On groups with an irredundant 7-cover

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Abstract

A cover for a group is a collection of proper subgroups whose union is the whole group. A cover is irredundant if no proper sub-collection is also a cover and is called maximal if all its members are maximal subgroups. For an integer \( n > 2 \), a cover with \( n \) members is called an \( n \)-cover. In this paper we determine groups with a maximal irredundant 7-cover with core-free intersection. The intersection of an irredundant \( n \)-cover is known to have index bounded by a function of \( n \), though in general the precise bound is not known. Here we prove that the exact bound is 81 when \( n = 7 \).

MSC: 20D60

1. Introduction and results

A covering or cover for a group \( G \) is a collection of subgroups of \( G \) whose union is \( G \). We use the term \( n \)-cover for a cover with \( n \) members. The cover is irredundant if no proper sub-collection is also a cover, and is called maximal if all its members are maximal subgroups of \( G \). A cover is called a core-free intersection if the core of its intersection is trivial. A cover is called a \( C_n \)-cover if it is a maximal irredundant \( n \)-cover with core-free intersection. We call a group \( G \) a \( C_n \)-group if \( G \) admits a \( C_n \)-cover.

Scorza [8] and Greco [6] determined the structure of all groups having an irredundant \( n \)-cover with core-free intersection for \( n = 3, 4 \) respectively. Bryce et al. [2] and Abdollahi et al. [1] characterized \( C_n \)-groups for \( n = 5, 6 \) respectively. Here we characterize \( C_7 \)-groups.

**Theorem A.** If \( G \) is a \( C_7 \)-group with core-free intersection \( D \), then \( G \) and \( D \) satisfy one of the following properties.

1. \( G \cong (C_2)^6 \) or \( C_2 \times Sym_4 \) and \( |D| = 1 \).
2. \( G \cong (C_3)^4 \), \( Sym_3 \times Sym_3 \) or \( (C_3)^3 \times C_2 \) and \( |D| = 1 \).
3. \( G \cong (C_3)^4 \times C_2 \) and \( |D| = 2 \).
4. \( G \cong Sym_4 \) or \( (C_2)^4 \times C_3 \) and \( |D| = 1 \).
5. \( G \cong (C_2)^4 \times Sym_3 \) and \( |D| = 2 \).

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(6) \( G \cong (C_2)^6 \times C_3 \) and \( |D| = 3 \).

(7) \( G \cong (C_2)^6 \times \text{Sym}_3 \) and \( |D| = 6 \).

Neumann [7] proved that if \( G \) has an irredundant \( n \)-cover then the index of the intersection of the cover in \( G \) is bounded by a function of \( n \). Tomkinson [10] improved that bound and gave estimates for \( f(n) \), the largest index \( |G : D| \) over all groups \( G \) having an irredundant \( n \)-cover with intersection \( D \). He suggested that the lower bound

\[
g(n) = \begin{cases} 
4 \cdot 3^{(2n-3)/3} & \text{if } n = 3k \\
3^{2n-1} & \text{if } n = 3k + 1 \\
16 \cdot 3^{(2n-5)/3} & \text{if } n = 3k + 2 
\end{cases}
\]

for \( f(n) \), gives in fact the value of \( f(n) \).

In [8,6,2], the value of \( f(n) \) was obtained for \( n = 3, 4, 5 \), namely \( f(3) = g(3) \), \( f(4) = g(4) \) and \( f(5) = g(5) \), respectively. Also Abdollahi et al. [1] have shown that \( f(6) = g(6) \). Here, using the list of all \( C_7 \)-groups and some further works we are able to prove that \( f(7) = 81 \). This coincides with Tomkinson’s lower bound \( g(7) \).

**Theorem B.** \( f(7) = 81 \).

For other aspects of covering groups by subgroups, especially for abelian groups, the reader may refer to [9], where some covering problems are closely related to combinatorial problems, including the so-called additive basis conjecture, the three-flow conjecture and a conjecture of Alon, Jaeger and Tarsi about nowhere zero vectors.

Throughout the paper for any \( C_7 \)-group \( G \), we always assume that \( \{M_i \mid 1 \leq i \leq 7\} \) is a \( C_7 \)-cover with intersection \( D = \bigcap_{i=1}^7 M_i \). Note that by [7], \( C_7 \)-groups are finite and by [2, Lemma 2.2(a)] every \( C_7 \)-group is a finite \( \{2,3,5\} \)-group.

Let us give an outline of the proofs of our main results. The proof of **Theorem A** is similar to that of the characterization of \( C_5 \) and \( C_6 \)-groups given in [2] and [1], respectively. To characterize \( C_7 \)-groups, we distinguish between three cases: nilpotent, semisimple and non-semisimple groups, where by a semisimple group we mean a group having no non-trivial normal abelian subgroup. The semisimple case cannot simply occur in the characterization of \( C_5 \)-groups in [2], since these groups are soluble by [2, Lemma 2.2(a)] and Burnsides’ \( p^aq^b \) theorem. The techniques used in the proof of **Theorem A** are more or less similar to those in the characterization of \( C_6 \)-groups [1].

The proof of **Theorem B** is similar to that of [2] and to the proof of Theorem D of [1]. Note that in this method, one should at least know the value of \( f(7) \) on the class of \( C_7 \)-groups as well as the value of \( f(n) \) for \( n \leq 6 \).

We use the usual notations; for example, \( C_n \) denotes the cyclic group of order \( n \), \( (C_n)^j \) is the direct product of \( j \) copies of \( C_n \), and the core of a subgroup \( H \) of \( G \) is denoted by \( H_G \). Recall that a group \( G \) is a direct product of a family of groups \( \{G_i \mid i \in I\} \) if there exists a family of normal subgroups \( \{N_i \mid i \in I\} \) of \( G \) such that \( \cap_{i \in I} N_i = 1 \) and \( G/N_i \cong G_i \) for all \( i \in I \). We denote \( \{1,2,\ldots,7\} \) by \( [7] \) and for each \( m \in [7] \), \( [7]^m \) will denote the set of all subsets of \( [7] \) of size \( m \).

2. Nilpotent \( C_7 \)-groups

The main result of this section is the characterization of all nilpotent \( C_7 \)-groups. Before stating the main result, we quote Lemma 2.2 of [2] that will be used in the proofs repeatedly, sometimes without reference.

**Lemma 2.1** (See Lemma 2.2 of [2]). Let \( \Gamma = \{A_i : 1 \leq i \leq m\} \) be an irredundant covering of a group \( G \) whose intersection of the members is \( D \).

(a) If \( p \) is a prime number, \( x \) a p-element of \( G \) and \( |\{i : x \in A_i\}| = n \), then either \( x \in D \) or \( p \leq m - n \).

(b) \( \bigcap_{i,j \neq i} A_i = D \) for all \( i \in \{1,2,\ldots,m\} \).

(c) If \( \bigcap_{i \in S} A_i = D \) and \( |S| = n \), then \( |\bigcap_{i \in T} A_i : D| \leq m - n - 1 \) whenever \( |T| = n - 1 \).

(d) If \( \Gamma \) is maximal and \( U \) is an abelian minimal normal subgroup of \( G \) such that \( |\{i : U \subseteq A_i\}| = n \), then either \( U \leq D \) or \( |U| \leq m - n \).

**Theorem 2.2.** Let \( G \) be a \( C_7 \)-group. If \( G \) is nilpotent, then \( G \cong (C_2)^6 \) or \( (C_3)^4 \). In particular, if \( (C_2)^6 = \langle a, b, c, d, e, f \rangle \), then \( \langle b, c, d, e, f \rangle, \langle a, c, d, e, f \rangle, \langle a, b, c, d, e, f \rangle, \langle a, b, c, d, f \rangle, \langle a, b, c, d, f \rangle, \langle a, b, c, d, e, f \rangle, \langle a, b, c, d, e, f \rangle \) provide a \( C_7 \)-cover for \( (C_2)^6 \) and if \( (C_3)^4 = \langle a, b, c, d \rangle \), then \( \langle a, c, d \rangle, \langle b, c, d \rangle, \langle a, b, c \rangle, \langle a, b, d \rangle, \langle a, c, d \rangle, \langle a, b, c^{-1}d \rangle \) are members of a \( C_7 \)-cover for \( (C_3)^6 \).
We first deal with the case in which $G$ is a $p$-group for some prime $p$. In this case the proof is similar to those of Lemma 2.1 and Proposition 2.2 of [1]. Then we argue as in the proof of Theorem A of [1] to prove that a nilpotent $\mathcal{C}_7$-group is a $p$-group for some prime $p$. \hfill \Box

Note that Theorem 2.2 proves $f(7) \geq 81$. In Section 5 – after we have completed the proof of Theorem A – we will show that $f(7) \leq 81$.

3. Semisimple groups

Recall that by a semisimple group we mean a group having no non-trivial normal abelian subgroup. The main result of this section is

**Theorem 3.1.** Semisimple groups do not have a $\mathcal{C}_7$-cover.

**Remark 3.2.** (1) The only primitive groups of degree 5 are $C_5$, $C_5 \times C_2$, $C_5 \times C_4$, $Alt_5$ and $Sym_5$.

(2) The only primitive groups of degree 6 are $Alt_5$, $Alt_6$, $Sym_5$ and $Sym_6$.

**Proof of Theorem 3.1.** Suppose, on the contrary, that $G$ is semisimple and $\{M_1, \ldots, M_7\}$ is a $\mathcal{C}_7$-cover with intersection $D$ for $G$. Let $|G : M_i| = \alpha_i$ such that \( \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \leq \alpha_7 \). Note that $G$ is a $(2, 3, 5)$-group and it follows from Lemma 3.1 of [11] that $\alpha_2 \leq 6$. Also

\[
\bigcap_{i \in S}(M_i)_G = 1 \quad \text{for every } S \in \{7\}^3, \tag{\ast}
\]

since the intersection $\bigcap_{i \in S}(M_i)_G$ contains no 5-element, and so it is a normal soluble subgroup of $G$, by Burnside’s $p^aq^b$ theorem.

As $\alpha_2 \leq 6$, we distinguish three cases: $\alpha_2 \leq 4$, $\alpha_2 = 5$ and $\alpha_2 = 6$. In the following we discuss only the first case $\alpha_2 \leq 4$; the others are similar. The main idea is to determine the minimal normal subgroups of $G$, and then prove that the product of all minimal normal subgroups has order larger than $|G|$, which is impossible. For this reason, we prove for a suitable set of $M_i$’s, each of them contains a minimal normal subgroup and the intersection of any two of such $M_i$’s is core-free.

So suppose that $\alpha_1 \leq \alpha_2 \leq 4$. Then $(M_1)_G \cap (M_2)_G$ is non-trivial by semisimplicity of $G$. By applying Lemma 3.2 of [11], we have $\alpha_3 \leq 5$. If $\alpha_3 < 5$ then by (\ast), $G$ can be embedded in $Sym_4 \times Sym_4 \times Sym_4$, which is impossible. Thus $\alpha_3 = 5$, and so by Lemma 3.2 of [11], $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 5$ and $M_i \cap M_j \leq M_1 \cup M_2$ for $3 \leq i < j \leq 7$. It follows that $M_i \cap M_j \leq M_1$ or $M_2$. Thus by (\ast),

\[
(M_i)_G \cap (M_j)_G = 1 \quad \text{for all distinct } i, j \in \{3, 4, 5, 6, 7\} \tag{\ast\ast}
\]

and so $G$ embeds into $Sym_5 \times Sym_5$. If $(M_1)_G = 1$, then $G$ is a primitive group of degree 5, and so $G \cong Sym_5$ or $Alt_5$ by Remark 3.2. But $Sym_5$ and $Alt_5$ cannot be covered by seven proper subgroups by Lemma 7 of [3]. Therefore $(M_i)_G$ is non-trivial for every $i \geq 3$. On the other hand every minimal normal subgroup of $G$ is isomorphic to $Alt_5$; for, if $U$ is a minimal normal subgroup of $G$, there exists an index $i \geq 3$ such that $U \neq M_i$, and so $U \cap (M_i)_G = 1$. Also we have $U := \frac{U(M_i)_G}{(M_i)_G} \leq (M_i)_G$ and $U \cong U$, which imply that $U \cong Alt_5$ by the semisimplicity of $G$ and Remark 3.2.

Since $(M_i)_G \neq 1$ for every $i \geq 3$, there exists a minimal normal subgroup $N_i \leq (M_i)_G$ of $G$. Now it follows from (\ast\ast) that $|N| = 60^5$, where $N = Dr_{i=3}^7 N_i$, which is a contradiction, since $60^5 \leq |N| \leq |G| \leq 120^2$. \hfill \Box

4. Proof of Theorem A

According to Theorems 2.2 and 3.1, to characterize all $\mathcal{C}_7$-groups we need only consider non-nilpotent non-semisimple $\mathcal{C}_7$-groups. Since $G$ is not semisimple, $G$ contains an abelian minimal normal subgroup $U$. Thus $U$ is a normal elementary abelian subgroup of $G$. By Lemma 2.1 (d), $U \cong C_2$, $C_2 \times C_2$, $C_3$ or $C_5$. Hence four cases arise, according to which one of the four latter elementary abelian groups is isomorphic to $U$. In this way, we encounter that $G$ may be isomorphic to a certain subdirect product of a set of primitive groups of degree $\leq 5$. We collect these cases in Lemma 4.1. These cases may occur, in which we have to determine, for future purpose, i.e., the proof of Theorem B, the size of the intersection $D$ of any $\mathcal{C}_7$-group of $G$. The computational group theory package GAP [5] will be used to determine $|D|$ in Lemma 4.1; in fact we simply test whether $G$ has any $\mathcal{C}_7$-cover and, if so, we find
Lemma 4.1. (1) The following are not $\mathcal{C}_7$-groups.
(a) Subdirect products of at most five symmetric groups $\text{Sym}_3$ of orders 108 or 18.
(b) Subdirect products of three $C_2$'s and one $\text{Sym}_3$.
(g) Subdirect products of $\text{Sym}_3$ and $\text{Alt}_4$.
(h) Subdirect products of two $C_2$'s and two $\text{Sym}_3$'s with non-trivial center.
(l) Subdirect products of $C_5$, $\text{Alt}_4$ and $C_3$, $\text{Sym}_4$.
(n) Subdirect products of $l$ cyclic groups $C_3$ and $k$ symmetric groups $\text{Sym}_3$, where $l \leq 3$ and $k \leq 2$.
(p) Subdirect products of two dihedral groups of order 10.
(2) The following are $\mathcal{C}_7$-groups, where $D$ denotes the intersection of an arbitrary $\mathcal{C}_7$-cover:
(a) Subdirect products of two alternating groups $\text{Alt}_4$ of order 48 with $|D| = 1$.
(b) Subdirect products of three symmetric groups $\text{Sym}_4$ of order 24, 96 and 384 with $|D| = 1$, 2 and 6, respectively.
(e) Among all subdirect products of two $C_2$'s and two $\text{Sym}_3$'s, only $\text{Sym}_3 \times \text{Sym}_3$ is a $\mathcal{C}_7$-group for which $D = 1$.
(f) Among all subdirect products of $\text{Sym}_3$ and $\text{Sym}_4$, there is only one $\mathcal{C}_7$-group isomorphic to $\text{Sym}_4$ and $D = 1$.
(m) The only subdirect products of two $C_2$'s and one $\text{Sym}_4$ which are $\mathcal{C}_7$-groups are $\text{Sym}_4$ and $C_2 \times \text{Sym}_4$. The intersection $D$ of an arbitrary $\mathcal{C}_7$-cover for $\text{Sym}_4$ and $C_2 \times \text{Sym}_4$ is trivial.
(n) The only subdirect products of one $C_2$ and two primitive groups of degree 4 (which are $\text{Alt}_4$ and $\text{Sym}_4$) are $\text{Sym}_4$ and $C_2 \times \text{Sym}_4$, where for both of them $D = 1$.
(p) Among all subdirect products of three $\text{Sym}_3$'s, $\text{Sym}_3 \times \text{Sym}_3$ is a $\mathcal{C}_7$-group with $D = 1$ and possibly a group of the form $(C_3)^3 \times C_2$ in $\mathcal{C}_7$ with $D = 1$.

Proof. The proof is similar to that of Lemma 4.1 of [1]. □

Lemma 4.2. If $G$ is a $\mathcal{C}_7$-group and $G$ has a minimal normal subgroup of order 2, then $G$ is isomorphic to one of the following groups: (1) $(C_2)^6$, (2) $C_2 \times \text{Sym}_4$. In both cases any $\mathcal{C}_7$-cover for $G$ has trivial intersection.

Proof. First note that $G$ is a $\{2, 3, 5\}$-group. Suppose that $U$ is a minimal normal abelian subgroup of $G$ and $|U| = 2$. Then by Lemma 2.1, $U$ is not contained in at least two $M_i$'s, say $U \not\subseteq M_6, M_7$. Since $U$ is central, $G = M_6 U = M_7 U, M_6, M_7 \leq G$ and

$$|G : M_6| = |G : M_7| = 2.$$ (1)

Assume that $U \not\subseteq M_\ell$ for some $\ell \in \{1, 2, 3, 4, 5\}$. (#)

Without loss of generality we may suppose that $\ell = 5$. Then

$$|G : M_5| = 2.$$ (*)

Therefore $G$ is a $\{2, 3\}$-group, and so $G$ is soluble. By Theorem 2.2, $M_5 \cap M_6 \cap M_7$ is non-trivial, and so there exists a minimal normal subgroup $V$ of $G$ such that $V \leq M_5 \cap M_6 \cap M_7$. It follows that $|V| \in \{2, 3, 4\}$. We distinguish the following three cases.

Case 1. If $|V| = 2$, then $V$ is not contained in at least two $M_i$'s, say $V \not\subseteq M_3, M_4$. This yields that $|G : M_3| = |G : M_4| = 2$. Therefore $G$ is a 2-group and by Theorem 2.2, $G \cong (C_2)^6$.

Case 2. If $|V| = 3$, then $V$ is not contained in at least three $M_i$'s, say $i = 2, 3, 4$. Thus $|G : M_i| = 3$ and $M_i \cap V = 1$ for every $i \in \{2, 3, 4\}$. If $K := M_6 \cap M_7 \cap (M_2)G \cap (M_3)G$ is non-trivial then there is a minimal normal subgroup $W$ contained in $K$, and it follows that $|W| \leq 3$.

Subcase 1. If $|W| = 3$ then by Lemma 2.1, $W$ is contained in none of $M_1, M_4, M_5$. Thus $|G : M_5| = 3$, which contradicts (*).

Subcase 2. If $|W| = 2$, then $|G : M_1| = 2$. It follows that every 3-element of $G$ belongs to $M_1 \cap M_5 \cap M_6 \cap M_7$. Suppose that $x$ is a 3-element in $(M_i)G$ for some $i \in \{2, 3, 4\}$. Then $x \in D$, which yields that $x \in D_G = 1$, and so $(M_i)G$ is a 2-group. Thus $(M_i)G$ is a Sylow 2-subgroup of $C_G(V) = V(M_i)G$ for every $i \in \{2, 3, 4\}$. Therefore
\[(M_2)_G = (M_3)_G = (M_4)_G.\] Hence we have \(M_5 \cap M_6 \cap M_7 \cap (M_2)_G = D_G = 1.\) This implies that \(G\) is a subdirect product of three cyclic groups \(C_2\) and one symmetric group \(\text{Sym}_3.\) But by Lemma 4.1(1)-b, such a group \(G\) cannot be a \(\mathcal{C}_7\)-group.

Therefore \(K = 1\) and \(G\) is a subdirect product of two cyclic groups \(C_2\) and two symmetric groups \(\text{Sym}_3.\) Since \(G\) has a non-trivial center, by Lemma 4.1(1)-h, \(G\) is not a \(\mathcal{C}_7\)-group.

**Lemma 4.3.** Let \(G\) be a group of order 162 that is a subdirect product of at most five symmetric groups \(\text{Sym}_3.\) Then \(G\) is a \(\mathcal{C}_7\)-group with \(|D| = 2.\)

**Proof.** We first prove that \(G\) is a \(\mathcal{C}_7\)-group, by explicitly providing a \(\mathcal{C}_7\)-cover for \(G.\) First note that \(G\) has a unique normal Sylow 3-subgroup \(P\) such that \(P \cong (C_3)^4.\) Take a \(\mathcal{C}_7\)-cover (see Theorem 2.2) for \(P,\) say \(P = \bigcup_{i=1}^7 K_i,\) where \(K_i\) is a normal subgroup of \(G\) of order 27. Now consider \(L_i = \langle K_i, \iota \rangle,\) where \(\iota\) is an element of order 2 in \(G.\) Then \(\{L_i : 1 \leq i \leq 7\}\) is a \(\mathcal{C}_7\)-cover for \(G.\)

Now consider an arbitrary \(\mathcal{C}_7\)-cover \(\{M_1, \ldots, M_7\}\) with intersection \(D\) for \(G.\) We shall prove \(|D| = 2.\) It is clear that \(|G : M_i| \in \{2, 3\}.\) Therefore we may assume that \(|G : M_i| = 3\) for all \(i \in \{2, \ldots, 7\}.\) Then \(P \cap M_i = (M_i)_G,\) and so \(P \cap D = P \cap \bigcap_{i=2}^7 M_i = \bigcap_{i=2}^7 (P \cap M_i) = D_G = 1.\) Thus \(D\) is a 2-group. Now it is enough to show that \(D \neq 1.\)

Suppose, for a contradiction, that \(D = 1.\) Then \(|\cap_{i \in T} M_i| \leq 2\) for each \(T \in [7]^5.\) We now distinguish the following cases:

**Case 1:** Suppose that \(P\) occurs in the \(\mathcal{C}_7\)-cover \(\{M_1, \ldots, M_7\}.\) Now by considering the subcases in which (i) two members of the cover are conjugate and (ii) no two member of the cover are conjugate, one can get a contradiction in each subcase.

**Case 2:** By Case 1, we may assume that \(|M_i| = 54\) for each \(1 \leq i \leq 7.\) Suppose there are two distinct \(i\) and \(j\) such that \(M_i\) and \(M_j\) are conjugate in \(G.\) As the core of the intersection of any five of the \(M_i\)’s is trivial, one can prove that this case cannot happen.

**Case 3:** Suppose that \(|M_i| = 54\) and \(M_i, M_j\) are not conjugate to each other for every two distinct \(i, j \in [7].\) Now one can get a contradiction, by considering the size of \(\cap_{i \in T} M_i\) \((T \in [7]^5)\) and using the facts that \(f(4) = 9, f(5) = 16\) and \(f(6) = 36.\)

**Lemma 4.4.** Let \(G\) be a group of order 324 that is a subdirect product of at most five symmetric groups \(\text{Sym}_3.\) Then \(G\) is not a \(\mathcal{C}_7\)-group.

**Proof.** There are three groups of order 324 up to isomorphism such that they are subdirect products of five symmetric groups \(\text{Sym}_3.\) Two of them are not \(\mathcal{C}_7\)-groups. This can be checked by a similar program in the proof of Lemma 4.1 of [1]. But that program cannot be applied for the third, because of loose enough time and deficit of memory.

Suppose \(G\) has a \(\mathcal{C}_7\)-cover \(\{M_1, \ldots, M_7\}\) with intersection \(D.\) By hypothesis, \(G\) has a unique Sylow 3-subgroup \(P\) which is an elementary abelian group. It follows that \(P \cap M_i \subseteq \langle P, M_i \rangle = G\) (for all \(i\) such that \(P \neq M_i\)), and so \(P \cap D = 1.\) Therefore \(D\) is a 2-group. By noting that every minimal normal subgroup of \(G\) is of order 3, it is not hard to prove that \(|D| = 4.\) Therefore \(|G : M_i| = 3\) for each \(i \in [7]\) and \(D = \cap_{i \in S} M_i\) for all \(S \in [7]^5.\) One can complete the proof in the following steps:

**Step 1:** \(M_i\) and \(M_j\) are not conjugate for some \(i, j \in [7],\) since otherwise \(|G : M_i \cap M_j| = 6,\) which would imply that \(|M_i \cap M_j| = 54.\) On the other hand \(D\) is a subgroup of \(M_i \cap M_j\) of order 4. This is a contradiction. It follows from [4, Theorem 16.2, p. 57] that \(G = M_i M_j,\) and so \(|G : M_i \cap M_j| = 9.\)
Step 2: Suppose there exists a subset $T \in [7]^4$ such that $D = \bigcap_{i \in T} M_i$. Since $|M_i \cap M_j \cap M_k : D| \leq 4$, $|M_i \cap M_j \cap M_k : D| = 1$ or 3. If $M_i \cap M_j \cap M_k = D$, then $|G : M_i \cap M_j \cap M_k| \leq 3^3 = 27$, a contradiction. Therefore $|M_i \cap M_j \cap M_k : D| = 3$, and so $|G : M_i \cap M_j \cap M_k| = 27$ for all distinct $i, j, k \in [7]$. It follows that $|G : M_i \cap M_j \cap M_k \cap M_t| = 27$ or 81 for all distinct $i, j, k, t \in [7]$.

Step 3: Let $x$ and $y$ denote the number of $S \in [7]^3$ such that $|\cap_{i \in S} M_i| = 4$ and $|\cap_{i \in S} M_i| = 12$, respectively. Then by the Inclusion–Exclusion Principle, we get $4x + 12y = 156$ and we also have $x + y = 35$. It follows that $x = 33$ and $y = 2$.

Step 4: We have $P = \bigcup_{i = 1}^7 (P \cap M_i)$ and $|P \cap M_i| = 27$. This implies that $P$ has a maximal 7-cover. But it can be easily checked by GAP [5] that every 7-cover including normal subgroups of $G$ of order 27 cannot form an irredundant cover for $P$. Also this cover cannot form an irredundant $n$-cover for $n = 5, 6$. Thus $P = \bigcup_{i \in T} (P \cap M_i)$ for some $T \in [7]^4$ is a maximal irredundant 4-cover. Since $f(4) = 9, |\cap_{i \in T}(P \cap M_i)| = 9$. But this is a contradiction, since by Step 2 we have $|\cap_{i \in T} M_i| = 4$ or 12. □

Lemma 4.5. Let $G$ be a $\mathcal{C}_7$-group. If $G$ contains a central subgroup of order 3, then $G \cong (C_3)^4$.

Proof. Suppose that $W$ is a central subgroup of order 3. Then $W$ is not contained in at least three $M_i$’s, say for $i = 1, 2, 3, \ldots$, and so $M_i \not\triangleleft G$, yielding that $G_{M_i} \cong D_3$. Therefore every 5-element of $G$ lies in $M_1 \cap M_2 \cap M_3$. It follows that $G$ contains no 5-element, and so $G$ is soluble. If $K = M_1 \cap M_2 \cap M_3$ is trivial, then $|G| \leq 27$ and $G$ is a 3-group, which contradicts Theorem 2.2. Thus $K$ contains a minimal normal subgroup $L$ of $G$, so that $|L| \in \{2, 3, 4\}$. By Lemma 4.2, $|L| \neq 2$. If $|L| = 4$, then $L \ntriangleleft M_i$ for $i \in \{4, 5, 6, 7\}$ and $M_1 \cap (M_4)_G = 1$. It follows that $G$ is a subdirect product of $C_3, Alt_4$ and $C_3, Sym_3$. This contradicts Lemma 4.1(1).

If $|L| = 3$, then $L$ is not contained in at least three $M_i$’s, say for $i = 4, 5, 6$. If $L$ is central, then by Theorem 2.2, $G \cong (C_3)^4$; otherwise we have $G_{(M_i)_G} \cong Sym_3$ for $i = 4, 5, 6$. Now since $K \cap (M_4)_G \cap (M_5)_G = 1$, the group $G$ is a subdirect product of at most three cyclic groups $C_3$ and at most two symmetric groups $Sym_3$, which contradicts Lemma 4.1(m). This completes the proof. □

Lemma 4.6. Let $G$ be a $\mathcal{C}_7$-group. Suppose that $G$ contains a normal subgroup of order 3. Then $G$ is isomorphic to one of the following groups, (here D is the intersection of any $\mathcal{C}_7$-cover for $G$):

1. $(C_3)^4$;
2. $Sym_3 \times Sym_3$;
3. $(C_3)^3 \times C_2$ (this case may not occur);
4. $(C_3)^4 \times C_2$ and $|D| = 2$. Moreover $|G : D| \leq 81$.

Proof. First, $G$ contains no normal subgroup of order 2 by the hypothesis and by Lemma 4.2. Suppose that $U$ is a minimal normal subgroup of order 3. By Lemma 2.1, $U \not\triangleleft M_i$ for $i = 1, 2, 3, \ldots$, say. Then $G = U M_i$ and $|G : M_i| = 3$ for $1 \leq i \leq 3$. Every 5-element of $C_G(U) = U(M_i)_G$ lies in $\cap_{i = 1}^3 (M_i)_G$, and so lies in $D_G = 1$. Thus $C_G(U)$ is a $\{2, 3\}$-group. It follows that $G$ is a $\{2, 3\}$-group, which implies that $G$ is soluble. If $Z(G) \neq 1$, then $G$ has a central subgroup $L$ of order 3. By Lemma 4.5, $G \cong (C_3)^4$. Thus we may assume that $Z(G) = 1$. Now if $\cap_{i = 1}^3 (M_i)_G = 1$, then $G$ is a subdirect product of three symmetric groups $Sym_3$. It follows from Lemma 4.1(2)-p that $G \cong Sym_3 \times Sym_3$ or $G \cong (C_3)^3 \times C_2$ with $D = 1$ in both cases. Therefore, in any case, $|G : D| \leq 81$. But this is a contradiction, since $Sym_3$ contains no normal subgroup of order 3.

If $|V| = 3$, then $|G : M_i| = 3$ and $C_G(V) = V(M_i)_G$ for $i = 4, 5, 6, 7$, say. Since $\cap_{i = 1}^3 (M_i)_G = 1$, $C_G(U) \cap C_G(V)$ is a 3-group. On the other hand we have $|G : C_G(U)| = |G : C_G(V)| = 2$, and so $|G : C_G(U) \cap C_G(V)| = 2$ or 4. Since $G$ does not contain any normal subgroup of order 2, $\cap_{i = 1}^3 (M_i)_G = 1$, and so $G$ is a subdirect product of five symmetric groups $Sym_3$. Therefore $G$ is supersoluble and $|G| = 3^t \cdot 2^k$, where $t \in \{2, 3, 4, 5\}$ and $k \in \{1, 2\} (\ast)$. Also $D = \cap_{i = 3}^M M_i$ is a 2-group (**); for, if $P$ is a unique normal Sylow 3-subgroup of $G$ (note that $G$ is supersoluble and it is well-known that a finite supersoluble group has a unique normal Sylow $p$-subgroup for the largest prime divisor $p$ of the order of the group), then $P \cap M_i \triangleleft (P, M_i) = G (i = 1, \ldots, 6)$, and so $P \cap D < G$. It follows that $P \cap D < D_G = 1$. We distinguish the following two cases:

Case 1: Suppose that $U \leq M_i$ for all $i \geq 4$. It follows from Lemma 3.2 of [11] that $M_2 \cap M_3 \leq \bigcup_{i = 4}^M M_i$. Thus $M_2 \cap M_3 \leq M_j$ for some $j \in \{4, 5, 6, 7\}$ or $M_2 \cap M_3 \leq M_j$ for each $4 \leq i$ $\leq 7$. The first case implies that...
Let $G$ be a group of order $|G|$, and so $|M_1 \cap M_2 \cap M_3| = |M_1 \cap M_2 \cap M_3| = |M_1 \cap M_2 \cap M_3| = |M_1 \cap M_2 \cap M_3| = 3$. It follows that $|G : D| \leq 3$. The second case implies that $|\Gamma| = |M_2 \cap M_3 : M_1| : 4 \leq 7$ is a 4-cover for $M_2 \cap M_3$. If $\Gamma$ is irredundant, then $|M_2 \cap M_3 : \cap_{i=1}^3 M_i| = |M_2 \cap M_3 : D| \leq f(4) = 9$, and so $|G : D| \leq 81$. If $\Gamma$ is redundant, then $M_2 \cap M_3$ has a 3-cover, and so $|M_2 \cap M_3 : M_2 \cap M_3 \leq M_1 \cap M_1 \cap M_1| \cap M_1 \cap M_1| = f(3) = 4$ for some distinct $i, j, k \in \{4, 5, 6, 7\}$. It follows that $|M_2 \cap M_3 : M_j \cap M_k| \leq 2$, and so $|M_2 \cap M_3 : D| \leq 8$. Thus $|G : D| \leq 72$. In this case, we have proved that $|G : D| \leq 81$. Hence $|G| \in \{18, 36, 54, 108, 162, 324\}$ by $(\ast)$ and $(\ast \ast)$. By 4.1(1)-a, $|G| \neq 18, 108$ and $|G| \neq 324$ by Lemma 4.4.

- If $|G| = 36$, then $G \cong Sym_3 \times Sym_3$.
- If $|G| = 54$, then $G = (C_3)^3 \times C_2 = \cap_{i=1}^3 M_i$.

Suppose that $D = \cap_{i=1}^7 M_i$. Since $G$ is supersoluble, $|G : M_i| = 2$ or $3$. Assume that $P$ is the normal Sylow 3-subgroup of $G$. Then $P \cap M_i \leq G$, and so $P \cap D \leq G$. It follows that $P \cap D = 1$, and so $D$ is a 2-group. Therefore $|D| = 1$ or 2. If $|D| = 2$, then 2 divides $|M_i|$ for each $i \in \{7\}$, and so $|G : M_i| = 3$. Therefore $P = \cup_{i=1}^7 (P \cap M_i) = \cup (M_i)_G$. Since $|P| = 27$, $\Gamma = \Gamma (M_i)_G : 1 \leq i \leq 7$ does not form a $C_n$-cover for $n = 5, 6, 7$. Hence $P = \cup_{i \in S} (M_i)_G$ for some $S \in \{7\}$. This implies that $|\cap_{i \in S} M_i| = 6$. By the Inclusion–Exclusion Principle, $|\cup_{i \in S} M_i| = (4 \times 18) - (6 \times 6) + (4 \times 6) - 6 = 54$ follows, a contradiction. Thus $|D| \neq 2$, and so $D = 1$.

- If $|G| = 162$, then it follows from Lemma 4.3 that $G \cong (C_3)^3 \times C_2$ and $|D| = 2$.

**Case 2:** Suppose that $U$ is contained in at most three $M_i$’s and by Case (1), we may assume that any minimal normal subgroup of $G$ is contained in at most three $M_i$’s. Then $\cap_{i \in T} (M_i)_G = 1$ for each $T \in \{7\}$. Since $C_G(U) = U (M_i)_G$, $C_G(U)$ is a 3-group, and so it is an elementary abelian group. Therefore $|G : C_G(U)| = 2$ since $Z(G) = 1$. Since $\cap_{i=1}^7 (M_i)_G = 1$, $G$ is a subdirect product of four symmetric groups $Sym_3$. Then $G \cong (C_3)^3 \times C_2$ or $(C_3)^4 \times C_2$.

**Lemma 4.7.** Let $G$ be a group of order 192 that is a subdirect product of three alternating groups $Alt_4$. Then $G$ is a $C_7$-group with $|D| = 3$, where $D$ is the intersection of any $C_7$-cover.

**Proof.** First, one may easily check that $G$ is a $C_7$-group by using the command `ConjugacyClassesMaximalSubgroups(G)` instead of `MaximalSubgroups(G)` in the GAP program used in Lemma 4.1 of [1] (the program to test having $C_7$-covers must be modified). But we cannot obtain all $C_7$-covers for $G$, since the number of maximal subgroups of $G$ is very large to run the program.

Now suppose that $\{M_1, \ldots, M_7\}$ is a $C_7$-cover with intersection $D$ for $G$. Note that $G$ has a unique normal Sylow 2-subgroup $P$ which is elementary abelian. We claim that $M_i \neq P$ for each $i \in \{7\}$. Suppose, for a contradiction, that $M_i = P = (C_2)^5$. Then $|M_1 \cap M_i| = 16$ for every $i \geq 2$. One can first prove that no two $M_i$’s are conjugate in $G$. Thus by [4, Theorem 16.2, p. 57] $G = M_1 M_j$, and so $|M_1 \cap M_j| = 12$ for all distinct $i, j \in \{2, \ldots, 7\}$. Also we have $M_i \cap M_j \neq M_k$ for every distinct $i, j, k \in \{2, \ldots, 7\}$. If $(M_i)_G \cap (M_j)_G \subseteq M_k$, then $K = M_1 \cap (M_i)_G \cap (M_j)_G \cap (M_k)_G \neq 1$, and so $K$ contains a minimal normal subgroup of order 3. A contradiction. Thus $G = (M_i \cap (M_j)_G)$, which implies that $4 = |G : M_k| = |M_1 \cap (M_j)_G : M_1 \cap M_j \cap M_k|$. Then $|M_1 \cap M_j \cap M_k| = 3$ for all distinct $i, j \in \{2, \ldots, 7\}$. Now it is easy to see that $|M_1 \cap M_j \cap M_k| = 4$ for all distinct $i, j \in \{2, \ldots, 7\}$. If follows that $|\cap_{i \in S} M_i| = 1$ for every $S \in \{7\}$, where $t \geq 5$, since 3 divides $|\cap_{i \in S} M_i|$. Now by applying the Inclusion–Exclusion Principle on $G = \cap_{i=1}^7 M_i$, one can get a contradiction.

Therefore $M_i \neq P$ and $|G : M_i| = 4$ for all $i \in \{7\}$. Since $P \cap D = \cap_{i=1}^7 (M_i)_G = D_G = 1$, $D$ is a 3-group. We now claim that $D \neq 1$. Suppose, on the contrary, that $|D| = |\cap_{i=1}^7 M_i| = |\cap_{i=2}^7 M_i| = 1$. Then $|\cap_{i \in T} M_i| \leq 2$ for every $T \in \{7\}$. If there exists $T \in \{7\}$ such that $|\cap_{i \in T} M_i| = 2$, then $\cap_{i \in T} M_i \leq P$, and so $2 = |\cap_{i \in T} M_i \cap P| = \cap_{i \in T} (M_i)_G$, a contradiction. Therefore $|\cap_{i \in T} M_i| = 1$ for each $T \in \{7\}$.

(*)

It follows from Lemma 2.1 that $|\cap_{i \in W} M_i| = 1$ or 3 for all sets $W \in \{7\}$. We complete the proof in the following three steps:
Step 1. If \( M_2 = M_1^i \) and \( M_3 = M_1^j \) for some \( x, y \in G \), then \((M_1)_G = (M_2)_G = (M_3)_G\), which implies that \((M_1)_G \cap (M_4)_G = 1\), a contradiction.

Step 2. If \( M_2 = M_1^i \) for some \( g \in G \) and \( M_i, M_j \) are not conjugate for all distinct \( i, j \geq 2 \), then \(|M_1 \cap M_2| = 16\) and \(|M_1 \cap M_i| = |M_2 \cap M_i| = |M_1 \cap M_j| = 12\) for \( i, j > 2 \). Now it is easy to see that \(|M_1 \cap M_2 = (M_1)_G = (M_2)_G\) and \(|M_1 \cap M_2 \cap M_i| = |(M_1)_G \cap (M_4)_G| = 4\) for \( i > 2 \). Since \(|M_1 \cap M_2 \cap M_3 \cap M_4| = 1\), \(|M_1 \cap M_j \cap M_k| \leq 4\). Thus \(|M_1 \cap M_j \cap M_k| = 3\) for all distinct \( i, j, k \in \{1, 3, \ldots, 7\} \) or \( \{2, \ldots, 7\} \). Now let \( x \) and \( y \) denote the number of sets \( S \in [7]^4 \) such that \(|\cap_{i \in S} M_i| = 3\) and \(|\cap_{i \in S} M_i| = 1\), respectively. By the Inclusion–Exclusion Principle, we have that \( 3x + y = 13 \). We also have \( x + y = 35 \), which gives us a contradiction. Hence \( M_i, M_j \) are not conjugate for all distinct \( i, j \geq 1 \), \(|M_i \cap M_j| = 12\) and \(|M_i| = 48\).

Step 3. If \(|\cap_{i \in x} M_i| = 3\) for all sets \( X \in [7]^4 \), then, by (\(*\)), \( M_i \cap M_j \neq M_k \) for all distinct \( i, j, k \in [7] \). If \(|M_i \cap M_j \cap M_k| = 6\) for some distinct \( i, j, k \in [7] \), then \(|M_i \cap M_j \cap M_k \cap M_l| = |M_i \cap M_j \cap M_k \cap M_s|\) for all distinct \( i, j, k, s \in [7] \) and this is the Sylow \( 3\)-subgroup of \( M_i \cap M_j \cap M_k \). Therefore \( D = M_i \cap M_j \cap M_k \cap M_s = M_i \cap M_j \cap M_k \), a contradiction. It follows that \(|M_i \cap M_j \cap M_k| = 3\) for all distinct \( i, j, k \in [7] \). This implies that \( D = M_i \cap M_j \cap M_k \cap M_l = M_i \cap M_j \cap M_k \cap M_s \), a contradiction. Thus there exists \( T \in [7]^3 \) such that \(|\cap_{i \in T} M_i| = 1\). This implies that \(|\cap_{i \in S} M_i| \leq 4\) for every \( S \in [7]^3 \). Thus \(|M_i \cap M_j \cap M_k| = 3\) and \(|M_i \cap M_j \cap M_k \cap M_l| = 1\) or \( 3\), for if \(|M_i \cap M_j \cap M_k \cap M_l| = 2\), then \(|M_i \cap M_j \cap M_k| = |M_i \cap M_j \cap M_k \cap M_l| = 4\) and \(|(M_1)_G \cap (M_4)_G|\). Since \(|M_i \cap M_j| = 12\), we have that \( M_i \cap M_j \cap M_k \cap M_l \cap M_j \cap M_j \) is the normal Sylow \( 2\)-subgroup of \( M_i \cap M_j \), a contradiction. Also note that
\[
|M_i \cap M_j \cap M_k| = 4 \iff |(M_1)_G \cap (M_4)_G \cap (M_4)_G| = 4,
\]
for all \( i, j, k \in [7] \). Now let \( x \) and \( y \) denote the number of \( S \in [7]^3 \) such that \(|\cap_{i \in S} M_i| = 3\) and \(|\cap_{i \in S} M_i| = 4\), respectively; and let \( z \) and \( w \) denote the number of \( S \in [7]^4 \) such that \(|\cap_{i \in S} M_i| = 1\) and \(|\cap_{i \in S} M_i| = 3\), respectively.

Now the Inclusion–Exclusion Principle implies that
\[
(3x + 4y) - (z + 3w) = 93 \quad \text{and} \quad x + y = z + w = 35.
\]
But by (\(\bullet\)) we have
\[
64 = |P| = \left| \bigcup_{i=1}^{7} (P \cap M_i) \right| = \left| \bigcup_{i=1}^{7} (M_i)_G \right| = 7 \times 16 - (21 \times 4) + (x + 4y) - 35 + 21 - 7 + 1.
\]
Therefore \( x + 4y = 56 \). It follows that \( x = 28 \) and \( y = 7 \), so \( z + 3w = 19 \). This is a contradiction since \( z + w = 35 \). \(\square\)

Lemma 4.8. Let \( G \) be a \( \mathcal{C}_7 \)-group and suppose that \( G \) contains a minimal normal subgroup of order 4 and none of order 2, or 3. Then \( G \) is isomorphic to one of the following groups: (1) \( \text{Sym}_4 \) and \( |D| = 1 \); (2) \( (C_2)^4 \times C_3 \) and \( |D| = 1 \); (3) \( (C_2)^4 \times \text{Sym}_3 \) and \( |D| = 2 \); (4) \( (C_2)^6 \times C_3 \) and \( |D| = 3 \); (5) \( (C_2)^6 \times \text{Sym}_3 \) and \( |D| = 6 \). Moreover, \( |G : D| \leq 64 \), where \( D \) is the intersection of any \( \mathcal{C}_7 \)-cover of \( G \).

Proof. First, \( G \) contains no normal subgroup of order 2 or 3 by hypothesis and Lemmas 4.2 and 4.6. Suppose that \( U \) is a minimal normal subgroup of order 4 and \( U \) is not contained in at least four \( M_i \)'s, say, \( U \not\subseteq M_4, M_5, M_6, M_7 \). So \( G = U M_7 \) and \( |G : M_i| = 4 \) for \( i = 4, 5, 6, 7 \). Thus \( G(U) = U(M_7)_G \) for \( 4 \leq i \leq 7 \), and so \( G(U) \) is a \( \{2, 3\} \)-group by Lemma 2.1. Since \( G \) does not contain any normal subgroup of order 2 or 3, \( \cap_{i=4}^{7} (M_i)_G = 1 \), and so \( G(U) \) is a 2-group, which implies that \( G(U) = (U(M_i)_G \cong (C_2)^n \) for some integer \( n \).

The group \( G \) is a \( \{2, 3\} \)-group, since \( G(U) \) embeds into \( \text{Sym}_3 \). Since \( \Phi(G) = 1 \) and \( U(M_i)_G \) is an abelian normal subgroup of \( G \), \( G = G(U) \times H \) such that \( H \cong C_3 \) or \( \text{Sym}_3 \). On the other hand we have \( G(M_7)_G \cong \text{Alt}_4 \) or \( \text{Sym}_4 \), and so \( G \) is a subdirect product of four alternating groups \( \text{Alt}_4 \) or four symmetric groups \( \text{Sym}_4 \). Now we claim that \( G \) is a subdirect product of three alternating groups \( \text{Alt}_4 \) or a subdirect product of three symmetric groups \( \text{Sym}_4 \).

If there exists a subset \( T \subset \{4, 5, 6, 7\} \) such that \( T = 3 \) and \( \cap_{i \in T} (M_i)_G \neq 1 \), then the claim holds. Assume that \( \cap_{i \in T} (M_i)_G \neq 1 \) for each subset \( T \subset \{4, 5, 6, 7\} \) and \( |T| = 3 \). Then \( |G : M_i| = 4 \) for every \( 1 \leq i \leq 7 \) and \( (M_i)_G \) is abelian for each \( i \in [7] \). Since \( G = (M_1)_G \cap (M_2)_G \cap (M_3)_G \cap (M_4)_G \) for every distinct \( i, j, k, t \in [7] \), we have \(|M_1)_G \cap (M_2)_G \cap (M_3)_G \cap (M_4)_G| = |G : M_4| = 4 \). Similarly we have \(|M_1)_G \cap (M_2)_G| = 16 \) and \(|M_4)_G| = 64 \) for every distinct \( i, j \in [7] \). Thus \(|G(U)| = 256 \). On the other hand we have \( G(U) \cap M_i = (M_i)_G \) for each
i ∈ [7], since $C_G(U) = C_G(V)$ for every minimal normal subgroup $V$ of $G$. Therefore $C_G(U) = \bigcup_{i=1}^{7} (M_i)_G$. By the Inclusion–Exclusion Principle, $|\bigcup_{i=1}^{7} (M_i)_G| = 232$. This is a contradiction. This completes the proof of the claim.

If there exists an $i$ such that $(M_i)_G = 1$ and $|G : M_i| = 4$, then $G \cong \text{Sym}_4$. If $G$ is a subdirect product of $\text{Alt}_4$ and $\text{Alt}_4$, then $|G| = 48$ and $|D| = 1$ by Lemma 4.1(2)-a. If $G$ is a subdirect product of $\text{Sym}_4$ and $\text{Sym}_4$, then $|G| = 96$ and $|D| = 2$ by Lemma 4.1(2)-b.

If $G = (C_2)^6 \rtimes C_3$, then $|D| = 3$ by Lemma 4.7.

If $G = (C_2)^6 \rtimes \text{Sym}_3$, then $|D| = 6$ by Lemma 4.1(2)-b, and so $|G : D| = 64$. □

Lemma 4.9. Let $G$ be a $\mathcal{C}_7$-group. Then $G$ does not contain any normal subgroup of order 5.

Proof. Suppose, for a contradiction, that $U$ is a normal subgroup of $G$ of order 5. Then $U$ is not contained in at least five $M_i$’s, say for $1 \leq i \leq 5$. Therefore $G = U M_1 U \cap M_i = 1$, $|G : M_i| = 5$ and $C_G(U) = U(M_i)_G$. Thus $G$ does not contain any normal subgroup of order 3 or 4, since otherwise there would exist at least three $M_i$’s of index 3 or four $M_i$’s of index 4. We may assume that $G$ contains no normal subgroup of order 2 by Lemma 4.2. $C_G(U)$ is a $(2,5)$-group since every 3-element of $C_G(U)$ lies in $\cap_{i=1}^{5} (M_i)_G$, and so lies in $D_G = 1$. Hence $G$ is a $(2,5)$-group since $G/\overline{C_G(U)}$ embeds into $C_4$. Since $G$ is soluble and $G$ contains no normal subgroup of order 2, 3 or 4, we have

$$\bigcap_{i \in S} (M_i)_G = 1 \text{ for every subset } S \subset \{1, \ldots, 5\} \text{ with } |S| = 3.$$  

(*)

Every 2-element of $C_G(U)$ lies in $\cap_{i=1}^{5} (M_i)_G = 1$. So $C_G(U)$ is the unique normal Sylow 5-subgroup of $G$. Thus $C_G(U)$ is an elementary abelian 5-group of rank at most 3. Also note that $G/\overline{C_G(U)}$ is a soluble primitive group of degree 5, and so $G/\overline{C_G(U)} \cong C_5 \times C_5 \times C_2$ or $C_5 \times C_4$. If $G/\overline{C_G(U)} \cong C_5$ for some $i \in \{1, \ldots, 5\}$, then $U$ is central, and so $G = U \times M_j$ for $j \in \{1, \ldots, 5\}$. It follows that $G$ is a 5-group by (*), which is a contradiction by Theorem 2.2. Thus $G/\overline{C_G(U)} \cong C_5 \times C_2$ or $C_5 \times C_4$.

If $N := (M_1)_G \cap (M_2)_G \neq 1$, then $N$ contains a normal subgroup of $G$ of order 5. It follows that $|G : M_i| = 5$ for $i \in \{3, \ldots, 7\}$. Now by (*), we can apply Lemma 3.2 of [11] such that $V_i := M_i$ for $i \in \{3, \ldots, 7\}$. Therefore we have $M_i \cap M_j \subseteq M_i \cup M_j$, which implies that $(M_i)_G \cap (M_j)_G = 1$ for every distinct $i, j \in \{3, \ldots, 7\}$ by (*).

Hence there exist distinct $i, j \in [7]$ such that $(M_i)_G \cap (M_j)_G = 1$, and so $G$ is a subdirect product of $H$ and $H$, where $H \cong C_5 \times C_2$ or $C_5 \times C_4$ and $G = (C_5 \times C_5) \times C_2$ or $(C_5 \times C_5) \times C_4$. By Lemma 4.1(1)-v.such groups do not have a $\mathcal{C}_7$-cover. □

5. The value of $f(7)$

Note that we already know (from Section 2) that $f(7) \geq 81$.

Proof of Theorem B. Suppose, on the contrary, that $G$ is a group with an irredundant 7-cover $C$ with core-free intersection $D$ such that $|G : D| > 81$.

By Theorem A, $C$ is not maximal. Suppose that $C$ is chosen from among such 7-covers of $G$ with as many maximal subgroups as possible. Let $C^*$ be a cover of $G$ that we get from $C$ by replacing one of its non-maximal subgroup by a maximal subgroup containing it. Let $D^*$ be the intersection of $C^*$. The cover $C^*$ is redundant; for, otherwise $D^* = D$ by Lemma 2.1, and so $(D^*)_G = 1$, while $C^*$ has more maximal subgroups than $C$ does. It follows that we may write $G = \bigcup_{i=1}^{7} A_i$, where $A_1$ is not maximal and if $A_i^*$ is a maximal subgroup containing it, then $C^* = \{A_1^*, A_2, \ldots, A_7\}$ is redundant as a cover of $G$. If $G$ is an irredundant union of six subgroups in $C^*$, we may suppose that

$$G = A_1^* \cup A_2 \cup \cdots \cup A_6,$$

since $A_1^*$ is certainly essential. If $D_1 = A_1^* \cap A_2 \cap \cdots \cap A_6$, then it follows from Theorem D of [1], that $|G : D_1| \leq 36$.

But $|D_1 : D| \leq 2$ by Lemma 2.1. Therefore $|G : D| \leq 72$, a contradiction.

If $G$ is an irredundant union of five subgroups from $C^*$, then we may suppose that

$$G = A_1^* \cup A_2 \cup \cdots \cup A_5.$$

If $D_1 = A_1^* \cap A_2 \cap A_3 \cap A_4$, then $|G : D_1| \leq f(5) = 16$ by Theorem 1.1 of [2]. We know $|D_1 : D| \leq 3! = 6$.

Since $|G : D| > 81$ and $G$ is a $(2,3)$-group, $|G : D_1| = 16$ and $|D_1 : D| = 6$. If there exists $i \in \{2, 3, 4, 5\}$ such that
\[ |G : A_i| = 8, \] then \( D_1 = A_i \cap A_j \) for some \( j \neq i \) and \( 2 \leq j \leq 5 \). It follows that \( D = \bigcap_{i=1}^{7} A_i = A_1 \cap A_i \cap A_j \cap A_6 \cap A_7 \), and so \( |D_1 : D| \leq 3 \), a contradiction. If there exists \( i \in \{2, 3, 4, 5\} \) such that \( |G : A_i| = 4 \), then \( |G : A_1 \cap A_j| = 8 \) or 16, and so \( A_i \cap A_j = A_j \cap A_k \) or \( A_i \cap A_j = D_1 \), a contradiction. Therefore \( |G : A_i| = 2 \) for each \( i \in \{2, 3, 4, 5\} \), and so \( |G : A_i^*| = 2 \). In particular, \( \frac{G}{D_1} \cong (C_2)^4 \). It is easy to see \( |A_1 \cap D_1 : D_1| = 2 \), and so \( |A_1 \cap D_1 : |A_1| | = |D_1 : D_1 \cap A_1| = 3 \). Thus 6 divides \( |G : A_1| \). Since \( |G : D| = 96 \), we have \( |G : A_1 \cap D_1| = 48 \); this yields that \( |G : A_1| \) divides 48. Therefore \( |G : A_1| \in \{6, 12, 24, 48\} \). Now it is not hard to get a contradiction by considering the possible sizes obtained for the index of \( A_1 \) in \( G \).

Now assume that \( G \) is an irredundant union of four subgroups in \( C^* \). We may suppose that

\[ \{ A_1^*, A_2, A_3, A_4 \} \]

is an irredundant cover for \( G \). If \( D_1 = A_1^* \cap A_2 \cap A_3 \cap A_4 = A_2 \cap A_3 \cap A_4 \), then \( |G : D_1| \leq 9 \) and \( |G : A_1^*| \leq 3 \) by [6]. Now we distinguish between the following three cases:

**Case 1:** Assume that \( |G : D_1| = 9 \) or 6. Then \( D_1 = A_1^* \cap A_1 \) for each \( i \in \{2, 3, 4\} \) and the set \( \{ A_1, D_1, A_1^* \cap A_5, A_1^* \cap A_6, A_1^* \cap A_7 \} \) is a cover for \( A_1^* \). By considering the irredundancy and redundancy of the latter cover and its subcovers, one can get a contradiction.

**Case 2:** Assume that \( |G : D_1| = 8 \). Then \( |G : A_1^*| = 2 \) and there is an \( i \in \{2, 3, 4\} \) such that \( A_i \) is not maximal. It follows that \( |G : A_i| = 4 \) or 8. But if \( |G : A_i| = 8 \), then \( D_1 = A_i \), which is impossible. Therefore \( |G : A_i| = 4 \), and so \( |G : A_i^* \cap A_i| = 4 \). Thus \( D_1 = A_1^* \cap A_i \). It follows that \( |A_1^* : D_1| \leq 3 \), and so \( |G : D_1| = 4 \), a contradiction.

Hence the set \( \{ A_1^*, A_2, A_3, A_4 \} \) is a redundant cover for \( G \), and so

\[ G = A_1^* \cup A_2 \cup A_3. \]

Then \( D_1 = A_1^* \cap A_2 = A_1^* \cap A_3 = A_1^* \cap A_2 \cap A_3 = A_2 \cap A_3 \) and \( |G : A_1^*| = 2 \). Thus

\[ D = \{ A_1, D_1, A_1^* \cap A_4, A_1^* \cap A_5, A_1^* \cap A_6, A_1^* \cap A_7 \} \]

is a cover for \( A_1^* \). If the cover \( D \) is irredundant, then \( |A_1^* : D| \leq 36 \), and so \( |G : D| \leq 72 \), a contradiction. Therefore \( D \) is redundant, and by considering the subsets of \( D \) which are covers for \( A_1^* \), one can get a contradiction. The proof is now complete. \( \square \)

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**References**