Minimal coverings of completely reducible groups
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Abstract. Let $G$ be a group that is a set-theoretic union of finitely many proper subgroups. Cohn defined $\sigma(G)$ to be the least integer $m$ such that $G$ is the union of $m$ proper subgroups. Determining $\sigma$ is an open problem for most non-solvable groups. In this paper we give a formula for $\sigma(G)$, where $G$ is a completely reducible group.

1. Introduction and results

Let $G$ be a group that is a set-theoretic union of finitely many proper subgroups and by a cover (or covering) of $G$ we mean any finite set of proper subgroups whose set-theoretic union is the whole group $G$. Cohn [4] defined $\sigma(G)$ to be the least integer $m$ (if it exists) such that $G$ has a covering with $m$ subgroups (we call any such covering minimal) and otherwise $\sigma(G) = \infty$. A result of Neumann [12] states that if $G$ is a union of $m$ proper subgroups, then the intersection of these subgroups is of finite index in $G$. It follows that in study of $\sigma(G)$, we may assume that $G$ is finite. It is an easy exercise that $\sigma(G)$ can never be 2, so $\sigma(G) \geq 3$. Groups that are the union of three proper subgroups, as $C_2 \times C_2$ is for example, are investigated in papers [6], [7], [14]. Also groups $G$ with $\sigma(G) \in \{3, 4, 5\}$ and $\sigma(G) = 6$ are characterized in [4] and [1], respectively. However Tomkinson [15] proved that there is no group with $\sigma(G) = 7$. Cohn [4] showed that for any prime power $p^a$ there exists

Mathematics Subject Classification: 20D60.
Key words and phrases: covering groups by subgroups, completely reducible groups, simple groups.

This work was in part supported by a grant from the Center of Excellence for Mathematics, University of Isfahan. The research of the first author was in part supported by a grant from IPM (No. 85200032).
a solvable group \( G \) with \( \sigma(G) = p^a + 1 \). In fact, Tomkinson [15] established that \( \sigma(G) - 1 \) is always a prime power for solvable groups \( G \). It is natural to ask what can be said about \( \sigma(G) \) for non-solvable groups. Bryce, Fedri and Serena begun this project in [3], where they calculated \( \sigma(G) \) for the linear groups \( G \in \{ \text{PSL}_2(q), \text{PGL}_2(q), \text{SL}_2(q), \text{PGL}_2(q) \} \). They obtained the formula \( \frac{1}{2}q(q + 1) \) for even prime powers \( q \geq 4 \) and the formula \( \frac{1}{2}q(q + 1) + 1 \) for odd prime powers \( q \geq 5 \). Moreover Lucido [10] studied this problem for the simple Suzuki groups and found that \( \sigma(\text{Sz}(q)) = \frac{1}{2}q^2(q^2 + 1) \), where \( q = 2^{2m+1} \).

Marótı [11] gave exact or asymptotic formulas for \( \sigma(\text{Sym}_n) \) and \( \sigma(\text{Alt}_n) \). In particular, it is shown in [11] that if \( n > 1 \) is odd, then \( \sigma(\text{Sym}_n) = 2^{n-1} \) unless \( n = 9 \) and \( \sigma(\text{Sym}_n) \leq 2^{n-2} \) if \( n \) is even. Also Marótı proved that if \( n \neq 7, 9 \), then \( \sigma(\text{Alt}_n) \geq 2^{n-2} \) with equality if and only if \( n \) is even but not divisible by 4.

Holmes in [8] obtained \( \sigma(S) \) for some sporadic simple groups \( S \). See also [9] for some related results. Thus the situation for non-solvable groups seems to be totally different from solvable ones.

A group \( G \) is called completely reducible if it is a direct product of simple groups. In the sequel a completely reducible group will be called a CR-group. Note that in a CR-group, every normal subgroup is a direct factor (see [13, Theorem 3.3.12]). A CR-group is centerless if and only if it is a direct product of non-abelian simple groups. A finite group \( G \) contains a normal centerless CR-subgroup which contains all normal centerless CR-subgroups; this subgroup is called the centerless CR-radical of \( G \). For more details concerning CR-groups, see [13, pp. 88–89]. In this paper we prove the following results.

**Theorem 1.1.** Let \( G \) be a finite group. If \( G = A_1 \times A_2 \times \cdots \times A_n \), where \( A_i \) is a non-abelian simple group for each \( i \), then \( \sigma(G) = \min\{\sigma(A_1), \ldots, \sigma(A_n)\} \).

**Theorem 1.2.** Let \( G \) be a finite CR-group. Then \( \sigma(G) = \min\{\sigma(R), \sigma(G/R)\} \), where \( R \) is the centerless CR-radical of \( G \).

## 2. Proofs

We begin with the following easy lemma.

**Lemma 2.1.** Let \( G \) be a finite non-cyclic group. If \( M \) is a maximal subgroup of \( G \) such that \( \sigma(G) < \sigma(M) \), then either \( M \) is a normal subgroup of \( G \) or \( |G : M| \leq \sigma(G) - 1 \).

**Proof.** Suppose that \( M \not\triangleleft G \). Then \( M \) has \( |G : M| \) conjugates in \( G \). There are maximal subgroups \( A_i \) of \( G \) for which \( G = \cup_{i=1}^{\sigma(G)} A_i \) and \( M = \cup_{i=1}^{\sigma(G)} (M \cap A_i) \).
Since \( \sigma(G) < \sigma(M) \), then there exists \( j \in \{1, \ldots, \sigma(G)\} \) such that \( M = M \cap A_j \).
Hence for every \( x \in G \), there exist \( i_x \in \{1, \ldots, \sigma(G)\} \) such that \( M^x = A_{i_x} \).
Therefore \( |G : M| \leq \sigma(G) \). Now since \( G \neq \bigcup_{g \in G} M^g \), \( |G : M| \leq \sigma(G) - 1 \). 

The following result which will be useful in the sequel, is a generalization of Lemma 4 of [4]. Its proof is similar to that of Lemma 4 of [4] and we give it for the reader’s convenience.

**Proposition 2.2.** Let \( G \) be a finite group such that \( G = H \times K \) for two subgroups \( H \) and \( K \) of \( G \). If every maximal subgroup of \( G \) contains either \( H \) or \( K \), then \( \sigma(G) = \min\{\sigma(H), \sigma(K)\} \).

**Proof.** Since every maximal subgroup \( M \) of \( G \) contains either \( H \) or \( K \), \( M \)
is equal to either \( H_0 \times K \) or \( H \times K_0 \), where \( H_0 \) is maximal in \( H \) and \( K_0 \) maximal in \( K \). Thus we may assume that \( G = (\bigcup_{i=1}^p H \times M_i) \bigcup (\bigcup_{j=1}^q N_j \times K) \), where \( p + q = \sigma(G) \), \( p, q \geq 0 \) and \( M_i \) is maximal in \( K \) and \( N_j \) is maximal in \( H \). Now we claim that one of \( p \) and \( q \) must be zero.

Let \( G_1 = \bigcup_{i=1}^p H \times M_i \) and \( G_2 = \bigcup_{j=1}^q N_j \times K \). If \( q \neq 0 \), then \( G_1 \neq G \) and so there exists an element \( a_2 \in G \setminus G_1 \). Therefore \( a_2 \notin M_i \) for all \( i \in \{1, \ldots, p\} \) and so \( aa_2 \notin G_1 \) for all \( a \in H \). Hence \( aa_2 \in G_2 \) for all \( a \in H \). Thus \( aa' \in G_2 \) for all \( a \in H \) and \( a' \in K \). Hence \( G_2 = G \) and \( p = 0 \).

Now if \( p = 0 \), then \( G = G_2 = (\bigcup_{j=1}^q N_j)K \), whence \( H = \bigcup_{j=1}^q N_j \). This implies that \( \sigma(H) \leq \sigma(G) = q \). Similarly if \( q = 0 \), then \( \sigma(K) \leq p = \sigma(G) \). But \( \sigma(G) \leq \min\{\sigma(H), \sigma(K)\} \) — see for example Lemma 2 in [4] — which gives the result. 

Recall that a finite group \( G \) is said to be primitive if it has a maximal subgroup \( M \) such that the core of \( M \) in \( G \), \( M_G = \cap_{g \in G} M^g \) is trivial. In this situation we call \( M \) a stabilizer of \( G \). We need the following trichotomy of R. Baer on primitive groups.

**Theorem 2.3 (Baer [2]).** Let \( G \) be a finite primitive group with a stabilizer \( M \). Then exactly one of the following three statements holds:

1. \( G \) has a unique minimal normal subgroup \( N \), this subgroup \( N \) is self-centralizing (in particular, abelian), and \( N \) is complemented by \( M \) in \( G \).
2. \( G \) has a unique minimal normal subgroup \( N \), this \( N \) is non-abelian, and \( N \) is supplemented by \( M \) in \( G \).
3. \( G \) has exactly two minimal normal subgroups \( N \) and \( N^* \), and each of them is complemented by \( M \) in \( G \). Also \( C_G(N) = N^* \), \( C_G(N^*) = N \) and \( N \cong N^* \cong NN^* \cap M \).
Remark 2.4 (see Example 15.3(3) in p. 54 of [5]). Let $G$ be a finite group. 
(1) If $M$ is a maximal subgroup of $G$, then $G/M$ is a primitive group.
(2) If $G$ is a non-abelian simple group, then $G \times G$ is a primitive group in which the diagonal subgroup $D = \{(g,g): g \in G\}$ is a stabilizer.

**Lemma 2.5.** Let $H$ and $K$ be non-abelian simple groups. If $G = H \times K$, then $\sigma(G) = \min(\sigma(H), \sigma(K))$.

**Proof.** If $H \cong K$, then $G \cong H \times H$ is a primitive group with stabilizer diagonal subgroup $D = \{(h,h): h \in H\}$. We have $D \cong H$ and $D$ is a maximal subgroup of $G$ which is not normal in $G$. If $\sigma(G) < \sigma(H) = \sigma(D)$, then by Lemma 2.1, $|G : D| \leq \sigma(G) - 1$. Since $|G : D| = |H|$, we have $|H| < \sigma(H)$ which is a contradiction. Thus $\sigma(G) \geq \sigma(H)$. Now the corollary to Lemma 2 of [4] completes the proof.

Thus we may assume that $H \not\cong K$. Then by Theorem 2.3 $G$ is not a primitive group and so $M_G$ is non-trivial for every maximal subgroup $M$ of $G$. Therefore $M_G = H$ or $M_G = K$ and so $H \leq M$ or $K \leq M$. The proof is now complete by Proposition 2.2.

**Proof of Theorem 1.1.** We argue by induction on $n$. If $n = 1$, then the result is clear and if $n = 2$, then the result follows from Lemma 2.5. So we may assume that $n \geq 3$. If there exist distinct $i, j \in \{1, \ldots, n\}$ such that $A_i \cong A_j$ and $i < j$, then $G \cong G_1 = N \times A_i \times A_j$, where

$$N = \prod_{k \in \{1, \ldots, n\} \setminus \{i,j\}} A_k.$$ 

Now consider $M = N \times D$, where $D = \{(a,a): a \in A_i\}$ is the diagonal subgroup of $A_i \times A_i$. Then $M$ is a maximal subgroup of $G_1$ which is not normal in $G_1$, since $D \not\trianglelefteq A_i \times A_i$. On the other hand, since $D \cong A_i$, by the induction hypothesis we have $\sigma(M) = \min\{\sigma(A_1), \ldots, \sigma(A_n)\}$. It follows from the corollary to Lemma 2 of [4] that $\sigma(G_1) \leq \sigma(M)$. Now suppose, aiming for a contradiction, that $\sigma(G_1) < \sigma(M)$. Then Lemma 2.1 implies that $|G_1 : M| < \sigma(G)$. Therefore $\sigma(G) > |A_i| > \sigma(A_i)$, which is the contradiction we sought. Hence $\sigma(G) = \sigma(M) = \min\{\sigma(A_1), \ldots, \sigma(A_n)\}$.

Now assume that $A_i \not\cong A_j$ for any two distinct $i, j \in \{1, \ldots, n\}$ and let $H = A_1 \times A_2 \times \cdots \times A_{n-1}$. We claim that every maximal subgroup $S$ of $G$ contains either $H$ or $A_n$. If $A_n \not\leq S$, then $A_n \not\leq S_G$ and so $S_G = A_{i_1} \times \cdots \times A_{i_k}$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n-1\}$. Since $G_{S_G}$ is a primitive group, Theorem 2.3 implies that $k = n - 1$ and so $S_G = H \leq S$. The proof is now complete by Proposition 2.2 and induction hypothesis. □
Let by hypothesis and Proposition 2.2, it is enough to show that every

In this case K contains either A or R. If A /∈ M, then A /∈ MG. Thus there exists a normal subgroup N of prime order such that N /∈ MG. Since G/MG is a primitive group and N/MG is a minimal normal subgroup of G/MG, it follows from Theorem 2.3 that G/MG contains a unique minimal normal abelian subgroup. If R /∈ MG, then there exists a non-abelian simple normal subgroup S ⊆ R of G such that S /∈ MG. Thus SMG is a minimal normal subgroup of G/MG, and so it is abelian, a contradiction. This implies that R ≤ MG ≤ M. Now the proof follows from Proposition 2.2.

**Proposition 2.6.** Let H be a finite CR-group whose center is of odd order and let Symn be the symmetric group of degree n ≥ 5. Then σ(H × Symn) = min{σ(H), σ(Symn)}.

**Proof.** By hypothesis and Proposition 2.2, it is enough to show that every maximal subgroup M of G = H × Symn contains either H or Symn. If H /∈ M, then H /∈ MG and so, as H is a CR-group, there exists a (non-abelian or abelian) simple normal subgroup S contained in H such that S /∈ MG. Therefore S ∩ MG = 1 and SMG ∼ S is a (simple) minimal normal subgroup of G/MG. Also MG ∩ Symn = 1, Altn or Symn.

We dismiss the first two of these possibilities.

(1) If MG ∩ Symn = 1, then Symn ∼ SMG ≤ G/MG. Since Altn ≤ Symn, K = MG/Alt is a minimal normal subgroup of G/MG. Now we claim that K ≠ SMG; if X = Alt, MG = SMG and each product is direct. Now C_G(MG) = Z(MG) ∩ Alt = Z(MG)S so C_(MG)' = Alt = S ≤ H, a contradiction. Since G/MG is primitive, Theorem 2.3 implies that C_G(SMG/MG) = K. Thus Symn ∼ MG/Symn ≤ K ∼ Alt, which is a contradiction.

(2) In this case MG ∩ Symn = Alt and so MG/Symn is a normal subgroup of order 2, therefore central in the primitive group G/MG. Thus by Theorem 2.3, K ∼ C_2. Since S ∼ SMG ≤ G/MG, we have that S ∼ C_2 and so the center of H is of even order, contradicting the hypothesis.

Hence MG ∩ Symn = Symn ≤ MG ≤ M. This completes the proof.

**Acknowledgment.** The authors are grateful to the referee for valuable suggestions.
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(Received October 10, 2006; revised December 22, 2006)