We associate a graph $C_G$ of a non locally cyclic group $G$ (called the non-cyclic graph of $G$) as follows: take $G \setminus \text{Cyc}(G)$ as vertex set, where \text{Cyc}(G) = \{x \in G \mid \langle x, y \rangle \text{ is cyclic for all } y \in G\}$ is called the cyclicizer of $G$, and join two vertices if they do not generate a cyclic subgroup. For a simple graph $\Gamma$, $\omega(\Gamma)$ denotes the clique number of $\Gamma$, which is the maximum size (if it exists) of a complete subgraph of $\Gamma$. In this paper we characterize groups whose non-cyclic graphs have clique numbers at most 4. We prove that a non-cyclic group $G$ is solvable whenever $\omega(C_G) < 31$ and the equality for a non-solvable group $G$ holds if and only if $G/\text{Cyc}(G) \cong A_5$ or $S_5$.

Keywords: Non-cyclic graph; diameter; domination number; solvable groups.

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1. Introduction and Results

Let $G$ be a non locally cyclic group. Following [2], the non-cyclic graph $C_G$ of $G$ is defined as follows: take $G \setminus \text{Cyc}(G)$ as vertex set, where \text{Cyc}(G) = \{x \in G \mid \langle x, y \rangle \text{ is cyclic for all } y \in G\}$, and join two vertices if they do not generate a cyclic subgroup. We call the complement of $C_G$, the cyclic graph of $G$, which has the same vertex set as $C_G$ and two distinct vertices are adjacent whenever they generate a cyclic subgroup. The cyclic graph of $G$ will be denoted by $\overline{C_G}$.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph $\Gamma$, we denote the sets of the vertices and the edges of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $d_{\Gamma}(v)$ of a vertex $v$ in $\Gamma$ is the number of edges incident to $v$ and if the graph is understood, then we denote $d_{\Gamma}(v)$ simply
by $d(v)$. The order of $\Gamma$ is defined $|V(\Gamma)|$. A graph $\Gamma$ is regular if $d(v) = d(w)$ for any two vertices $v$ and $w$. A subset $X$ of the vertices of $\Gamma$ is called a clique if the induced subgraph on $X$ is a complete graph. The maximum size of a clique in a graph $\Gamma$ is called the clique number of $\Gamma$ and is denoted by $\omega(\Gamma)$. If there exists a path between two vertices $v$ and $w$ in $\Gamma$, then $d_\Gamma(v, w)$ denotes the length of the shortest path between $v$ and $w$; otherwise $d_\Gamma(v, w) = \infty$. If the graph is understood, then we denote $d_\Gamma(v, w)$ simply by $d(v, w)$. The largest distance between all pairs of the vertices of $\Gamma$ is called the diameter of $\Gamma$, and is denoted by $\text{diam}(\Gamma)$. A graph $\Gamma$ is connected if there is a path between each pair of the vertices of $\Gamma$. So disconnected graphs have infinite diameter. For a graph $\Gamma$ and a subset $S$ of the vertex set $V(\Gamma)$, denote by $N_\Gamma[S]$ the set of vertices in $\Gamma$ which are in $S$ or adjacent to a vertex in $S$. If $N_\Gamma[S] = V(\Gamma)$, then $S$ is said to be a dominating set (of vertices in $\Gamma$). The domination number of a graph $\Gamma$, denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of the vertices in $\Gamma$. A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex at which both are incident. We denote the symmetric group on $n$ letters and the alternating group of degree $n$ by $S_n$ and $A_n$, respectively. Also $Q_8$ and $D_{2n}$ are used for the quaternion group with 8 elements and dihedral group of order $2n$ ($n > 2$), respectively.

The present work is a continuation of that of [2]. In Sec. 2, we study the diameter and domination number of the cyclic and non-cyclic graphs. In Sec. 3, we characterize all groups whose non-cyclic graphs have clique numbers $\leq 4$. In Sec. 4, we classify all groups whose non-cyclic graphs are planar or Hamiltonian. Finally in Sec. 5, we give a sufficient condition for solvability, by proving that a group $G$ is solvable whenever $\omega(C_G) < 31$. We also prove the bound 31 cannot be improved and indeed the equality for a non-solvable group $G$ holds if and only if $G/Cyc(G) \cong A_5$ or $S_5$.

2. On the Diameter and Domination Numbers of the Non-Cyclic Graph and its Complement

We first observe that to study the diameter of the non-cyclic graph, we may factor out the cyclicizer. Recall that if there exists a path between two vertices $v$ and $w$ in a graph $\Gamma$, then $d(v, w)$ denotes the length of the shortest path between $v$ and $w$; otherwise $d(v, w) = \infty$. The largest distance between all pairs of the vertices of $\Gamma$ is called the diameter of $\Gamma$, and is denoted by $\text{diam}(\Gamma)$. Thus if $\Gamma$ is disconnected then $\text{diam}(\Gamma) = \infty$.

**Lemma 2.1.** Let $G$ be a non locally cyclic group. Then $\overline{C_G}$ is connected if and only if $\overline{C_G}$ is connected, $\text{diam}(\overline{C_G}) = \text{diam}(\overline{C_G})$ and $\text{diam}(\overline{C_G}) = \text{diam}(\overline{C_G})$. Moreover, corresponding connected components of $\overline{C_G}$ and $\overline{C_G}$ have the same diameter when the component in $\overline{C_G}$ is not an isolated vertex.
Proof. It is enough to prove that \( x - y \) is an edge in \( \overline{C_G} \) if and only if \( \overline{x - y} \) is an edge in \( \overline{C_{\text{Cyc}(G)}} \), where \( \overline{\ } \) is the natural epimorphism from \( G \) to \( \text{Cyc}(G) \). If \( x - y \) is an edge in \( C_G \), then \( (x, y) \) is not cyclic. We have to prove that \( (\overline{x}, \overline{y}) \) is not cyclic. Suppose, for a contradiction, that \( (\overline{x}, \overline{y}) \) is cyclic. Then \( x = g_1c_1 \) and \( y = g_2c_2 \) for some \( g \in G \), \( c_1, c_2 \in \text{Cyc}(G) \) and integers \( i, j \). Thus

\[
\langle x, y \rangle = \langle g_1c_1, g_2c_2 \rangle \leq \langle g, c_1, c_2 \rangle.
\]

Now since \( \langle c_1, c_2 \rangle = \langle c \rangle \), for some \( c \in \text{Cyc}(G) \), it follows that \( \langle g, c_1, c_2 \rangle \) is cyclic. Therefore \( (x, y) \) is cyclic, a contradiction.

Now if \( \overline{x - y} \) is an edge in \( \overline{C_{\text{Cyc}(G)}} \), then \( (\overline{x}, \overline{y}) \) is not cyclic; and since \( (\overline{x}, \overline{y}) \) is a homomorphic image of \( (x, y) \), \( (x, y) \) is not cyclic. This completes the proof. \( \blacksquare \)

Lemma 2.2. Let \( G \) be a finite non-cyclic group and let \( x, y \in G \setminus \text{Cyc}(G) \). Then \( d_{\text{Cyc}}(x, y) = 3 \) if and only if \( G = \text{Cyc}(x) \cup \text{Cyc}(y) \). Moreover, \( (x, y) \) is cyclic and for all \( t \in \text{Cyc}(x) \setminus \text{Cyc}(y) \) and for all \( s \in \text{Cyc}(y) \setminus \text{Cyc}(x) \), \( (t, s) \) is not cyclic.

Proof. The proof is contained in that of [2, Proposition 3.2]. \( \blacksquare \)

Recall that if \( G \) is a non locally cyclic group, then two distinct vertices are adjacent in the cyclic graph \( \overline{C_G} \) if and only if they generate a cyclic group.

Lemma 2.3. Let \( G \) be a non locally cyclic group. Then \( \text{diam}(\overline{C_G}) \neq 1 \). In other words, \( \overline{C_G} \) cannot be isomorphic to a complete graph.

Proof. If \( \text{diam}(\overline{C_G}) = 1 \), then every two elements of \( G \) generates a cyclic group. This is equivalent to \( G \) being locally cyclic, a contradiction. \( \blacksquare \)

Proposition 2.4. (1) If \( G \) is a non locally cyclic group such that either \( \text{Cyc}(G) \neq 1 \) or \( \overline{C_G} \) is connected and \( \text{Cyc}(G) = 1 \), then \( G \) is either a torsion group or a torsion free group.

(2) If \( A \) is an abelian torsion-free non locally cyclic group, then \( \overline{C_A} \) is disconnected.

(3) There are torsion-free simple groups \( H \) such that \( \text{diam}(\overline{C_H}) = \text{diam}(\overline{C_H}) = 2 \).

Proof. (1) If \( \text{Cyc}(G) \neq 1 \), then the proof follows from Lemma 2.3 of [2]. Thus assume that \( \overline{C_G} \) is connected and \( \text{Cyc}(G) = 1 \). If there were elements of infinite order and non-trivial elements of finite order, then connectivity would guarantee some pair of these would be adjacent in \( \overline{C_G} \), which is not possible. This proves (1).

(2) Suppose, for a contradiction, that \( \overline{C_A} \) is connected. Note that any two adjacent vertices \( a, b \) satisfy \( \langle a \rangle \cap \langle b \rangle \neq 1 \). Since \( A \) is torsion-free and \( \overline{C_A} \) is connected, it follows that \( \langle a \rangle \cap \langle b \rangle \neq 1 \) for any two non-trivial elements \( a, b \) of \( A \). Fix a non-trivial element \( a \in A \), then it is easy to see that the map \( f \) defined from \( A \) to the additive group \( \mathbb{Q} \) of rational numbers by \( f(x) = mx/n \), where \( x^n = a^m \), is a group monomorphism, where \( m \) and \( n \) are integers such that \( 1 \neq x^m = a^n \in \langle x \rangle \).
Lemma 2.5. Let $G$ be a finite non-cyclic group of prime power order. Then $\overline{G}$ is disconnected.

Proof. Suppose, for a contradiction, that $\overline{G}$ is connected. By Lemma 2.1, we may assume that $\text{Cyc}(G) = 1$. Now we prove that $G$ has only one subgroup of prime order. Suppose that there are two elements $a$ and $b$ of prime order. Since $\text{Cyc}(G)$ is connected, there exists a sequence $x_1, \ldots, x_n$ of elements of $G \setminus \text{Cyc}(G)$ such that

$$\langle a, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{n-1}, x_n \rangle, \langle x_n, b \rangle \quad (2.1)$$

are all cyclic. Since a cyclic group of $p$-power order has only one subgroup of order $p$, and both $\langle a \rangle$ and $\langle b \rangle$ are subgroups of order $p$ of the cyclic groups (1), we have that $\langle a \rangle = \langle b \rangle$. Now let $A$ be the only subgroup of prime order in $G$, which can be generated by $x$ and let $y$ be any non-trivial element of $G$. Then $\langle x, y \rangle = \langle y \rangle$. This shows that $x \in \text{Cyc}(G)$, which is impossible. This completes the proof.

Lemma 2.6. Let $G$ be a non locally cyclic group. If $\text{diam}(\overline{G}) = 3$ then $\overline{G}$ is connected and $\text{diam}(\overline{G}) \in \{2, 3\}$.

Proof. Let $x, y \in G \setminus \text{Cyc}(G)$ such that $d_{\overline{G}}(x, y) = 3$. By Lemma 2.2, we have $G = C_x \cup C_y$, where $C_x = \text{Cyc}_G(x)$ and $C_y = \text{Cyc}_G(y)$. Now let $a$ and $b$ be two distinct elements of $G \setminus \text{Cyc}(G)$. If $K = \langle a, b \rangle$ is cyclic, then $d_{\overline{G}}(a, b) = 1$. Suppose that $K$ is non-cyclic. We may assume without loss of generality that $a \in C_x \setminus C_y$ and $b \in C_y \setminus C_x$. (otherwise $d_{\overline{G}}(a, b) = 2$ as either $a - x - b$ or $a - y - b$ is a path of length two in $\overline{G}$). In this case, $a - x - y - b$ is a path of length 3 in $\overline{G}$. Now Lemma 2.3 completes the proof.

We have checked by GAP [14], that for each finite non-cyclic group $G$ of order at most 100, the following holds

$$\text{diam}(\overline{G}) = 3 \Leftrightarrow \text{diam}(\overline{G}) = 3.$$

We were unable to prove the equality $\text{diam}(\overline{G}) = 3$ in Lemma 2.6 for all non locally cyclic groups $G$. So we may pose the following question:

Question 2.7. In Lemma 2.6, for which non locally cyclic group $G$ does the equality $\text{diam}(\overline{G}) = 3$ hold?
The following is an example of a finite non-cyclic group $G$ with $\text{diam}(C_G) = 2$ and $\text{diam}(\overline{C_G}) = 4$. Let $G = C_2 \times F$ be the direct product of a cyclic group of order 2 generated by $z$ say, with a Frobenius group $F$ of order $6 \cdot 7$ (which is not the dihedral group $D_{42}$). A Sylow 3-subgroup (there are seven of these) is cyclic of order 3, and if $P$ and $Q$ are two distinct ones, then $C_G(P) \cap C_G(Q) = \langle z \rangle$. In particular, if $x$ and $y$ are two non-central elements of $G$, then $x$ fails to centralize at least 6 Sylow 3-subgroups, and then $x$ and $y$ together fail to centralize at least 5 of these. Thus, the distance $d_{C_G}(x,y)$ in the non-cyclic graph is at most 2. On the other hand, if exactly one of these elements, say $x$, is central (so $x = z$), then choose a Sylow 3-subgroup $P$ which does not centralize $y$, and then choose an element $g$ of order 6 in the centralizer $C_G(P)$. Then we have the path $x - g - y$ of length 2 in the non-cyclic graph of $G$. This establishes $\text{diam}(C_G) = 2$.

In the cyclic graph $\overline{C_G}$, every element has distance at most 2 from the central element $z$. Certainly, elements of odd order are directly adjacent to $z$, elements of even order $\neq 2$ are connected to elements of odd order, so have distance $\leq 2$ from $z$, while a non-central involution centralizes some (unique) Sylow 3-subgroup of $G$ so that it too has distance $\leq 2$ from $z$. Hence $\text{diam}(\overline{C_G}) \leq 4$. It remains to find two elements $u$ and $v$ whose distance is exactly 4 in the cyclic graph $\overline{C_G}$.

Choose $u$ and $v$ to be non-central involutions centralizing two distinct Sylow 3-subgroups, say $P = \langle g \rangle$ and $Q = \langle h \rangle$, respectively. Certainly $u - g - z - h - v$ is a path of length 4 in the cyclic graph $\overline{C_G}$, and we argue that there is no shorter path from $u$ to $v$. Clearly, $u \neq v$ and $u, v$ are not adjacent in the cyclic graph. Moreover, $C_G(u) \cap C_G(v) = \langle z \rangle$ shows that there is no path of length 2 from $u$ to $v$ (as neither $\langle u, z \rangle$ nor $\langle v, z \rangle$ is cyclic). Furthermore, if $x$ is any element adjacent to $u$, then $\langle x \rangle$ is either $P$ or $P(u)$. Therefore, in any path from $u$ to any other element, say $u - x - \cdots$ we may replace $x$ by an appropriate generator of $P$. If this path ends at $v$, then the ending $\cdots y - v$ may be adjusted similarly so that $y$ is a generator of $Q$. As $\langle x, y \rangle$ is not cyclic, the total length of the path is $\geq 4$.

Two other examples are SmallGroup(48,11) and SmallGroup(48,12) in GAP [14]. It is also checked that for all non-cyclic groups $G$ of order at most 100, either $C_G$ is disconnected or $\text{diam}(C_G) \in \{3, 4\}$.

Lemma 2.8. Let $G$ be a non locally cyclic group. Then

1. $\gamma(\overline{C_G}) \geq 2$. The equality holds if and only if $\text{diam}(C_G) = 3$.
2. $\gamma(C_G) = 1$ if and only if $\text{Cyc}(G) = 1$ and there is an element $x$ of order 2 such that $\text{Cyc}_G(x) = \langle x \rangle$.
3. If either $G = E \ast H$ is the free product of a non-trivial elementary abelian 2-group $E$ with an arbitrary group $H$ such that either $|E| > 2$ or $|H| > 1$; or $G$ has an abelian 2'-subgroup $A$ and an element $x$ of order 2 such that $G = A(x)$ and $a^2 = a^{-1}$ for all $a \in A$, then $\gamma(C_G) = 1$. 
Since every non-trivial element of $E = \mathbb{Z}$

Each dominating set for elements $g^x$ is contained in a (unique) maximal cyclic subgroup. Replacing each $i$ by an arbitrary non-trivial element of $E$, we have that $\langle a \rangle$ is cyclic for all $a \in G$ and so $x \in \text{Cyc}(G)$, a contradiction.

Now suppose that $\gamma(G) = 2$. Then there exist two distinct vertices $x$ and $y$ of $G$ such that for every vertex $a \notin \{x, y\}$, either $\langle a, x \rangle$ or $\langle a, y \rangle$ is cyclic. This implies that $G = \text{Cyc}_G(x) \cup \text{Cyc}_G(y)$. Now Lemma 2.2 and [2, Proposition 3.2] complete the proof.

(2) Suppose that $C_G$ has a dominating singleton set $\{x\}$. Since $\langle x, x^{-1} \rangle$ is trivially cyclic, $x = x^{-1}$ and so $x^2 = 1$.

If $t \in \text{Cyc}(G)$ and $t \neq 1$, then $\langle tx, x \rangle$ is cyclic. It follows that $tx = x$ and so $t = 1$. Thus $\text{Cyc}(G) = 1$.

If $c \in \text{Cyc}_G(x)$ and $c \neq x$, then $\langle c, x \rangle$ is cyclic. This implies that $c \in \text{Cyc}(G) = 1$ and so $\text{Cyc}_G(x) = \langle x \rangle$.

For the converse, it is enough to note that for all $a \in G \setminus \{1, x\}$, $\langle a, x \rangle$ is not cyclic. This shows that $\{x\}$ is a dominating set for $C_G$ and so $\gamma(G) = 1$.

(3) Suppose that $G = E \ast H$ is the free product of an elementary abelian 2-group $E$ with an arbitrary group $H$ such that either $|E| > 2$ or $|H| > 1$. Let $x$ be an arbitrary non-trivial element of $E$. Then the centralizer $C_G(x)$ of $x$ in $G$ is equal to $E$ and since $E$ is elementary abelian, we have that $\text{Cyc}_E(x) = \langle x \rangle$. It follows that $\text{Cyc}_G(x) = \langle x \rangle$. Now by part (2) it is enough to show that $\text{Cyc}(G) = 1$. If $Z(G) = 1$, then obviously $\text{Cyc}(G) = 1$. If $Z(G) \neq 1$, then $|H| = 1$ as $|E| > 2$. Thus $G = E$ and since $|E| > 2$, there are two non-trivial distinct elements $a$ and $b$ in $E$. Since every non-trivial element of $E$ has order 2, $\text{Cyc}_E(g) = \langle g \rangle$ for all non-trivial elements $g \in E$. Thus

\[ \text{Cyc}(G) = \text{Cyc}(E) \leq \langle a \rangle \cap \langle b \rangle = 1, \]

as required.

It is straightforward to see that if $G$ is of second type, then the singleton $\{x\}$ is a dominating set for $C_G$.

3. Finite Groups Whose Non-Cyclic Graphs Have Small Clique Numbers

In this section we characterize groups whose non-cyclic graphs have clique numbers at most 4. If $\{x_1, x_2, \ldots, x_n\}$ is a maximal clique for the finite group $G$ then each $x_i$ is contained in a (unique) maximal cyclic subgroup. Replacing each $x_i$ by a generator of this maximal cyclic subgroup does no harm, and the resulting collection of cyclic subgroups $\langle x_1 \rangle, \ldots, \langle x_n \rangle$ is a complete list of all the maximal cyclic subgroups, by [2, Theorem 4.7].

Lemma 3.1. Let $G$ be a non locally cyclic group. Then $\omega(C_G) \geq 3$. 
Proof. Since $G$ is not locally cyclic, there exists two elements $x$ and $y$ such that $\langle x, y \rangle$ is not cyclic. Thus $\{x, y, xy\}$ is a clique in $C_G$. This completes the proof. \hfill \Box

Lemma 3.2. Let $G$ be a non locally cyclic group whose non-cyclic graph has no infinite clique. Then $\omega(\Gamma_G)$ is finite and $\omega(C_G) = \omega(C_{\omega(C_G)})$.

Proof. It follows from [2, Theorem 4.2 and Lemma 2.3-(2)]. \hfill \Box

Thus by [2, Lemma 3.2 and Lemma 2.3-(2)], to characterize groups $G$ with finite fixed $\omega(C_G)$, it is enough to characterize finite ones with trivial cyclicizers.

We use the following result in the proof of Theorems 3.12 and 5.5.

Lemma 3.3. Let $G$ be a non locally cyclic group such that $\omega(C_G)$ is finite. If $N$ is a normal subgroup of $G$ such that $G/N$ is not locally cyclic, then $\omega(C_{\bar{G}}) \leq \omega(C_G)$, with equality if and only if $N \leq \text{Cyc}(G)$.

Proof. Let $\omega(C_G) = n$ and $\bar{G} = G/N$. If $L/N = \text{Cyc}(\bar{G})$, then $\text{Cyc}(G)N \leq L$ and so by Lemma 3.2

$$\omega(C_{\bar{G}}) = \omega(C_{\bar{G}}^L) \leq \omega(C_{\bar{G}}^{L/N}) \leq \omega(C_G) = \omega(C_{\omega(C_G)})$$

Since by [2, Theorem 4.2] $G/\text{Cyc}(G)$ is finite, without loss of generality, we may assume that $G$ is finite.

Clearly $\omega(C_{G/N}) \leq \omega(C_G)$. Now suppose that $\omega(C_{G/N}) = \omega(C_G)$. Then there exist elements $y_1, \ldots, y_n \in G$ such that $M = \{y_iN \mid i = 1, \ldots, n\}$ is a clique of $C_G$. Choose now a maximal cyclic subgroup $C_i$ of $G$ containing $y_i$ ($C_i$ is in fact uniquely determined by $y_i$). There is no harm in replacing each $y_i$ by a generator $x_i$ of $C_i$. Now it follows from [2, Theorem 4.7] that $C_1, \ldots, C_n$ are all the maximal cyclic subgroups of $G$. Consider an arbitrary element $a \in N$. Then

$$\{x_1, x_2, \ldots, x_n, ax_1\}$$

is not a clique of $C_G$. Since $M$ is a clique for $C_G$, it follows that

$$\langle x_1, ax_1 \rangle = \langle a, x_1 \rangle$$

is cyclic for all $a \in N$.

This says that $a \in \langle a, x_1 \rangle = C_1 = \langle x_1 \rangle$ for all $a \in N$. But $x_1$ may be replaced by any of the $x_i$, and we conclude that $N \leq \bigcap_{i=1}^n C_i = \text{Cyc}(G)$. This completes the proof. \hfill \Box

Throughout for a prime number $p$ we denote by $\nu_p(G)$ the number of subgroups of order $p$ in a group $G$. It is well-known that $\nu_p(G) \equiv 1 \mod p$ for a finite group $G$, whenever $p$ divides $|G|$.

Lemma 3.4. Let $G$ be a finite group with trivial cyclicizer. Then for any prime divisor $p$ of $|G|$, $\nu_p(G) \leq \omega(C_G)$. If $p_1, \ldots, p_k$ are distinct prime numbers such that $G$ has no element of order $p_ip_j$ for all distinct $i, j$, then $\sum_{i=1}^k \nu_{p_i}(G) \leq \omega(C_G)$. 
**Proof.** Let \( C_1, \ldots, C_{\nu_p(G)} \) be all the subgroups of order \( p \) of \( G \). If \( c_i \) is a generator of \( C_i \), then \( \{ c_1, \ldots, c_{\nu_p(G)} \} \) is a clique in \( C_G \). Thus \( \nu_p(G) \leq \omega(C_G) \), as required. To prove the second part, for every \( i \in \{ 1, \ldots, k \} \) and every subgroup of order \( p_i \), take a generator of the subgroup, then the set consisting of these generators is a clique in \( C_G \). This completes the proof. \( \square \)

**Lemma 3.5.** Let \( G \) be a finite group \( G \) with trivial cyclicizer. Let \( p \) be a prime number such that \( p^{k-1} < \omega(C_G) \leq p^k \), for some \( k \in \mathbb{N} \).

1. For every \( p \)-element \( x \) of \( G \), we have \( x^{p^{k-1}} \in Z(G) \).
2. If \( k = 1 \), then \( G \) has no non-trivial \( p \)-element.
3. No Sylow \( p \)-subgroup of \( G \) is cyclic of order greater than \( p^{k-1} \).

**Proof.** (1) Let \( n = \omega(C_G) \) and suppose that \( x^{p^{k-1}} \neq 1 \). The goal is to show that every \( y \in G \) centralizes \( x^{p^{k-1}} \). This is obvious, if \( y \in \langle x \rangle \) so assume \( y \notin G - \langle x \rangle \). Then with \( X = \{ x \} \cup \langle x \rangle y \), it is clear that \( |X| \geq n + 1 \). Next, as some two element subset of \( X \) generates a cyclic group, there are only two cases to consider. If one of these two elements is \( x \), then the cyclic subgroup in question is \( \langle x, x^i y \rangle = \langle x, y \rangle \), so clearly \( y \in C_G(x) \subseteq C_G(x^{p^{k-1}}) \). If on the other hand the elements are \( x^j y \) and \( x^{j'} y \), then since

\[
\langle x^j y, x^{j'} y \rangle = \langle x^j y (x^{j'} y)^{-1}, x^j y \rangle = \langle x^{-1}, x^j y \rangle,
\]

we conclude that \( x^j y \in C_G(x^{i-j}) \subseteq C_G(x^{p^{k-1}}) \), so clearly \( y \) belongs to this last set as well.

(2) In the proof of part (1), put \( k = 1 \). Since \( i - j < p \), \( \text{gcd}(i - j, p) = 1 \) and so \( \langle x, y \rangle \) is cyclic for all \( y \in G \). Thus \( x \in \text{Cyc}(G) = 1 \). This completes the proof of part (2).

(3) Suppose, for a contradiction, that \( G \) has a cyclic Sylow \( p \)-subgroup of order greater than \( p^{k-1} \). Then by part (1), \( x^{p^{k-1}} \in Z(G) \) for every \( p \)-element of \( x \) of \( G \). Since Sylow \( p \)-subgroups of \( G \) are cyclic, it follows that \( \langle x^{p^{k-1}}, y \rangle \) is cyclic for all \( y \in G \). This implies that \( x^{p^{k-1}} \in \text{Cyc}(G) = 1 \) for all \( p \)-elements of \( x \) in \( G \), which gives a contradiction. \( \square \)

For a group \( G \), we denote the non-commuting graph of \( G \) by \( A_G \). This is the graph whose vertex set is \( G \setminus Z(G) \) and two vertices \( x \) and \( y \) are adjacent if \( xy \neq yx \).

This graph was studied in [1] and [9].

**Lemma 3.6.** Let \( G \) be an abelian group. Then \( \omega(C_G) = 3 \) if and only if \( G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus T \), where \( T \cong \text{Cyc}(G) \) is a locally cyclic torsion group in which all elements have odd order.

**Proof.** Suppose that \( \omega(C_G) = 3 \) and \( \hat{G} = G/\text{Cyc}(G) \). Since \( \text{Cyc}(G) = 1 \), then by Lemma 3.5, \( \hat{G} \) is a 2-group. Thus \( \hat{G} \cong \mathbb{Z}_{2^n_1} \oplus \cdots \oplus \mathbb{Z}_{2^n_k} \), and as \( \omega(C_G) = 3 \), we
have \( k = 2 \) and \( \alpha_1 = \alpha_2 = 1 \). Therefore \( G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Now it follows from parts (4) and (5) of [2, Lemma 2.3] that \( G \) is torsion. If \( \text{Cyc}(G) \) contains an element of order 2, then \( G \) contains a subgroup isomorphic to either \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), which is not possible. Thus all elements of \( \text{Cyc}(G) \) have odd order and so \( \text{Cyc}(G) \) is the 2'-primary component of \( G \) and so \( G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \text{Cyc}(G) \).

The converse is clear.

**Theorem 3.7.** Let \( G \) be a non locally cyclic group. Then \( \omega(C_G) = 3 \) if and only if \( G/\text{Cyc}(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

**Proof.** Suppose that \( \omega(C_G) = 3 \). By Lemma 3.2, we may assume that \( \text{Cyc}(G) = 1 \). Also it follows from Lemma 3.5, that \( G \) is a 2-group, and by Lemma 3.6, we may assume that \( G \) is a non-abelian group. Now since \( \omega(C_G) \geq \omega(A_G) \), we have \( \omega(A_G) = 3 \). Thus by a well-known result (see e.g. [4, Lemma 2.4-(3)] and [7, Theorem 2]), we have \( G/Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Since \( G \) is nilpotent, \( Z(G) \neq \text{Cyc}(G) = 1 \). Then there exists an element \( a \in G \) such that \( H = \langle a, Z(G) \rangle \) is not cyclic. Now since \( H \) is abelian and since \( \omega(C_H) \leq \omega(C_G) = 3 \), it follows from Lemma 3.6 that \( H \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \). Thus \( Z(G) \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). If \( Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), then taking \( a' \in G \setminus Z(G) \), again \( \langle a', Z(G) \rangle \) is abelian, and by a similar argument, it is isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), which is a contradiction. Thus \( Z(G) \cong \mathbb{Z}_2 \), and so \( |G| = 8 \). Therefore \( G \cong D_8 \) or \( Q_8 \), but \( \text{Cyc}(Q_8) \neq 1 \). Thus \( G \cong D_8 \), another contradiction, as \( \omega(C_{D_8}) = 5 \); since if \( D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle \), then \( \langle a, b, ab, a^2b, a^3b \rangle \) is a clique in \( C_{D_8} \).

The converse is clear.

**Lemma 3.8.** Let \( G \) be a group of size \( p^n \) and exponent \( p \), where \( p \) is a prime number and \( n > 1 \) is an integer. Then \( \omega(C_G) = \frac{p^n - 1}{p - 1} \).

**Proof.** For \( x, y \in G \), \( \langle x, y \rangle \) is not cyclic if and only if \( \langle x \rangle \neq \langle y \rangle \). Thus \( \omega(C_G) \) is equal to the number of subgroups of \( G \) of order \( p \). This completes the proof.

**Lemma 3.9.** Let \( G \) and \( H \) be two finite non-cyclic groups such that \( \gcd(|G|, |H|) = 1 \). If \( \omega(C_G) = n \) and \( \omega(C_H) = m \) are finite, then \( \omega(C_{G \times H}) \geq nm \).

**Proof.** Let \( \{g_1, \ldots, g_n\} \) and \( \{h_1, \ldots, h_m\} \) be two clique sets in \( G \) and \( H \), respectively. Now it is easy to see that the set

\[
\{(g_i, h_j) | i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}
\]

is a clique in \( C_{G \times H} \). This completes the proof.

**Lemma 3.10.** Let \( G \) be an abelian group such that \( \omega(C_G) = 4 \). Then \( G/\text{Cyc}(G) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

**Proof.** By [2, Lemmas 3.2 and 3.5 and Lemma 2.3-(2)], \( H = G/\text{Cyc}(G) \) is an abelian \( \{2, 3\} \)-group with Sylow \( p \)-subgroups of exponent \( p \) and \( \omega(C_H) = 4 \) and...
\text{Cyc}(H) = 1. \text{ Thus}
\[ H \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3. \]

Since \( \omega(C_H) = 4 \) and \( \text{Cyc}(H) = 1 \), it follows from Lemmas 3.8 and 3.9 that \( k = 0 \) and \( \ell = 2 \), that is, \( H \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), as required. \hfill \Box

\textbf{Lemma 3.11.} \textit{There is no non-cyclic group \( G \) of order 27 with \( \omega(C_G) = 4 \).}

\textbf{Proof.} Suppose, for a contradiction, that \( G \) is a group of order 27 with \( \omega(C_G) = 4 \). It follows from Lemmas 3.10 and 3.8 that \( G \) is a non-abelian group of exponent 9. Therefore
\[ G \cong \langle c, d \mid c^9 = d^3 = 1, d^{-1}cd = c^4 \rangle. \]
Now the set \( \{c, d, cd, c^{-1}d, cd^{-1}\} \) is a clique. This completes the proof. \hfill \Box

\textbf{Theorem 3.12.} \textit{Let \( G \) be a non locally cyclic group. Then \( \omega(C_G) = 4 \) if and only if \( G/\text{Cyc}(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) or \( S_3 \).}

\textbf{Proof.} Suppose that \( \omega(C_G) = 4 \). Then by Lemma 3.2 we may assume that \( \text{Cyc}(G) = 1 \). Also it follows from Lemma 3.5 that \( G \) is a \{2,3\}-group. By Lemma 3.10, we may assume that \( G \) is non-abelian. Now since \( 4 = \omega(C_G) = \omega(A_G) \), we have \( \omega(A_G) = 3 \) or \( 4 \).

If \( \omega(A_G) = 3 \), then \( G/Z(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2 \). In this case by an argument similar to the proof of Theorem 3.7 we obtain a contradiction. So \( \omega(A_G) = 4 \), and by [4, Lemma 2.4-(4)] and [7, Theorem 5], we have \( G/Z(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) or \( S_3 \). By Lemma 3.11, Sylow 3-subgroups of \( G \) are of order 3 or 9. By Lemma 3.3, \( Z(G) \) is cyclic. Therefore \( Z(G) \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \).

(1) Let \( G/Z(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \). Then \( \text{gcd}(|G/Z(G)|, |Z(G)|) = 1 \), as Sylow subgroups of \( G \) are of order at most 9. Thus \( Z(G) = \text{Cyc}(G) = 1 \) and so \( G \) is abelian, a contradiction.

(2) Let \( G/Z(G) \cong S_3 \).

(I) If \( Z(G) \cong \mathbb{Z}_3 \), then \( |G| = 18 \), and since \( G \) is not abelian \( G \cong D_{18}, \langle c, d, e \mid c^3 = d^3 = e^2 = 1, cd = dc, c^e = c^{-1}, d^c = d^{-1} \rangle \), or \( \mathbb{Z}_3 \times S_3 \). The first two have trivial centers, so \( G \cong \mathbb{Z}_3 \times S_3 \). But if \( \mathbb{Z}_3 = \langle z \rangle \) and \( S_3 = \langle a, b \mid a^3 = b^2 = 1, a^b = a^{-1} \rangle \), then \( \{a, b, ab, a^2b, za\} \) is a clique, which is a contradiction.

(II) Let \( Z(G) \cong \mathbb{Z}_2 \). Then \( |G| = 12 \), and since \( G \) is non-abelian \( G \cong A_4, D_{12} \) or \( \langle a, b \mid a^2 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle \). But \( G \not\cong A_4 \), since \( Z(A_4) = 1 \). If \( G \cong D_{12} = \langle c, d \mid c^6 = 1 = d^2, dcd^{-1} = c^{-1} \rangle \), then \( \{c, d, cd, c^2d, c^3d\} \) is a clique, a contradiction. If \( G \cong \langle a, b \mid a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle \), then \( \{a, b, ab, a^{-1}b, a^2b\} \) is a clique, a contradiction.

This completes the proof. \hfill \Box
4. Planar and Hamiltonian Non-Cyclic Graphs

We were unable to decide whether the non-cyclic graph of a finite group is Hamiltonian or not. On the other hand, since the non-cyclic graph of a finite group is so rich in edges, it is hard to believe it is not Hamiltonian.

The following result reduces the verification of being Hamiltonian of the non-cyclic graph of a finite group $G$ to that of the graph $C_G/C_G(G)$.

**Lemma 4.1.** Let $G$ be a finite non-cyclic group such that $C_G/C_G(G)$ is Hamiltonian. Then $C_G$ is also Hamiltonian.

**Proof.** By hypothesis there exists a cycle

$$a_1 - a_2 - \cdots - a_n = a_1$$

in $C_G/G$, such that

$$\frac{G}{C_G(G)} \backslash \frac{C_G}{C_G(G)} = \{a_1, a_2, \ldots, a_n\}.$$ 

Let $C_G(G) = \{c_1, \ldots, c_k\}$. By $\ast$, $a_i c_j$ is adjacent to $a_i c_{j+1}$ in $C_G$ for all $i \in \{1, \ldots, n\}$ and all $j, \ell \in \{1, \ldots, k\}$, where indices of $a$'s are computed modulo $n$. Thus

$$a_1 c_1 - \cdots - a_n c_1 - a_1 c_2 - \cdots - a_n c_2 - \cdots - a_1 c_k - \cdots - a_n c_k - a_1 c_1$$

is a Hamilton cycle in $C_G$. This completes the proof.

In the following result we give a large family of finite groups with Hamiltonian non-cyclic graphs.

**Proposition 4.2.** Let $G$ be a finite non-cyclic group such that

$$|G| + |C_G(G)| > 2|C_G(x)|$$

for all $x \in Z(G) \setminus C_G(G)$. Then $C_G$ is Hamiltonian. In particular, $C_G$ is Hamiltonian whenever $Z(G) = C_G(G)$.

**Proof.** First note that the degree of any vertex $x$ in $C_G$ is equal to $|G\setminus C_G(x)|$. We now prove that $|G\setminus C_G(x)| > \frac{|G| - |C_G(G)|}{2}$ for all $x \in G\setminus C_G(G)$. Suppose, for a contradiction, that $|G\setminus C_G(x)| \leq \frac{|G| - |C_G(G)|}{2}$ for some $x \in G\setminus C_G(G)$. It follows that

$$2|C_G(x)| \geq |G| + |C_G(G)|.$$ 

It now follows from $\ast$ that $2|C_G(x)| \geq |G| + |C_G(G)|$. Since $|C_G(x)|$ divides $|G|$, we have $|G| = |C_G(x)|$ and so $x \in Z(G)$. Now $\ast$ contradicts our hypothesis, as $x$ belongs to $Z(G)\setminus C_G(G)$. Therefore $d(x) > (|G| - |C_G(G)|)/2$ for all vertices $x$ of $C_G$. Hence by Dirac’s theorem [5, p. 54], $C_G$ is Hamiltonian.
The inequality stated in Proposition 4.2 does not hold in general. For example if $G = C_4 \times S_3$, and $x \in C_4$ is an element of order 3, then it is easy to see that $|Cyc_G(x)| = 24$ and as $C cyc(G) = 1$, we see that the inequality does not hold.

Proposition 4.3. Let $G$ be a non locally cyclic group. Then $C_G$ is planar if and only if $G$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $S_3$ or $Q_8$.

Proof. It is easy to see that the non-cyclic graphs of the groups stated in the lemma are all planar. Now suppose that $C_G$ is planar. Since the complete graph of order 5 is not planar, we have $\omega(\Gamma_G) < 5$. Thus $G/Cyc(G)$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or $S_3$, where $p \in \{2, 3\}$, by Theorems 3.7 and 3.12. Now we prove that $|Cyc(G)| \leq 2$.

Suppose, for a contradiction, that $|Cyc(G)| > 2$ and consider a finite subset $C$ of Cyc(G) with $|C| = 3$. Let $x$ and $y$ be two adjacent vertices in $C_G$. Put $T = Cx \cup Cy$. Now the induced subgraph $C_T$ of $C_G$ by $T$ is a planar graph. On the other hand, $C_T$ is isomorphic to the bipartite graph $K_{3,3}$, a contradiction, since $K_{3,3}$ is not planar. If $G/Cyc(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $Q_8$. If $G/Cyc(G) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, then there are two adjacent vertices $x$ and $y$ such that the orders of $x$Cyc(G) and $y$Cyc(G) are both 3. Let $I = \{x, x^{-1}, y\}$ and $J = \{xy, x^{-1}y, (xy)^{-1}\}$. In the non-cyclic graph $C_G$ every vertex of $I$ is adjacent to every vertex of $J$. Therefore $C_G$ contains a copy of $K_{3,3}$, a contradiction.

If $G/Cyc(G) \cong S_3$, then there are vertices $a$ and $b$ such that $a \neq a^{-1}$ and both $a$ and $a^{-1}$ are adjacent to each vertex in $\{b, ab, a^2b\}$. Now suppose, for a contradiction, that Cyc(G) contains a non-trivial element $c$. Then $\{a, a^{-1}, ac\}$ and $\{b, ab, a^2b\}$ are the parts of a subgraph of $C_G$ isomorphic to $K_{3,3}$, a contradiction. Therefore, in this case, Cyc(G) = 1 and so $G \cong S_3$. This completes the proof.

5. A Solvability Criterion and New Characterizations for the Symmetric and Alternating Groups of Degree 5

We need the following result in the proof of Theorem 5.3 below.

Proposition 5.1 [3, Proposition 2.6]. Let $p$ be a prime number, $n$ a positive integer and $r$ and $q$ be two odd prime numbers dividing respectively $p^n + 1$ and $p^n - 1$. Then the number of Sylow $r$-subgroups (respectively, $q$-subgroups) of $L_2(p^n)$ is $p^n(p^n - 1)/2$ (respectively, $p^n(p^n + 1)/2$). Also any two distinct Sylow $r$-subgroups or $q$-subgroups have trivial intersection.

Proof. The proof follows from Theorems 8.3 and 8.4 in [8, Chap. II]. For a complete proof see the proof of [3, Proposition 2.6].

Lemma 5.2. Let $G$ be one of the following groups:

- $L_2(2^p)$, $p = 4$ or a prime; $L_2(3^p)$, $L_2(5^p)$, $p$ a prime; $L_2(p)$, $p$ a prime $\geq 7$; $L_3(3), L_3(5)$; PSU(3, 4) (the projective special unitary group of degree 3 over the finite field of order $4^2$) or Sz(2^p), $p$ an odd prime. Then $\omega(C_G) > 31$. 

Proof. For every group listed above we find a set $S$ of prime numbers $p$ for which Lemma 3.4 is applicable.

For every prime number $p$ and every integer $n > 0$, we have that the number of Sylow $p$-subgroups of $L_2(p^n)$ which are elementary abelian, is $p^n + 1$ and any two distinct Sylow $p$-subgroups have trivial intersection (see [8, Chap. II Theorem 8.2(b), (c)]). It follows that $\nu_p(L_2(p^n)) = (p^n + 1)(p^n - 1)/(p - 1)$. Thus among the projective special linear groups, we only need to investigate the following groups: $L_3(3)$, $L_3(5)$, $L_2(p)$ for $p \in \{7, 11, 13, 17, 19, 23, 29\}$. Now if in Proposition 5.1, we take $q = 3$ for $L_2(13)$ and $L_2(19)$; and $r = 3$ for $L_2(11)$, $L_2(17)$, $L_2(23)$ and $L_2(29)$; Then by Lemma 3.4 we are done in these cases. Therefore we must consider the groups $L_2(7)$, $L_3(3)$, $L_3(5)$, $PSU(3, 4)$ or $Sz(2^p)$, $p$ an odd prime.

If $G = L_2(7)$, then $|G| = 2^3 \times 3 \times 7$ and $G$ has no element of order $3 \times 7$. Now it follows from Lemma 3.4, that $\nu_3(G) + \nu_7(G) \leq w(C_G)$. Now by Proposition 5.1, we have $\nu_3(G) = 28$ and $\nu_7(G) = 8$ and so $w(C_G) \geq 36$. If $G = L_3(3)$, then $|G| = 2^4 \times 3^3 \times 13$ and $G$ has no element of order $3 \times 13$. Thus $\nu_3(G) = 1 + 3k$ and $\nu_{13}(G) = 1 + 3\ell$, for some $k > 0$ and $\ell > 0$. Since 14 does not divide $|G|$ and no non-abelian simple group contains a subgroup of index less than 5, $\nu_{13}(G) \geq 27$ and $\nu_3(G) \geq 7$. Now it follows from Lemma 3.4 that $w(C_G) \geq 34$. If $G = L_3(5)$, then $|G| = 2^5 \times 3 \times 5^2 \times 31$. Thus $\nu_{31}(G) = 1 + 31k$, for some $k > 0$ and so $\nu_{31}(G) > 31$. If $G = PSU(3, 4)$, then $|G| = 2^6 \times 3 \times 5^2 \times 13$ (see [8, Theorem 10.12(d) of Chap. II] and note that $PSU(3, 4)$ is the projective special unitary group of degree 3 over the finite field of order $4^2$). Therefore $\nu_{31}(G) = 1 + 3k$ for some $k > 0$ and since 14 does not divide $|L|$, $\nu_{31}(L) > 26$. If $G = Sz(2^p)$ ($p$ an odd prime), then it follows from Theorem 3.10 (and its proof) of [8, Chap. XI] that $\nu_2(G) \geq 2^{2p} + 1 \geq 65$. This completes the proof.

Theorem 5.3. Let $G$ be a non locally cyclic group.

1. If $w(C_G) = 31$, then $G$ is simple if and only if $G \cong A_5$.
2. If $w(C_G) \leq 30$, then $G$ is solvable.

Proof. (1) It follows from [2, Theorem 4.2] that $G/\text{Cyc}(G)$ is finite. Thus, if $G$ is simple, it is finite. Now suppose, for a contradiction, that there exists a non-cyclic finite simple group $K$ with $w(C_K) = 31$ which is not isomorphic to $A_5$. Let $T$ be such a group of least order. Thus every proper non-abelian simple section of $K$ is isomorphic to $A_5$. Therefore by [6, Proposition 3] $T$ is isomorphic to one of the groups in the statement of Lemma 5.2, which is impossible. This implies that $G \cong A_5$.

Now we prove that $w(C_{A_5}) = 31$. Note that the order of an element of $A_5$ is 2, 3 or 5; $A_5$ has five Sylow 2-subgroups, ten Sylow 3-subgroups and six Sylow 5-subgroups; and any two distinct Sylow subgroups has trivial intersection. Now consider the set of all non-trivial 2-elements of $A_5$ and select one group generator from each Sylow $p$-subgroup for $p \in \{3, 5\}$. Then the union $\mathcal{C}$ of these sets is of size 31 and every two distinct element of $\mathcal{C}$ generate a non-cyclic subgroup. On the
other hand, since $A_5$ is the union of its Sylow subgroups, it is easy to show that every clique set of $C_{A_5}$ is of size at most 31. This completes the proof of (1).

(2) It follows from [2, Theorem 4.2] that $G/\text{Cyc}(G)$ is finite and by Lemma 3.2, we may assume that $G$ is finite. Let $K$ be a counter-example of the least order. Thus every proper subgroup of $K$ is solvable and $G$ is a non-abelian simple group. That is to say, $K$ is a minimal simple group. Thus according to Thompson’s classification of the minimal simple groups in [15], $K$ is isomorphic to one of the following: $L_2(p)$ for some prime $p \geq 5$, $L_2(2p)$ or $L_2(3p)$ for some prime $p \geq 3$, $Sz(2p)$ for some prime $p \geq 3$, or $L_3(3)$.

By Lemma 5.1 and part (1) we have that $\omega(C_K) \geq 31$ and this contradicts the hypothesis. Hence $G$ is solvable.

\begin{lemma}
Let $G$ be either $A_5$ or $S_5$. Then $\omega(C_G) = 31$. Moreover, every non-trivial element of $G$ is contained in a maximum clique of $C_G$.
\end{lemma}

\begin{proof}
Clearly $\omega(C_{S_5}) \geq w(C_{A_5})$. Let $C$ be the clique found in Theorem 5.3 for $C_{A_5}$. It is easy to see (e.g. by GAP [14]) that

$$G = \bigcup_{x \in C} \text{Cyc}_G(x)$$

and $\text{Cyc}_G(x) = \text{Cyc}_G(a)$ for all non-trivial $a \in \text{Cyc}_G(x)$. (*)

Thus $G$ is the union of 31 cyclic subgroups and so $\omega(C_{S_5}) \leq 31$. It follows that $\omega(C_G) = 31$. The second part follows easily from (*). This completes the proof.
\end{proof}

\begin{theorem}
Let $G$ be a non-solvable group. Then $\omega(C_G) = 31$ if and only if $G/\text{Cyc}(G) \cong A_5$ or $S_5$.
\end{theorem}

\begin{proof}
If $G/\text{Cyc}(G) \cong A_5$ or $S_5$, then it follows from Lemmas 3.2 and 5.4 that $\omega(C_G) = 31$.

Suppose that $\omega(C_G) = 31$. By [2, Theorem 4.2] $G/\text{Cyc}(G)$ is finite. Thus we may assume without loss of generality that $G$ is finite and $\text{Cyc}(G) = 1$ and so we have to prove $G \cong A_5$ or $S_5$.

Let $S$ be the largest normal solvable subgroup of $G$ (here, for the existence of $S$ we use the finiteness of $G$). Then $\tilde{G} = G/S$ has no non-trivial abelian normal subgroup. Let $R$ be the product of all minimal normal non-abelian subgroups of $\tilde{G}$. It follows from [3, Lemma 2.1] and Theorem 5.3 that $\tilde{R} \cong A_5$. Since $C_{\tilde{G}}(R) = 1$, we have that $\tilde{G}$ is isomorphic to a subgroup of $S_5$. It follows that $\tilde{G} \cong A_5$ or $S_5$. Now it follows from Lemmas 3.3 and 5.4, that $S \leq \text{Cyc}(G) = 1$. This completes the proof.
\end{proof}

We remark here that there are solvable groups $G$ for which $\omega(C_G) = 31$; for example, by Lemma 3.8, we may take $G$ to be either the elementary abelian 2-group of rank 5 or the elementary abelian 5-group of rank 3.

We end the paper with the answer of following question posed in [2].
Question 5.6 [2, Question 2.4]. Let $G$ be a torsion free group such that $\text{Cyc}(G)$ is non-trivial. Is it true that $G$ is locally cyclic?

In [12, Theorem 31.4] Ol’shanskii has constructed a non-abelian torsion-free group $G$ all of whose proper subgroups are cyclic and it is central extension of an infinite cyclic group $Z$ by an infinite group of bounded exponent. Since the group $G$ is 2-generated, it is not locally cyclic. Also $Z \leq \text{Cyc}(G)$, for if $z \in Z$ and $a \in G$, then $\langle z, a \rangle$ is abelian, and as $G$ is not abelian, $\langle z, a \rangle \neq G$. Thus $\langle z, a \rangle$ is cyclic. Hence the answer of [2, Question 2.4] is negative.

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