1-FACTORIZATIONS OF CAYLEY GRAPHS

A. ABDOLLAHI

Abstract. In this note we prove that all connected Cayley graphs of every finite group $G$ are 1-factorizable, where $G$ is any non-trivial group of 2-power order and $H$ is any group of odd order.

1. Introduction and Results

Let $G$ be a non-trivial group, $S \subseteq G \setminus \{1\}$ and $S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph $\Gamma(S; G)$ of the group $G$ with respect to the set $S$ has the vertex set $G$ and the edge set $\{(g, sg) : g \in G, s \in S \cup S^{-1}\}$.

A $j$-factor of a graph is a spanning subgraph which is regular of valence $j$. In particular, a 1-factor of a graph is a collection of edges such that each vertex is incident with exactly one edge. A 1-factorization of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A 1-factorization of a regular graph of valence $v$ is equivalent to a coloring of the edges in $v$ colors (coloring each 1-factor a different color). This enables us to use a very helpful result: Any simple, regular graph of valence $v$ can be edge-colored in either $v$ or $v + 1$ colors. This is a specific case of Vizing's theorem (see [2, pp. 245-249]).

We study the conjecture that says all Cayley graphs $\Gamma(S; G)$ of groups $G$ of even order are 1-factorizable whenever $G = \langle S \rangle$. There are some partial results on this conjecture obtained by Stong [1]. Here we prove

Theorem. Let $H$ be a finite group of odd order and let $Q$ be a finite group of order $2^k$ ($k > 0$). Then the Cayley graph $\Gamma(S; Q \times H)$ is 1-factorizable for all generating sets $S$ of $Q \times H$.

As a corollary we prove that all connected Cayley graphs of every finite nilpotent group of even order are 1-factorizable which has been proved by

2000 Mathematics Subject Classification. 05C25, 05C70.
Key words and phrases. 1-factorizations; Cayley graphs; Nilpotent groups.
This research was in part supported by a grant from IPM (No. 852200032). The author thanks the Center of Excellence for Mathematics, University of Isfahan.

ARS COMBINATORIA 86 (2008), pp. 120-131
Stong in [1, Corollary 2.4.1] only for Cayley graphs on minimal generating sets.

2. Proof of the Theorem

We need the following lemma whose proof is more or less as Lemma 2.1 of [1] with some modifications.

**Lemma 2.1.** Let $H$ be a finite group of odd order. Then the Cayley graph $\Gamma(S: \mathbb{Z}_2 \times H)$ is 1-factorizable, for any generating set $S$ of $\mathbb{Z}_2 \times H$ containing exactly one element of even order.

**Proof.** Let $a$ be the only element of $S$ of even order. Then $a = z \cdot h$, where $z \in \mathbb{Z}_2$ and $h \in H$ and $z$ of order 2. If $a^2 = 1$, then $h = 1$ and $S \setminus \{a\} \subseteq H$ and so $a x^{-1} = x$ for all $x \in S \cap H$. Thus, in this case, Theorem 2.3 of [1] completes the proof. Therefore we may assume that $a^2 \neq 1$. Let $\Gamma' = \Gamma(S \setminus \{a\} : \mathbb{Z}_2 \times H)$ and $\Gamma_1$ and $\Gamma_2$ be the induced subgraphs of $\Gamma'$ on the sets $H$ and $zH$, respectively. It can be easily seen that the map $x \mapsto z \cdot x$ is an graph isomorphism from $\Gamma_1$ to $\Gamma_2$. By Vizing’s theorem the edges in both $\Gamma_1$ and $\Gamma_2$ can be edges-colored in the same manner in $[S \setminus \{a\}] + 1$ colors (by “the same manner” we mean that the edge $\{h_1, h_2\}$ in $\Gamma_1$ has “the same” color as $\{z h_1, z h_2\}$ in $\Gamma_2$, and vice versa). Then all that remains to be done is to color the edges from $H$ to $zH$, that is the following two ‘disjoint’ 1-factors of $\Gamma(S : \mathbb{Z}_2 \times H)$ (here we use $a^2 \neq 1$):

\[
\{\{z, z x\} \mid x \in H\} \quad \text{and} \quad \{\{z, a^{-1} x\} \mid x \in H\}.
\]

(note that the edges of $\Gamma(S : \mathbb{Z}_2 \times H)$ are exactly the edges of $\Gamma_1$, $\Gamma_2$ and those in the above 1-factors). Now since both $z \in H$ and $z \cdot x \in zH$ have edges (in $\Gamma_1$ and $\Gamma_2$, respectively) of the same $[S \setminus \{a\}]$ colors to them, there are ‘two’ colors (note that here we again use $a^2 \neq 1$) that can be used to color 1-factors in ($\ast$). This completes the proof. $\square$

**Proof of the Theorem.** Let $G = Q \times H$ and $S$ be any generating set of $G$. We argue by induction on $|S|$. If $|S| = 1$, then $G$ is a cyclic group of even order and Corollary 2.3.1 of [1] completes the proof. Now assume that $|S| > 1$ and for any non-trivial group $Q_1$ of 2-power order and subgroup $H_1$ of $H$ the Cayley graph $\Gamma(S_1 : Q_1 \times H_1)$ is 1-factorizable for any generating set $S_1$ of $Q_1 \times H_1$ with $|S_1| \leq |S|$. Since the set of elements of odd order in $G$ is the subgroup $H$ and $G = \langle S \rangle$, $S$ has at least one element $a$ of even order. First assume that $S$ has another element distinct from $a$ of even order. Consider the subgroup $G_1$ generated by $S \setminus \{a\}$ of $G$. Then $G_1 = Q_1 \times H_1$ for some subgroups $Q_1 \leq Q$ and $H_1 \leq H$ such that $Q_1 \neq 1$. Therefore the induction hypothesis implies that $\Gamma(S \setminus \{a\} : G_1)$ has a 1-factorization. Since $\Gamma(S \setminus \{a\}, G)$ consists of disjoint copies of $\Gamma(S \setminus \{a\} : G_1)$ which are 1-factorizable, $\Gamma(S \setminus \{a\}, G)$ has a 1-factorization. Now since the only element of $S \setminus \{a\}$ has even order, Lemma 2.2 of [1] shows that $\Gamma(S : G)$ is 1-factorizable.

Hence we may assume that $a$ is the only element of $S$ of even order. Since $a = a_1 a_2$ for some $a_1 \in Q$ and $a_2 \in H$, we have

\[
G = \langle S \rangle = \langle S \setminus \{a\}, a_1 a_2 \rangle = \langle a_1 \rangle \times \langle S \setminus \{a\}, a_2 \rangle.
\]

It follows that $Q = \langle a_1 \rangle$. Consider the subgroup $N = \langle a_1^2 \rangle$. Then $N$ is a normal subgroup of $G$ such that $N \cap S = \varnothing$. It is easy to see that when $s, t \in S$ with $s \neq t^{-1}$, neither $st$ nor $st^{-1}$ belongs to $N$. Now by Lemma 2.4 of [1], it is enough to show that $\Gamma(S_N : G_N)$ is 1-factorizable. Since $G_N \cong \mathbb{Z}_2 \times H$, it follows from Lemma 2.1 that $\Gamma(S_N : G_N)$ is 1-factorizable. This completes the proof. $\square$

**Corollary 2.2.** If $G$ is a finite nilpotent group of even order, then $\Gamma(S : G)$ is 1-factorizable for all generating sets $S$ of $G$.

**Proof.** It follows from the Theorem and the fact that every finite nilpotent group is the direct product of its Sylow subgroups. $\square$

**References**


**Department of Mathematics, University of Isfahan, Isfahan 81746-71441, Iran, and Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran.**

E-mail address: a.abbolhasani@math.ui.ac.ir