RIGHT 4-ENGL ELE ME NTS OF A GROUP

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We prove that the set of right 4-Engel elements of a group $G$ is a subgroup for locally nilpotent groups $G$ without elements of orders 2, 3 or 5; and in this case the normal closure $(x)^G$ is nilpotent of class at most 7 for each right 4-Engel elements $x$ of $G$.

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1. Introduction and Results

Let $G$ be any group and $n$ a nonnegative integer. For any two elements $a$ and $b$ of $G$, we define inductively $[a, n] b$ the $n$-Engel commutator of the pair $(a, b)$, as follows:

$$[a, 0] b = a, \quad [a, 1] b = a^{-1} b^{-1} a b \quad \text{and} \quad [a, n] b = [[a, n-1] b], b \quad \text{for all } n > 0.$$ 

An element $x$ of $G$ is called right $n$-Engel if $[x, n] g = 1$ for all $g \in G$. We denote by $R_n(G)$ the set of all right $n$-Engel elements of $G$. A group $G$ is called $n$-Engel if $G = R_n(G)$. It is clear that $R_1(G) = Z(G)$ is the center of $G$ and Kappe [5] proved $R_2(G)$ is a characteristic subgroup of $G$. Macdonald [6] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. Nickel [8] generalized Macdonald’s result to all $n \geq 3$. Although Macdonald’s example shows that $R_3(G)$ is not in general a subgroup of $G$, Heineken [4] has already shown that if $A$ is the subset of a group $G$ consisting of all elements $a$ such that $a^{±1} \in R_3(G)$, then $A$ is a subgroup if either $G$ has no element of order 2 or $A$ consists only of elements having finite odd order. Newell [7] proved that the normal closure of every right 3-Engel element is nilpotent of class at most 3. In Sec. 2, we prove that if $G$ is a 2-group, then $R_3(G)$ is a subgroup of $G$. Nickel’s example shows that the set
of right 4-Engel elements is not a subgroup in general (see also the first example in Sec. 4 of [1]). In Sec. 3, we prove that if $G$ is a locally nilpotent $\{2, 3, 5\}'$-group, then $R_4(G)$ is a subgroup of $G$.

Traustason [11] proved that any locally nilpotent 4-Engel group $H$ is Fitting of degree at most 4. This means that the normal closure of every element of $H$ is nilpotent of class at most 4. More precisely he proved that if $H$ has no element of order 2 or 5, then $H$ has Fitting degree at most 3. Now by a result of Havas and Vaughan-Lee [3], one knows any 4-Engel group is locally nilpotent and so Traustason’s result is true for all 4-Engel groups. In Sec. 3, by another result of Traustason [12] we show that the normal closure of every right 4-Engel element in a locally nilpotent $\{2, 3, 5\}'$-group, is nilpotent of class at most 7.

Throughout the paper, we have frequently used $\texttt{nq}$ package of Nickel [9] which is implemented in GAP [10]. All given timings were obtained on an Intel Pentium 4-1.70GHz processor with 512 MB running Red Hat Enterprise Linux 5.

2. Right 3-Engel Elements

Throughout, for any positive integer $k$ and any group $H$, $\gamma_k(H)$ denotes the $k$th term of the lower central series of $H$. The main result of this section implies that $R_3(G)$ is a subgroup of $G$ whenever $G$ is a $2'$-group. Newell [7] proved that

Theorem 2.1. Let $G = \langle a, b, c \rangle$ be a group such that $a, b \in R_3(G)$. Then

(1) $\langle a, c \rangle$ is nilpotent of class at most 5 and $\gamma_5(\langle a, c \rangle)$ has exponent 2.
(2) $G$ is nilpotent of class at most 6.
(3) $\gamma_5(G)/\gamma_6(G)$ has exponent 10. Furthermore $[a, c, b, c, c] \in \gamma_6(G)$.
(4) $\gamma_6(G)$ has exponent 2.

Theorem 2.2. Let $G$ be a group such that $\gamma_5(G)$ has no element of order 2. Then $R_3(G)$ is a subgroup of $G$.

Proof. Let $a, b \in R_3(G)$ and let $c$ be an arbitrary element of $G$. Thus

(1) $[a, c, c, c] = 1$.
(2) $[b, c, c, c] = 1$.

Since by our assumption $\gamma_5(G)$ has no element of order 2, it follows from Theorem 2.1 parts (1), (3), and (4), respectively that

(3) the subgroup $\langle a, c \rangle$ is nilpotent of class at most 4.
(4) $[a, c, b, c, c] = 1$.
(5) the subgroup $\langle a, b, c \rangle$ is nilpotent of class at most 5.
To prove $R_3(G)$ is a subgroup, we have to show that both $a^{-1}$ and $ab$ belong to $R_3(G)$. We first prove that $a^{-1} \in R_3(G)$. It easily follows from (1) and (3) that:
\[
[a^{-1}, c, c, c] = [a, c, c, c]^{-1} = 1.
\]
Therefore $a^{-1} \in R_3(G)$.

We now show that $ab \in R_3(G)$.
\[
[ab, c, c, c] = [a, c][a, b][b, c, c, c] = [a, c][a, b][b, c, c, c, c] \quad \text{by (5)}
\]
\[
= [a, c, c][a, c, b, c][b, c, c, c] \quad \text{by (1) and (5)}
\]
\[
= [a, c, c, b, c, c, c] \quad \text{by (2)}
\]
\[
= 1 \quad \text{by (4)}
\]
This completes the proof.

Now we give a proof of Theorem 2.2 by using \texttt{nq} package of Nickel [9] which is implemented in GAP [10]. Note that the knowledge of Theorem 2.1 is crucial in the following proof. The package \texttt{nq} has the capability of computing the largest nilpotent quotient (if it exists) of a finitely generated group with finitely many identical relations and finitely many relations. For example, if we want to construct the largest nilpotent quotient of a group $G$ as follows
\[
\langle x_1, \ldots, x_n \mid r_1(x_1, \ldots, x_n) = \cdots = r_m(x_1, \ldots, x_n) = 1, w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1 \rangle,
\]
where $r_1, \ldots, r_m$ are relations on $x_1, \ldots, x_n$ and $w(x_1, \ldots, x_n, y_1, \ldots, y_k) = 1$ is an identical relation in the group $\langle x_1, \ldots, x_n \rangle$, one may apply the following code to use the package \texttt{nq} in GAP:

```gap
LoadPackage("nq"); # nq package of Werner Nickel #
F:=FreeGroup(n+k);
L:=F/[r1(F.1,..,F.n),..,rm(F.1,..,F.n),w(F.1,..,F.n,F.(n+1),..,F.(n+k))];
H:=NilpotentQuotient(L,[F.(n+1),..,F.(n+k)]);
```

Note that we need to construct the free group of rank $n+k$ because as well as the $n$ generators for $G$ we also have an identical relation with $k$ free variables.

Note that the function \texttt{NilpotentQuotient(L)} attempts to compute the largest nilpotent quotient of $L$ and it will terminate only if $L$ has a largest nilpotent quotient.

**Second Proof of Theorem 2.2.** By Theorem 2.1, we know that $(x, y, z)$ is nilpotent if $x, y \in R_3(G)$ and $z \in G$. We now construct the largest nilpotent group $H = \langle a, b, c \rangle$ such that $a, b \in R_3(H)$ and $c \in H$, by \texttt{nq} package.
LoadPackage("nq");
F:=FreeGroup(4); a1:=F.1; b1:=F.2; c1:=F.3; x:=F.4;
L:=F/[LeftNormedComm([a1,x,x,x]),LeftNormedComm([b1,x,x,x])];
H:=NilpotentQuotient(L,[x]);
a:=H.1; b:=H.2; c:=H.3; d:=LeftNormedComm([a^{-1},c,c,c,c]);
e:=LeftNormedComm([a*b,c,c,c,c]); Order(d); Order(e);
C:=LowerCentralSeries(H); d in C[5]; e in C[5];

Then if we consider the elements 
\[ d = [a^{-1},c,c,c] \] and \[ e = [ab,c,c,c] \] of \( H \), we can see by above command in GAP that \( d \) and \( e \) are elements of \( \gamma_5(H) \) and have orders 2 and 4, respectively. So, in the group \( G \), we have \( d = e = 1 \). This completes the proof. \( \Box \)

Note that, the second proof of Theorem 2.2 also shows the necessity of assuming that \( \gamma_5(G) \) has no element of order 2.

3. Right 4-Engel Elements

Our main result in this section is to prove the following.

**Theorem 3.1.** Let \( G \) be a \( \{2,3,5\}^{'} \)-group such that \( \langle a,b,x \rangle \) is nilpotent for all \( a,b \in R_4(G) \) and any \( x \in G \). Then \( R_4(G) \) is a subgroup of \( G \).

**Proof.** Consider the “freest” group, denoted by \( U \), generated by two elements \( u,v \) with \( u \) a right 4-Engel element. We mean this by the group \( U \) given by the presentation

\[
\langle u,v \mid [u,4,x] = 1 \text{ for all words } x \in F_2 \rangle,
\]

where \( F_2 \) is the free group generated by \( u \) and \( v \). We do not know whether \( U \) is nilpotent or not. Using the nq package shows that the group \( U \) has a largest nilpotent quotient \( M \) with class 8. By the following code, the group \( M \) generated by a right 4-Engel element \( a \) and an arbitrary element \( c \) is constructed. We then see that the element \( [a^{-1},c,c,c] \) of \( M \) is of order \( 375 = 3 \times 5^3 \). Therefore, the inverse of a right 4-Engel element of \( G \) is again a right 4-Engel element. The following code in GAP gives a proof of the latter claim. The computation was completed in about 24s.

\[
F:=FreeGroup(3); a1:=F.1; b1:=F.2; x:=F.3;
U:=F/[LeftNormedComm([a1,x,x,x,x])];
M:=NilpotentQuotient(U,[x]);
a:=M.1; c:=M.2;
h:=LeftNormedComm([a^{-1},c,c,c,c]);
Order(h);
\]

We now show that the product of every two right 4-Engel elements in \( G \) is a right 4-Engel element. Let \( a,b \in R_4(G) \) and \( c \in G \). Then we claim that

\[
H = \langle a,b,c \rangle \text{ is nilpotent of class at most } 7. \quad (*)
\]
Corollary 3.3. Let \( G \) be a \( \{2,3,5\}' \)-group such that \( \langle a, b, x \rangle \) is nilpotent for all \( a, b \in R_4(G) \) and for any \( x \in G \). Then \( R_4(G) \) is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of group \( G \) is nilpotent of class at most 7.

Proof. By Theorem 3.1, \( R_4(G) \) is a subgroup of \( G \) and so it is a 4-Engel group. In [12], it is shown that every locally nilpotent 4-Engel \( \{2,3,5\}' \)-group is nilpotent of class at most 7. Therefore, \( R_4(G) \) is nilpotent of class at most 7. Since \( R_4(G) \) is a normal set, the second part follows easily.

Therefore, to prove that the normal closure of any right 4-Engel element of a \( \{2,3,5\}' \)-group \( G \) is nilpotent, it is enough to show that \( \langle a, b, x \rangle \) is nilpotent for all \( a, b \in R_4(G) \) and for any \( x \in G \).

Corollary 3.3. In any \( \{2,3,5\}' \)-group, the normal closure of any right 4-Engel element is nilpotent if and only if every 3-generator subgroup in which two of the generators can be chosen to be right 4-Engel, is nilpotent.

Proof. By Corollary 3.2, it is enough to show that a \( \{2,3,5\}' \)-group \( H = \langle a, b, x \rangle \) is nilpotent whenever \( a, b \in R_4(H) \), \( x \in H \) and both \( \langle a \rangle^H \) and \( \langle b \rangle^H \) are nilpotent. Consider the subgroup \( K = (a)^H (b)^H \) which is nilpotent by Fitting’s theorem. Now we prove that \( K \) is finitely generated. We have \( K = \langle a, b \rangle^{(x)} \) and since \( a \) and \( b \) are both right 4-Engel, it is well-known that

\[
\langle a \rangle^{(x)} = \langle a, a^x, a^{x^2}, a^{x^3} \rangle \quad \text{and} \quad \langle b \rangle^{(x)} = \langle b, b^x, b^{x^2}, b^{x^3} \rangle.
\]
and so

$$K = \langle a, a^x, a^{x^2}, a^{x^3}, b, b^x, b^{x^2}, b^{x^3} \rangle.$$  

It follows that $H$ satisfies maximal condition on its subgroups as it is (finitely generated nilpotent)-by-cyclic. Now by a famous result of Baer \[2\] we have that $a$ and $b$ lie in the $(m + 1)$th term $\zeta_m(H)$ of the upper central series of $H$ for some positive integer $m$. Hence $H/\zeta_m(H)$ is cyclic and so $H$ is nilpotent. This completes the proof.

We conclude this section with the following interesting information on the group $M$ in the proof of Theorem 3.1. In fact, for the largest nilpotent group $M = \langle a, b \rangle$ relative to $a \in R_4(M)$, we have that $M/T$ is isomorphic to the largest (nilpotent) 2-generated 4-Engel group $E(2, 4)$, where $T$ is the torsion subgroup of $M$ which is a $\{2, 3, 5\}$-group. Therefore, in a nilpotent $\{2, 3, 5\}$-group, a right 4-Engel element with an arbitrary element generate a 4-Engel group. This can be seen by comparing the presentations of $M/T$ and $E(2, 4)$ as follows. One can obtain two finitely presented groups $G_1$ and $G_2$ isomorphic to $M/T$ and $E(2, 4)$, respectively by GAP:

\begin{verbatim}
MoverT:=FactorGroup(M,TorsionSubgroup(M));
E24:=NilpotentEngelQuotient(FreeGroup(2),4);
iso1:=IsomorphismFpGroup(MoverT);iso2:=IsomorphismFpGroup(E24);
G1:=Image(iso1);G2:=Image(iso2);
\end{verbatim}

Next, we find the relators of the groups $G_1$ and $G_2$ which are two sets of relators on 13 generators by the following command in GAP:

\begin{verbatim}
r1:=RelatorsOfFpGroup(G1);r2:=RelatorsOfFpGroup(G2);
\end{verbatim}

Now, save these two sets of relators by LogTo command of GAP in a file and go to the file to delete the terms as

\begin{verbatim}
<identity ...>
\end{verbatim}

in the sets r1 and r2. Now call these two modified sets R1 and R2. We show that $R_1=R_2$ as two sets of elements of the free group $f$ on 13 generators $f_1, f_2, ..., f_{13}$.

\begin{verbatim}
f:=FreeGroup(13);
f1:=f.1;f2:=f.2;f3:=f.3;f4:=f.4;f5:=f.5;f6:=f.6;
f7:=f.7;f8:=f.8;f9:=f.9;f10:=f.10;f11:=f.11;f12:=f.12;f13:=f.13;
\end{verbatim}

Now by Read function, load the file in GAP and type the simple command $R_1=R_2$. This gives us true which shows $G_1$ and $G_2$ are two finitely presented groups with the same relators and generators and so they are isomorphic. We do not know if there is a guarantee that if someone else does as we did, then he/she finds the same relators for Fp groups $G_1$ and $G_2$, as we have found. Also we remark that using function IsomorphismGroups to test if $G_1 \cong G_2$, did not give us a result in less than 10 h and we do not know whether this function can give us a result or not.
We summarize the above discussion as following.

**Theorem 3.4.** Let $G$ be a nilpotent group generated by two elements, one of which is a right 4-Engel element. If $G$ has no element of order 2, 3, or 5, then $G$ is a 4-Engel group of class at most 6.

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**References**