \alpha_i$, where $\alpha_1 \leq \cdots \leq \alpha_r$. By Lemma 2.4, $r = 5, 6$ or 7.

Now suppose that $r = 5$. Then $\frac{|G|}{|Z(G)|} \leq f(5) = 16$. If $\alpha_2 \leq 3$, then $|\text{cent}(G)| = 4$ or 5 by Proposition 2.5(a)(2), a contradiction. Therefore $\alpha_2 = \cdots = \alpha_5 = 4$
by Lemma 2.1. If \( \alpha_1 = 2 \), then \(|G/Z(G)| \leq 8\) by Proposition 2.5(a)(2). Now, using Theorems 2 and 4 of [5] it is clear that \(|G/Z(G)| \neq 4\) or 6. It follows from Lemma 3.2 that \(G/Z(G) \cong C_2 \times C_2 \times C_2\). If \( \alpha_1 = 3 \), then \(|G/Z(G)| = 12\) and so \(G/Z(G) \cong A_4\) or \(D_{12}\) by Lemma 2.9, and since \( \alpha_1 = 4 \) for \( i = 2, 3, 4, 5 \), we have \(G/Z(G) \cong A_4\). Assume that \( \alpha_1 = 4 \). Proposition 2.5(c) implies that there exists a proper centralizer \(C_G(x)\) which is not abelian and contains at least two of \(C_G(x_i)\)s for \( 1 \leq i \leq 5 \).

Thus \( G = \cup_{i \in T} C_G(x_i) \cup C_G(x) \) for some \( T \subset \{1, 2, 3, 4, 5\} \) such that \(|T| \leq 3 \). It follows that \( \alpha_5 \leq 3 \) for some \( i \in T \), which is impossible.

Suppose that \( r = 6 \). Then \( \alpha_2 \leq 5 \) and so \( \alpha_2 = 4 \) by Lemma 3.1 and Proposition 2.5(a)(2). If \( \alpha_1 = 2 \), then \(|G/Z(G)| = 8\) from which it follows that \(G/Z(G) \cong C_2 \times C_2 \times C_2\), a contradiction. If \( \alpha_1 = 3 \), then \(|G/Z(G)| = 12\). It follows from Lemma 2.9 that \(G/Z(G) \cong A_4\) or \(D_{12}\), which is impossible since \( \alpha_1 = 3 \). Thus we have an irredundant 7-cover in this case. Therefore \( \alpha_1 = 4 \) and \( \alpha_i \geq 4 \) for \( i \in \{1, \ldots, 6\} \).

On the other hand there exists a proper centralizer \(C_G(x)\) which is not abelian and contains at least three of \(C_G(x_i)\)s for \( 1 \leq i \leq 6 \) by Proposition 2.5(b). Therefore \( G = \cup_{i \in T} C_G(x_i) \cup C_G(x) \) for some \( T \subset \{1, \ldots, 6\} \) with \(|T| \leq 3 \) which implies that there exists at least an index \( i \in T \) such that \( \alpha_i \leq 3 \), giving a contradiction.

Finally suppose that \( r = 7 \). Then \( \{C_G(x_i) : 1 \leq i \leq 7\} \) is a partition with kernel \(Z(G)\) for \( G \) by Remark 2.1 and Proposition 2.6. By Lemma 2.1, we have \( \alpha_2 \leq 6 \). Now by Proposition 2.5(a)(2) we have that \( \alpha_2 \geq 4 \) and so \( \alpha_2 = 4 \) or 6 by Lemma 3.1. Now assume that \( \alpha_3 = 4 \). If \( \alpha_1 = 2 \), then \(|G/Z(G)| \leq 8\) and so \(|G/Z(G)| = 8\). Thus by Lemma 3.2, \(G/Z(G) \cong C_2 \times C_2 \times C_2\) which is not possible, since \(C_2 \times C_2 \times C_2\) does not have an irredundant 7-cover with \( \alpha_1 = 2 \). If \( \alpha_1 = 3 \), then \(|G/Z(G)| = 12\) by Proposition 2.5(a)(2). It follows from Lemma 2.9 that \(G/Z(G) \cong A_4\) or \(D_{12}\). This is impossible since every irredundant cover of \(A_4\) with \( \alpha_1 = 3 \) has five members and \(D_{12}\) has an element of order 6 which implies that \( \alpha_1 = 2 \). Thus \( \alpha_1 = 4 \) and so \(|G/Z(G)| \leq \alpha_1 \alpha_2 \leq 16\). It follows that \(|G/Z(G)| = 8, 12 \) or 16. Since \( \alpha_1 = \alpha_2 = 4 \), \(G/Z(G) \cong D_{12}\). Hence by Lemma 3.2 and Lemma 2.9,

\[
\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \quad \text{or} \quad A_4,
\]

as required.

Now suppose that \( \alpha_2 = 6 \). Then Lemma 2.1 implies that \( \alpha_i = 6 \) for \( i \in \{2, \ldots, 7\} \). Since \(G/Z(G) = \bigcup_{i=1}^{7} C_G(x_i)\) and \(|C_G(x_i) \cap C_G(x_j)| = 1\) for any two distinct \( i, j \in \{1, \ldots, 7\} \), we have \(|G/Z(G)| = 6 \alpha_1\). If \( \alpha_1 = 2 \), then \(|G/Z(G)| = 12\) and so \(G/Z(G) \cong D_{12}\) or \(A_4\). Now since \(A_4\) does not contain a subgroup of index 2, we
have

\[ \frac{G}{Z(G)} \cong D_{12}, \] as required

If \( \alpha_1 = 3 \), then \( |\frac{G}{Z(G)}| = 18 \) and \( K := \frac{C_G(x_1)}{Z(G)} \) is a unique subgroup of \( \frac{G}{Z(G)} \) of order 6. Therefore there exists an element \( y \in \frac{Z(G)}{Z(G)} \) such that \(|y| = 2 \) and so \( \langle y \rangle \) is a subgroup of \( Z(\frac{G}{Z(G)}) \). Thus \( \frac{G}{Z(G)} = \langle y \rangle P \), where \( P \) is a normal Sylow 3-subgroup of \( G \). Hence \( G \) is abelian and we have \( G \cong C_2 \times C_3 \times C_3 \) which is not possible, since by Remark 2.2, \( C_2 \times C_3 \times C_3 \) is not capable. Finally if \( \alpha_1 = 4 \) or 6, then \( |\frac{G}{Z(G)}| = 24 \) or 36 which is impossible by Lemma 3.3. This completes the proof.

Proof of Theorem C. Suppose that \( G \) is a finite 8-centralizer group. Then by Theorem B, \( \frac{G}{Z(G)} \) is isomorphic to \( A_4, D_{12}, \) or \( C_2 \times C_2 \times C_2 \). Now the result follows since \( |\text{cent}(A_4)| = 6 \) and \( |\text{cent}(D_{12})| = 5 \).

We know that for a finite group \( G \) with \( \frac{G}{Z(G)} \cong A_4 \) or \( D_{12} \), we have \(|\text{cent}(G)| = 6 \) or 8 (see [2, Theorem 3.6] and Proposition 2.2). For the case \( \frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \), we also have:

**Proposition 3.4.** Let \( G \) be a finite group. If \( \frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \), then \(|\text{cent}(G)| = 6 \) or 8.

Proof. Suppose that \( \{x_1, \ldots, x_r\} \) is a set of pairwise non-commuting elements of \( G \) having maximal size. Then Remark 2.1 yields that \( \{\frac{C_G(x_i)}{Z(G)} \mid 1 \leq i \leq r\} \) forms an irredundant \( r \)-cover with trivial intersection for \( \frac{G}{Z(G)} \). Now the group \( C_2 \times C_2 \times C_2 \) is covered by at most seven proper subgroups and so \( r \leq 7 \). Now we claim that every proper centralizer of \( G \) is abelian. Since \( \frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \), \( \frac{C_G(x)}{Z(G)} \cong C_2 \) or \( C_2 \times C_2 \) for every \( x \in G \setminus Z(G) \) and so \( C_G(x) \) is abelian. Thus Lemma 2.6 implies that \(|\text{cent}(G)| = r + 1 \) and so \(|\text{cent}(G)| \leq 8 \). Now by Theorems 2 and 4 of [5] and Theorem A, we have \(|\text{cent}(G)| \neq 4, 5 \) or 7, which completes the proof.

Finally we note that there exist groups \( G_i \) for \( i = 1, 2 \) of order 64 such that \( \frac{G_i}{Z(G_i)} \cong C_2 \times C_2 \times C_2 \), \(|\text{cent}(G_1)| = 6 \) and \(|\text{cent}(G_2)| = 8 \) (for example, \( G_1 = \text{SmallGroup}(64, 60) \) and \( G_2 = \text{SmallGroup}(64, 73) \) in [8]). Also for \( G = \text{SmallGroup}(24, 3) \) in [8] we have \( \frac{G}{Z(G)} \cong A_4 \) and \(|\text{cent}(G)| = 8 \) (this is also pointed out in [2, Remark 3.7]).
GROUPS WITH SPECIFIC NUMBER OF CENTRALIZERS

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(A. Abdollahi) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, AND, INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS (IPM), TEHRAN, IRAN
E-mail address: a.abdollahi@math.ui.ac.ir

(S.M. Jafarian Amiri) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, AND, INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS (IPM), TEHRAN, IRAN
E-mail address: sm.jafarian@sci.ui.ac.ir

(A. Mohammadi Hassanabadi) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, AND, INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS (IPM), TEHRAN, IRAN
E-mail address: ammohaha@yahoo.com