3-GENERATOR GROUPS WHOSE ELEMENTS COMMUTE WITH THEIR ENDOMORPHIC IMAGES ARE ABELIAN

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A group in which every element commutes with its endomorphic images is called an E-group. Our main result is that all 3-generator E-groups are abelian. It follows that the minimal number of generators of a finitely generated non-abelian E-group is four.

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1. INTRODUCTION AND RESULTS

A group in which each element commutes with its endomorphic images is called an E-group. It is known that an E-group is a 2-Engel group, and thus it is nilpotent, of nilpotent class at most 3. All abelian groups are trivially E-group, non-abelian E-groups of class 2 exist (see, e.g., Caranti, 1985; Caranti et al., 1987) and examples of E-groups of class 3, asked by Caranti (The Kourovka Notebook, 2002, Problem 11.46a), are not known. The first examples of non-abelian E-groups are due to Faudree (1971). Faudree’s examples are 4-generator.

Our main result is to prove that 4 is the minimal number of generator of a non-abelian E-group.

Theorem 1.1. Every 3-generator E-group is abelian.

The unexplained notation follows that of Abdollahi et al. (2008). In Abdollahi et al. (2008, Theorem 1.1) we showed that a finite 3-generator E-group is nilpotent of class at most 2, and it is proved in Abdollahi et al. (2008, Theorem 1.3) that an infinite, 3-generator E-group is abelian. Thus, to prove Theorem 1.1, we are left with ruling out the case of a finite p-group, which is a 3-generator E-group of class 2. To prove the latter, the main ingredients are the following:

1) Theorems 2.2 and 2.5. In these theorems, we classify 3-generator pE-groups by introducing the groups G(p, r, t, [tij]).

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(2) The result of Morigi (1995) concerning $p$-groups with an abelian automorphism group, for $p$ odd, (Theorem 4.1) and an adaptation to the case $p = 2$ (Proposition 4.2).

(3) Lemma 2.7 in which we have proved a dichotomy for endomorphisms of a 3-generator $p\mathcal{E}$-group: they are either central automorphism or their images are contained in the center.

2. CLASSIFICATION OF 3-GENERATOR $p\mathcal{E}$-GROUPS

In Abdollahi et al. (2008, Theorem 2.10), a complete classification for all 3-generator $p\mathcal{E}$-groups for $p > 2$ is given. Here we classify 3-generator 2\mathcal{E}-groups (Theorems 2.2 and 2.5, below). We also determine all $p\mathcal{E}$-groups whose derived subgroups are cyclic (Theorem 2.4, below).

Remark 2.1. We know that a finite $pE$-group is a $p\mathcal{E}$-group (Malone, 1977); but the converse is false in general (Abdollahi et al., 2008, Remark 2.2). Besides $pE$-groups whose existence of class 3 is unknown, there exist $p\mathcal{E}$-groups of class 3. Thanks to the nq package of Nickel which is available in GAP (2005), one can construct the largest (with respect to the size) 2-Engel 9-generator group $G = \langle x_1, \ldots, x_9 \rangle$ of exponent 27 with the following relations:

$$\begin{align*}
    x_1^3 &= [x_2, x_3][x_4, x_5][x_6, x_7][x_8, x_9], \\
    x_2^3 &= [x_1, x_3][x_4, x_6][x_5, x_7][x_8, x_9], \\
    x_3^3 &= [x_1, x_2][x_4, x_7][x_5, x_6][x_8, x_9], \\
    x_4^3 &= [x_1, x_5][x_2, x_6][x_3, x_7][x_{10}, x_9], \\
    x_5^3 &= [x_1, x_4][x_2, x_8][x_3, x_9][x_6, x_7], \\
    x_6^3 &= [x_1, x_9][x_2, x_8][x_3, x_7][x_5, x_6], \\
    x_7^3 &= [x_1, x_8][x_3, x_9][x_5, x_7][x_6, x_9], \\
    x_8^3 &= [x_1, x_9][x_3, x_6][x_5, x_7][x_2, x_4], \\
    x_9^3 &= [x_1, x_8][x_3, x_6][x_5, x_7][x_2, x_4].
\end{align*}$$

Now it can be easily seen by GAP (2005), that we have $|G| = 3^{34}$, $|G'| = 3^{75}$, $|Z(G)| = 3^{39}$, $\exp(G) = 3$, $G' = Z_3^3(G) \cong C_9 \times C_3$, and $\Omega_1(G') = \gamma_3(G) = Z(G) \cong C_3^9$.

Since every commutator $[x_i, x_j]$ appears only once in the above relations, it follows that

$$\langle x_1^3, \ldots, x_9^3 \rangle = \langle x_1^3 \rangle \times \cdots \times \langle x_9^3 \rangle.$$

Therefore, $|G^3| = |\langle x_1^3, x_2^3, \ldots, x_9^3 \rangle G^3| = 3^{45}$ and so by regularity, $|\Omega_1(G)| = |G : G^3| = 3^{39}$. Hence $\Omega_1(G) = \gamma_3(G) = Z(G)$ and $G$ is a 3\mathcal{E}-group of class 3. We were unable to show whether $G$ is an $E$-group or not.

Theorem 2.2. Let $G$ be a non-abelian 3-generator $p\mathcal{E}$-group, $\exp(G) = p'$, $\exp(G') = p^r$, and $(p > 2$ or $(p = 2$ and $\exp(G') \neq 2^r))$. Then $|G| = p^{3(r+1)}$ and $G$ has the presentation

$$\langle x, y, z \mid x^{p^{r+1}} = y^{p^{r+1}} = z^{p^{r+1}} = [x^{p^r}, y] = [x^{p^r}, z] = [y^{p^r}, z] = [z^{p^r}, x] = [z^{p^r}, y] = 1, [x, y] = x^{p^{r+1}} y^{p^{r+1} z^{p^{r+1}}} x^{p^{r+1}}, [x, z] = x^{p^{r+1}} y^{p^{r+1} z^{p^{r+1}}} z^{p^{r+1}}, [y, z] = x^{p^{r+1}} y^{p^{r+1} z^{p^{r+1}}} z^{p^{r+1}} \rangle.$$


where \( 1 \leq t \leq r \) and \([t_{ij}] \in \text{GL}(3, \mathbb{Z}_p)\). Moreover every group with the above presentation is a \( p\mathbb{Z}\)-group.

**Proof.** For the case \( p > 2 \), the proof is the same as the proof of Theorem 2.10 of Abdollahi et al. (2008). For the other case, we need some modifications in the proof of the first case because of technical details. However, we give the following proof covering both cases. By Abdollahi et al. (2008, Theorem 2.9), \( \text{cl}(G) = 2 \). Suppose that \( \frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle \), for some \( a, b, c \in G \) such that \( |aZ(G)| = |bZ(G)| = p^t \) and \( |cZ(G)| = p^s \) for some integer \( s \), \( 0 \leq s \leq t \). Then clearly \( G' = \langle [a, b], [a, c], [b, c] \rangle \), \( [[a, b]] \leq p', [[a, c]] \leq p', \) and \( [[b, c]] \leq p' \). Therefore \( |G'| \leq p^{t+2s} \). For all \( x, y \in G \), we have \((xy)^{p^s} = x^{p^s}y^{p^s}[y, x]^{p^{s-1}} = x^{p^s}y^{p^s}\).

It follows that the map \( x\Omega_r(G) \mapsto x^{p^s} \) is an isomorphism from \( \frac{G}{\Omega_r(G)} \) to \( G^{p^s} \). Thus \( |G : \Omega_r(G)| = |G^{p^s}| \). Then \( |G| = |\Omega_r(G)||G^{p^s}| \leq |Z(G)||G'| \) and so \( |G : Z(G)| \leq |G'| \). Hence \( p^{2r+s} \leq p^{t+2s} \) and \( t \leq s \). It follows that \( s = t \), \( |G'| = \left| \frac{G}{\Omega_r(G)} \right| = p^{3r} \) and \( G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle \). We have \( G = \langle a, b, c \rangle \) (since \( \frac{G}{\Omega_r(G)} \cong C_p \times C_p \times C_p \)).

Now, since \( G^{p^r} \leq G' \) and \( |G'| = |G : Z(G)| \leq |G : \Omega_r(G)| = |G|^p \), we have \( G' = G^{p^r} \). By Abdollahi et al. (2008, Lemma 2.4), \( \exp(G) = p^{r+1} \) and since \( G' = G^{p^r} \) is an abelian group of order \( p^{3t} \), it follows that \( G^{p^r} = \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \), and \( |a| = |b| = |c| = p^{r+1} \). Also since \( G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \leq \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle \), it is not hard to see that \( \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \) and so \( p^{3r} = |\langle a^{p^r}, b^{p^r}, c^{p^r} \rangle| \leq |G^{p^r}| \leq |\Omega_r(G)| \leq |Z(G)| = |G : G'| \leq p^{3r} \).

It follows that \( G^{p^r} = \Omega_r(G) = Z(G) \) and so \( |G| = p^{3(t+1)} \). Since \( G' = G^{p^r} \), there exists a \( 3 \times 3 \) matrix \( T = [t_{ij}] \in \text{GL}(3, \mathbb{Z}_p) \) such that
\[
[a, b] = a^{p^{rt_11}}b^{p^{rt_12}}c^{p^{rt_13}}, \quad [a, c] = a^{p^{rt_{21}}}b^{p^{rt_{22}}}c^{p^{rt_{23}}}, \quad [b, c] = a^{p^{rt_{31}}}b^{p^{rt_{32}}}c^{p^{rt_{33}}},
\]
and every element of \( G \) can be written as \( a^ib^jc^k \) for some \( i, j, k \in \mathbb{Z} \), and
\[
(a^ib^jc^k)(a^{i'}b^{j'}c^{k'}) = a^{i+j-j'p^{rt_{11}}-i'p^{rt_{31}}-j'p^{rt_{31}}}b^{i+j'-i'p^{rt_{12}}-i'p^{rt_{22}}-j'p^{rt_{22}}-j'p^{rt_{32}}}c^{k+j^'+j'^{p^{rt_{13}}}+i'p^{rt_{32}}-i'p^{rt_{32}}-j'p^{rt_{33}}}
\]
Now consider \( \tilde{G} = \mathbb{Z}_{p^{rt_1}} \times \mathbb{Z}_{p^{rt_2}} \times \mathbb{Z}_{p^{rt_3}} \) and define the following binary operation on \( \tilde{G} \):
\[
(i, j, k)(i', j', k') = (i + i' - j'p^{rt_{11}} - i'p^{rt_{21}} - j'p^{rt_{31}}),
\]
\[
 j + j' - i'p^{rt_{12}} - i'p^{rt_{22}} - j'p^{rt_{22}} - i'p^{rt_{32}}, k + k' - i'p^{rt_{13}} - i'p^{rt_{32}} - j'p^{rt_{33}}.
\]
It is easy to see that, with this binary operation, \( \tilde{G} \) is a group and \( G \cong \tilde{G} \). Now one can easily see that the group \( G \) has the required presentation. \( \square \)

**Notation.** For any prime number \( p \), and integers \( r, t \) with \( 1 \leq t \leq r \) and \([t_{ij}] \in \text{GL}(3, \mathbb{Z}_p)\), we write \( G(p, r, t, [t_{ij}]) \) to denote the group \( G \) with the presentation given in Theorem 2.2.
Lemma 2.3. Let $G$ be a finite nilpotent group of class 2. If $G$ is 2-generator, then $|G| = |G'|^2|Z(G)|$.

Proof. Let $G = \langle a, b \rangle$, $H = \langle a \rangle Z(G)$ and $K = \langle b \rangle Z(G)$. Then $H$ and $K$ are normal subgroups of $G$. We see that $G = HK$ and $H \cap K = Z(G)$. If $|aZ(G)| = n$, then $[a, b]^p = 1$ and since $G' = \langle [a, b] \rangle$, so $|G'|$ divides $n$. Therefore $|G'|$ divides $\frac{n}{|Z(G)|}$. Similarly $|G'|$ divides $\frac{k}{|Z(G)|}$. It follows that $|G'|^2|Z(G)|$ divides $|G|$. On the other hand, we have

$$ |G : Z(G)| = |G : C_G(a) \cap C_G(b)| \leq |G : C_G(a)||G : C_G(b)| \leq |G'|^2. $$

Hence $|G| = |G'|^2|Z(G)|$. \hfill $\square$

Theorem 2.4. Let $G$ be a non-abelian $p\mathbb{E}$-group with cyclic derived subgroup. Then $G$ is isomorphic to $Q_8 \times C_2^n$, for some non-negative integer $n$.

Proof. Since $G$ is a $p$-group and $G'$ is cyclic, there exist elements $a, b \in G$ such that $G' = \langle [a, b] \rangle$. Let $H = \langle a, b \rangle$, $\exp\left(\frac{G}{H}\right) = p'$ and $\exp(G') = p$. By Lemma 2.3,

$$ |H'|^2 = |H : Z(H)| \leq |H : Z(G) \cap H| = |HZ(G) : Z(G)| \leq |G : Z(G)|. $$

Therefore $|G| \geq |G'|^2|Z(G)|$. If $p > 2$, then by regularity, $|G| = |G'|^2|\Omega_2(G)| \leq |G'|^2|Z(G)|$. This implies that $G$ is abelian, a contradiction. Thus $p = 2$. Since $G'$ is cyclic, we have $a^{2^s}, b^{2^s} \in G'$ we have $\langle a^{2^s} \rangle \leq \langle b^{2^s} \rangle$ or $\langle b^{2^s} \rangle \leq \langle a^{2^s} \rangle$. We may assume that $a^{2^s} = b^{2^s}$ for some integer $s$. It follows that $(ab^{-r})^{2^s+1} = 1$ and so $(ab^{-r})^{2^s} = 1$. Thus $1 \leq (ab^{-r})^{2^s} = [a, b]^{2^s}$ and so $t = 1$. If $r \geq 2$, then $(ab^{-r})^{2^s} = 1$ and so $ab^{-r} \in Z(G)$ implies that $[a, b] = 1$, a contradiction. Thus $r = 1$ and $G^2 = G'$. It follows that $a^2 = b^2 = [a, b]$ and so $H \cong Q_8$. Now we claim that $G = HC_G(H)$ and $C_G(H)$ is an elementary abelian 2-group. Assume on the contrary that there exists an element $g \in G$ such that $g \notin HC_G(H)$. Then $g^2 \neq 1$ and $g^2 = a^2 = b^2$ and so $(ga)^2 = [g, a]$. If $[g, a] = 1$, then $ga \in Z(G) \leq C_G(H)$ and $g \in HC_G(H)$, a contradiction. Therefore $[g, a] \neq 1$ and $[g, a] = [a, b]$. Similarly $[g, b] = [a, b]$. Then $ga$ and $gb$ commute with $a$ and $b$. Thus $g \in HC_G(H)$, a contradiction.

Next suppose that there exists $x \in C_G(H)$ such that $x^2 \neq 1$. We have $x^2 = a^2$ and $(xa)^2 = 1$. Then $xa \in Z(G)$ and so $1 = [xa, b] = [a, b]$ which is impossible. Hence our claim is proved. Also we have $H \cap C_G(H) = Z(H) = \langle a^2 \rangle$ and so $C_G(H) = \langle a^2 \rangle \times E$ for some elementary abelian 2-group $E$. Hence $G$ is isomorphic to $H \times E$ and the proof is complete. \hfill $\square$

Now we complete the classification of $3$-generator $p\mathbb{E}$-groups.

Theorem 2.5. Let $G$ be a non-abelian 3-generator $2\mathbb{E}$-group such that $\exp\left(\frac{G}{G'}\right) = \exp(G') = 2'$. Then $G$ is isomorphic to one of the following groups:

(i) $Q_8 \times C_2$;
(ii) $\langle x, y, z | x^4 = y^4 = [y, z] = 1, x^2 = z^2 = [x, y], (xz)^2 = y^2 \rangle$;
(iii) $\langle x, y, z | x^4 = z^4 = [y, z] = 1, x^2 = y^2 = [x, y], [x, z] = z^2 \rangle$;
(iv) $G(2, r, r, [t_{ij}])$ where $[t_{ij}] \in GL(3, \mathbb{Z}_{p'})$. 


Proof. Suppose that \( G = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle \), for some \( a, b, c \in G \), where \( |aZ(G)| = |bZ(G)| = |cZ(G)| = 2^s \) and \( 0 \leq s \leq r \). If \( s = 0 \), then \( G' \) is cyclic and so by Theorem 2.4, \( G \) is isomorphic with \( Q_8 \times C_2 \). Therefore, we may assume that \( s \geq 1 \). Clearly, we have \( G' = \langle [a, b], [a, c], [b, c] \rangle \).\(^*\) Since \( a^{2^r} = b^{2^r} = 1 \) and \( G' = \langle a, b \rangle \), we may assume that \( a^{2^{s+1}} = b^{2^{s+1}} \) for some integer \( k \). It follows that \( (ab^{-1})^2 \in \Omega_r(G) \leq Z(G) \) and so \( |a, b|^{2^1} = [a, ab^{-1}]^{2^1} = 1 \). Therefore, \( \exp(G') \leq 2^r \). Thus \( r = s, |G/Z(G)| = 2^{3r} \) and \( |a| = |b| = |c| = 2^r \). Since \( 2^{3r} = |G : Z(G)| \leq |G : \Omega_r(G)| \leq 2^{3r} \) we have \( G' = Z(G) = \Omega_r(G) \). Now the map \( x \Omega_{r+1}(G) \rightarrow x^{2^r+1} \) is an isomorphism from \( \frac{G}{\Omega_{r+1}(G)} \) to \( G^{2^r+1} \). It follows that

\[
|G| = |\Omega_{r+1}(G)| \cdot |G^{2^r+1}| \leq |\Omega_{r+1}(G)| \cdot |\Omega_r(G)| \cdot |\Omega_r(G)| \cdot (G')^2 | \leq 8|Z(G)||G|^2\]

and so \( |(G')^2| \geq 2^{3r-3} \). Suppose that \( G' \cong C_{2^r} \times C_{2^r} \times C_{2^r} \) where \( 0 \leq v \leq u \leq r \). If \( u = 0 \), then \( |(G')^2| = 2^{v+u-2} \geq 2^{r-2} \). Therefore in this case \( r = 1 \), \( |G| = 2^5 \), and so by GAP (2005) one can easily see that \( G \) has a presentation as in either (ii) or (iii). Then we may assume that \( v \geq 1 \) and \( |(G')^2| = 2^{v+u-v-3} \) which implies that \( u = v = r \), \( |G'| = 2^{3r}, |G| = 2^{6r} \). It follows that \( G' = \langle [a, b] \rangle \langle [a, c] \rangle \langle [b, c] \rangle \).

Now we claim that \( G^{2^r} = G' \). If \( r = 1 \), then \( |G| = 64 \) and by GAP (2005) it can be seen that there exist exactly four \( 2\)-groups \( G \) such that \( \exp(G') = \exp(G) = 2 \) satisfying \( G^{2^r} = G' \). Thus we may assume that \( r > 1 \). We prove that \( \langle a^{2^r}, b^{2^r}, c^{2^r} \rangle \rangle = \langle a^{2^r} \rangle \times \langle b^{2^r} \rangle \times \langle c^{2^r} \rangle \). If \( a^{2^rm} b^{2^rn} c^{2^rl} = 1 \) then \( a^{2^rm} b^{2^rn} c^{2^rl} = 1 \) and so \( a^{2^rm} b^{2^rn} c^{2^rl} \in Z(G) \). This implies that \( 2^r \) divide integers \( 2m, 2n, 2l \). Therefore, all integers \( m, n, \) and \( l \) are even and so \( a^{2^rm} b^{2^rn} c^{2^rl} \in \Omega_r(G) = Z(G) \). It follows that \( a^{2^rm} = b^{2^rn} = c^{2^rl} = 1 \). Hence \( \langle a^{2^r}, b^{2^r}, c^{2^r} \rangle = \langle a^{2^r} \rangle \times \langle b^{2^r} \rangle \times \langle c^{2^r} \rangle \cong C_{2^r} \times C_{2^r} \times C_{2^r} \) and so \( G^{2^r} = G' \). A proof similar to the last part of the proof of Theorem 2.2, gives that \( G \) is isomorphic to \( G(2, r, r, [t_{ij}]) \) for some matrix \( [t_{ij}] \in GL(3, \mathbb{Z}_{2^r}) \). This completes the proof. \( \square \)

Remark 2.6. It is not hard to see that groups (i)–(iii) in Theorem 2.5 are not \( E \)-groups.

Lemma 2.7. Let \( G \) be a finite 3-generator \( pE \)-group and \( \pi \in \text{End}(G) \).

(i) If \( \pi \in \text{Aut}(G) \), then \( \pi \) is a central automorphism.

(ii) If \( \pi \notin \text{Aut}(G) \), then \( \text{Im} \pi \leq Z(G) \), where \( \text{Im} \pi \) denotes the image of \( \pi \).

Proof. Suppose that \( G \) is non-abelian, \( \exp(G') = p' \) and \( \exp(\langle G \rangle) = p' \). By Theorems 2.2 and 2.5 and Remark 2.6, there exist elements \( a, b, c \in G \) such that \( G = \langle a, b, c \rangle, |a| = |b| = |c| = p^{r+1}, G' = Z(G) = \Omega_r(G), \) and

\[
G^{p'} = G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle, \quad |[a, b]| = |[a, c]| = |[b, c]| = p'.
\]

Now we prove that \( C_G(g) = \langle g \rangle Z(G) \) for each \( g \in \{a, b, c\} \). By symmetry between \( a, b, \) and \( c \), it is enough to show this claim for \( g = a \). Let \( x \in C_G(a) \). Then there exist integers \( i, n, m \) and an element \( w \in Z(G) \) such that \( x = a^i b^m c^m w \). Since \( [x, a] = 1 \), we have \( [b, a]^n [c, a]^m = 1 \) and so \( n = m = 0 \) (mod \( p' \)). Therefore, \( x = a^i w_a \) for some \( w_a \in Z(G) \), as required. Therefore, \( a^i = a^i w_a, b^m = b^m w_b, \) and \( c^m = c^m w_c, \) where \( 0 \leq i, j, k \leq p' - 1 \) and \( w_a, w_b, w_c \in Z(G) \).
Now the proof may be completed by applying the same methods used in Section 4 of Caranti (1985) concerning indecomposable \( pE \)-groups. But since these latter results are only stated for odd \( p \) in Caranti (1985), we prefer to complete the proof for the reader's convenience.

From \( [(ab)^s, ab] = 1 \) and \( [(ac)^s, ac] = 1 \), it follows respectively that \( i = j \) and \( i = k \). Also from the equality \( G'' = G' \), we have \( a^{p^r} = [a, b]^{p^r} [b, c]^{p^r} [a, c]^{p^r} \) where \( s, k, l \) are integers. Thus \( (a^s)^{p^r} = [a^s, b^{p^r}] [b^s, c^{p^r}] [a^s, c^{p^r}] \), and we obtain \( a^{p^r/3} = a^{p^r/3} \). Therefore, \( i^2 \equiv i \) (mod \( p' \)) and so \( i = 1 \) or \( i = 0 \). If \( i = 1 \), then \( z \) is a central automorphisms of \( G \). If \( i = 0 \), then image \( z \) is in the center of \( G \). This completes the proof.

\[ \square \]

3. A MATRIX FORMULATION FOR A MAP TO BE AN ENDOHMORPHISM OF CERTAIN \( E \)-GROUPS

Lemma 3.1. Let \( G = G(p, r, t, [t_{ij}]) = \langle a, b, c \rangle \), where \( p > 2 \) or \( (p = 2 \) and \( t \neq r) \), \( T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^r}) \), and let \( A \) be the above matrix. Then the map \( \alpha \) defined by

\[ a^2 = a_1^i b_1^{j_1} c_1^{k_1} z_1, \quad b^2 = a_2^i b_2^{j_2} c_2^{k_2} z_2, \quad c^2 = a_3^i b_3^{j_3} c_3^{k_3} z_3, \]

where \( i_1, j_1, \ldots, k_3 \) are integers and \( z_1, z_2, z_3 \in Z(G) \), can be extended to an endomorphism of \( G \) if and only if the equality \( T A = (adj A) T \) holds in the ring of matrices on \( \mathbb{Z}_{p^r} \).

**Proof.** Since \( \exp(G) = p^{r+t} \) and \( \exp(G^t) = p' \), we have \( x^{p^{r+t}} = [x^{p'}, y] = 1 \) for all \( x, y \in G \). Then \( \alpha \) can be extended to an endomorphism of \( G \) if and only if

\[ [a^2, b^2] = (a^2)^{p^{r+t_1}} (b^2)^{p^{r+t_2}} (c^2)^{p^{r+t_3}}, \quad [a^2, c^2] = (a^2)^{p^{r+t_1}} (b^2)^{p^{r+t_2}} (c^2)^{p^{r+t_3}}, \quad [b^2, c^2] = (a^2)^{p^{r+t_1}} (b^2)^{p^{r+t_2}} (c^2)^{p^{r+t_3}}. \]

Since \( (xy)^{p'} = x^{p'} y^{p'} \) for all \( x, y \in G \) and \( G^{p'} = \langle a^{p'} \rangle \times \langle b^{p'} \rangle \times \langle c^{p'} \rangle \cong C_{p'} \times C_{p'} \times C_{p'} \), it follows that the following equality in the ring of matrices on \( \mathbb{Z}_{p^r} \) holds if and only if \( \alpha \) can be extended to an endomorphism of \( G \):

\[
\begin{pmatrix}
    \begin{pmatrix}
        t_{11} & t_{21} & t_{31} \\
        t_{12} & t_{22} & t_{32} \\
        t_{13} & t_{23} & t_{33}
    \end{pmatrix}
    &
    \begin{pmatrix}
        i_1 j_2 - j_1 i_2 & i_1 j_3 - j_1 i_3 & i_2 j_3 - j_2 i_3 \\
        i_1 k_2 - k_1 i_2 & i_1 k_3 - k_1 i_3 & i_2 k_3 - k_2 i_3 \\
        j_1 k_2 - k_1 j_2 & j_1 k_3 - k_1 j_3 & j_2 k_3 - k_2 j_3
    \end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]
4. PROOF OF THE MAIN RESULT

**Theorem 4.1** (The Main Result of Morigi, 1995). For an odd prime, there exists no finite non-abelian 3-generator p-group having an abelian automorphism group.

Although the above theorem is false for $p = 2$, it is true for certain 2-groups.

**Proposition 4.2.** There exists no finite non-abelian 3-generator 2-group $G$ having an abelian automorphism group such that $\exp(G') = 2^t$, $\exp(G) = 2^{2t}$, and $t > 1$.

**Proof.** The same proof as that of Theorem 4.1 works for this proposition. □

**Proof of Theorem 1.1.** As we mentioned in Section 1, it is enough to show that every 3-generator $pE$-groups is abelian. Suppose, for a contradiction, that $G$ is a non-abelian 3-generator $pE$-group. By Theorems 2.2 and 2.5 and Remark 2.6, there exist elements $a, b, c \in G$ such that $G = G(p, t, t, [t_i]) = \langle a, b, c \rangle$, where $T = [t_i] \in GL(3, \mathbb{Z}_p)$.

Case I: $p > 2$; or $p = 2$ and $t \neq r$. Let $H = G(p, t, t, [t_i]) = \langle x, y, z \rangle$.

We claim that every automorphism of $H$ is central. If $\beta \in \text{Aut}(H)$, then

$$x^\beta = x^{i_1}y^{i_2}z^{i_3}, \quad y^\beta = x^{i_4}y^{i_5}z^{i_6}, \quad z^\beta = x^{i_7}y^{i_8}z^{i_9},$$

where $x_1, z_2, z_3 \in Z(H)$ and $i_1, i_2, \ldots, i_9 \in \{0, \ldots, p^t - 1\}$. If $A = \left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \\ i_7 & i_8 & i_9 \end{array} \right)$, by Lemma 3.1 we have $TA = (adj A)T$. Now we define the map $x$ on $G$ by

$$a^x = a^{i_1}b^{i_2}c^{i_3}, \quad b^x = a^{i_4}b^{i_5}c^{i_6}, \quad c^x = a^{i_7}b^{i_8}c^{i_9}.$$

By Lemma 3.1, $x$ can be extended to an endomorphism of $G$ and by Lemma 2.7, $x$ is a central automorphism or $\text{Im} x \subseteq Z(G)$. If $x$ is a central automorphism of $G$, then $a^{-1}a^x \in Z(G)$ and $a^{-1}b^{i_2}c^{i_3}Z(G) = Z(G)$. Since $\frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle$ and $|aZ(G)| = |bZ(G)| = |cZ(G)| = p^t$, we have $i_1 = 1, j_1 = 0, k_1 = 0$. Similarly, $b^{-1}b^x \in Z(G)$ and $c^{-1}c^x \in Z(G)$. It follows that $A$ is the identity matrix and so $\beta$ is a central automorphism of $H$. If $\text{Im} x \subseteq Z(G)$, then we similarly obtain that $A$ is the zero matrix and so $\text{Im} \beta \subseteq Z(H)$, a contradiction.

Therefore, all the automorphisms of $H$ are central so that they fix the elements of $H' = Z(H)$. If $\phi, \psi \in \text{Aut}(H)$, then $h^{\phi \psi} = h^{\phi \psi}$ for every $h \in \{x, y, z\}$. Hence $\text{Aut}(H)$ is abelian which contradicts Theorem 4.1 or Proposition 4.2 except when $p = 2$ and $t = 1$. In this case, $|H| = 64$, and it can be easily checked by GAP (2005) that there exist no 2$\mathbb{Z}$-group of order 64 having an abelian automorphism group, a contradiction.

Case II: $p = 2$ and $t = r$. By Lemma 2.7, every automorphism of $G$ is central, and so $\text{Aut}(G)$ is abelian (since $G' = Z(G)$). As in Case I, we reach to a contradiction. This completes the proof. □
We end the article with a result which generalizes Abdollahi et al. (2008, Theorem 2.9).

**Theorem 4.3.** There exists no \( p \mathcal{C} \)-group of class 3 such that \( G = \langle x_1, x_2, \ldots, x_n \rangle \), and for every \( i \in \{1, 2, \ldots, n\} \), the set \( \{ [x_i, x_j, x_k] | 1 \leq j < k \leq n, j \neq i \neq k \} \) is a linearly independent subset of the elementary abelian 3-group \( \gamma_3(G) \).

**Proof.** Suppose, for a contradiction, that \( G \) is a \( p \mathcal{C} \)-group of class 3. Let \( \exp(G) = 3^r \) and \( H = (G')^3 \gamma_3(G) \). Note that, by Abdollahi et al. (2008, Lemma 2.4), \( [H, G] = H^{3r} = 1 \). Modulo \( H \), we have that

\[
x_1^{3r} = [x_1, x_2]^{m_2}[x_1, x_3]^{m_3} \cdots [x_1, x_n]^{m_n} \prod_{2 \leq i < j \leq n} [x_i, x_j]^{m_{ij}}
\]

for some integers \( m_2, m_3, \ldots, m_n, m_{ij} \in \{-1, 0, 1\} \). Since \( [x_1, x_1^{3r}] = 1 \), we have

\[
\prod_{2 \leq i < j \leq n} [x_i, x_j]^{m_{ij}} = 1.
\]

Now it follows from the hypothesis that \( t_{ij} = 0 \) for all \( i, j \). Similarly, modulo \( H \), we have \( x_2^{3r} = [x_2, x_1]^{m_1} [x_2, x_3]^{k_3} \cdots [x_2, x_n]^{k_n} \), where \( m_1, k_3, \ldots, k_n \in \{-1, 0, 1\} \). Since \( [x_1^{3r}, x_2] = [x_2^{3r}, x_1]^{-1} \), we have \( k_3 = m_3, \ldots, k_n = m_n \). By a similar argument one can see that, modulo \( H \),

\[
x_i^{3r} = \prod_{j=1}^n [x_i, x_j]^{m_j} \quad \text{for all } i \in \{1, 2, \ldots, n\}.
\]

Therefore, \( [x_i, x_j]^{3r} = \prod_{k=1}^n [x_i, x_k, x_j]^{m_k} \) for all \( i, j \in \{1, 2, \ldots, n\} \). It follows that

\[
x_i^{3r} = (x_i^{3r})^{3r} = \prod_{j=1}^n [x_i, x_j]^{3r m_j} = \prod_{j=1}^n \prod_{k=1}^n [x_i, x_k, x_j]^{m_j m_k} = 1
\]

for all \( i \in \{1, 2, \ldots, n\} \). Hence \( G^{3r} = 1 \), contradicting Abdollahi et al. (2008, Lemma 2.4). This completes the proof. \( \square \)

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