ENGEL ELEMENTS IN GROUPS

ALIREZA ABDOLLAHI

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
Email: a.abdollahi@math.ui.ac.ir

Abstract

We give a survey of results on the structure of right and left Engel elements of a group. We also present some new results in this topic.

1 Introduction

Let $G$ be any group and $x, y \in G$. Define inductively the $n$-Engel left normed commutator

$$[x, n y] = [x, y, \ldots, y]_n$$

of the pair $(x, y)$ for a given non-negative integer $n$, as follows:

$$[x, 0 y] := x, \quad [x, 1 y] := [x, y] = x^{-1}y^{-1}xy =: x^{-1}x^y,$$

and for all $n > 0$

$$[x, n y] = [[x, n-1 y], y].$$

An element $a \in G$ is called left Engel whenever for every element $g \in G$ there exists a non-negative integer $n = n(g, a)$ possibly depending on the elements $g$ and $a$ such that $[g, n a] = 1$. For a positive integer $k$, an element $a \in G$ is called a left $k$-Engel element of $G$ whenever $[g, k a] = 1$ for all $g \in G$. An element $a \in G$ is called a bounded left Engel element if it is left $k$-Engel for some $k$. We denote by $L(G)$, $L_k(G)$ and $\overline{L}(G)$, the set of left Engel elements, left $k$-Engel elements, bounded left Engel elements of $G$, respectively.

In definitions of various types of left Engel elements $a$ of a group $G$, we observe that the variable element $g$ appears on the left hand side of the element $a$. So similarly one can define various types of right Engel elements $a$ in a group by letting the variable element $g$ to appear (in the $a$-Engel commutator of $a$ and $g$) on the right hand side of the element $a$. Therefore, an element $a \in G$ is called right Engel whenever for every element $g \in G$ there exists a non-negative integer $n = n(g, a)$ such that $[a, n g] = 1$. For a positive integer $k$, an element $a \in G$ is called a right $k$-Engel element of $G$ whenever $[a, k g] = 1$ for all $g \in G$. An element $a \in G$ is called a bounded right Engel element if it is right $k$-Engel for some $k$. We will denote $R(G)$, $R_k(G)$ and $\overline{R}(G)$, the set of right Engel elements, right $k$-Engel elements, bounded right Engel elements of $G$, respectively.

All these subsets are invariant under automorphisms of $G$. 
Groups in which all elements are left Engel are called Engel groups and for a given positive integer $n$, a group is called $n$-Engel if all of whose elements are left $n$-Engel elements. It is clear that

$$G = L(G) \iff G = R(G) \quad \text{and} \quad G = L_n(G) \iff G = R_n(G).$$

A group is called bounded Engel if it is $k$-Engel for some positive integer $k$. Note that

$$L_1(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots \subseteq L(G) \quad \text{and} \quad \overline{L}(G) = \bigcup_{k \in \mathbb{N}} L_k(G)$$

$$R_1(G) \subseteq R_2(G) \subseteq \cdots \subseteq R_n(G) \subseteq \cdots \subseteq R(G) \quad \text{and} \quad \overline{R}(G) = \bigcup_{k \in \mathbb{N}} R_k(G)$$

As stated in [43, p. 41 of Part II], the major goal of Engel theory can be stated as follows: to find conditions on a group $G$ which will ensure that $L(G)$, $\overline{L}(G)$, $R(G)$ and $\overline{R}(G)$ are subgroups and, if possible, coincide with the Hirsch–Plotkin radical, the Baer radical, the hypercenter and the $\omega$-center respectively. So let us put forward the following question.

**Question 1.1** For which groups $G$ and which positive integers $n$

1. $L(G)$ is a subgroup of $G$?
2. $R(G)$ is a subgroup of $G$?
3. $\overline{L}(G)$ is a subgroup of $G$?
4. $\overline{R}(G)$ is a subgroup of $G$?
5. $L_n(G)$ is a subgroup of $G$?
6. $R_n(G)$ is a subgroup of $G$?

In the next sections we shall discuss Question 1.1 on various classes of groups $G$ and small positive integers $n$ and we also study many new questions extracted from it.

The author has tried the present survey to be complete, but needless to say that it does not contain all results on ‘the Engel structure’ of groups. Most of results before 1970 was already surveyed in [43, chapter 7 in Part II] and here we have only sorted them as ‘left/right’ results into separate sections. The latter reference is still more comprehensive than ours for results before 1970.

As I believe the following famous sentence of Hilbert [22], I have had a tendency to write any question (not only ones which are very difficult!) that I have encountered.

“As long as a branch of science offers an abundance of problems, so long it is alive; a lack of problems foreshadows extinction or the cessation of independent development.”
2 Interaction of Right Engel Elements with Left Engel Elements

Baer [6, Folgerung N and Folgerung A] proved that in groups with maximal condition on subgroups and in hyperabelian groups, a right Engel element is a left Engel element. The answer to the following question is still unknown.

**Question 2.1 (Robinson [44, p. 370])** Is it true that every right Engel element of any group is a left Engel element?

Heineken’s result [19] gives the famous relation between left and right Engel elements of an arbitrary group; it can be read as follows: the inverse of a right Engel element is a left one.

**Theorem 2.2 (Heineken [19])** In any group $G$,
1. for any two elements $x, g \in G$ and all integers $n \geq 1$, $[x, g^{n+1}] = [g^{-x}, g]^n$.
2. $R(G)^{-1} \subseteq L(G)$.
3. $R_n(G)^{-1} \subseteq L_{n+1}(G)$.
4. $R(G)^{-1} \subseteq L(G)$.

**Proof** All parts follows easily from 1. We may write

$$[x, g^{n+1}] = [x, g]_{n, g} = [x^{-1}g^{-1}xg, g] = [(g^{-1})^x, g, g] = [(g^{-1})^x, g, g]^{n-1}g = [(g^{-1})^x, g, g]^{n-1}g$$

Now for instance, if $g \in G$ such that $g^{-1} \in R_n(G)$, then $(g^{-1})^x \in R_n(G)$ for any $x \in G$ and so $[(g^{-1})^x, g, g] = 1$ which implies that $[x, g^{n+1}] = 1$, by part 1.

Apart from Heineken’s result, we do not know of other inclusions holding between Engel subsets in an arbitrary group. We do not even know the answer of the following question.

**Question 2.3**
1. For which integers $n \geq 1$, $R_n(G) \subseteq L(G)$ for any group $G$?
2. Is it true that $R(G)^{-1} \subseteq L(G)$ for any group $G$?

The answer of Questions 2.1 and 2.3 are known for many classes of groups (that we shall see them), but I do not know of even a “knock” on the general case. Let us do that for part 1 of Question 2.3. Suppose $n > 0$ is an integer such that $R_n(G) \subseteq L(G)$ for any group $G$. We show that there is an integer $k > 0$ depending only on $n$ such that $R_n(G) \subseteq L_k(G)$ for any group $G$. Let $G_n$ be the group given by the following presentation

$$\langle x, y \mid [x, y, X] = 1 \text{ for all } X \in \langle x, y \rangle \rangle.$$
Thus, it follows that if $G$ is a group generated by two elements $a$ and $b$ such that $a \in R_n(G)$, then there is a group epimorphism $\varphi$ from $G$ onto $G$ such that $x^\varphi = a$ and $y^\varphi = b$. Since by assumption $R_n(G) \subseteq L(G)$, we have $[y^\varphi, x^\varphi] = 1$ for some $k$ depending only on $n$. This implies that

$$1 = [y^\varphi, x^\varphi] = [y^\varphi, x^\varphi] = [y, x]^k = [b, a].$$

Therefore, to confirm validity of part 2 of Question 2.3, one should find for any positive integer $n$, an integer $k$ such that $R_n(G) \subseteq L_k(G)$ for all groups $G$. So let us put forward the following question.

**Question 2.4** For which positive integers $n$, there exists a positive integer $k$ such that $R_n(G) \subseteq L_k(G)$ for all groups $G$?

To refute part 2 of Question 2.3 which is a question between bounded and unbounded Engel sets, one has to answer positively the following question on bounded Engel sets.

**Question 2.5** Is there a positive integer $n$ such that for any given positive integer $k$ there is a group $G_k$ with $R_n(G_k) \not\subseteq L_k(G_k)$?

What we know about other studies on relations between left and right Engel elements are mostly on the ‘negative side’. The following example of Macdonald bounds Heineken’s result “$R_n^{-1} \subseteq L_{n+1}$”.

**Theorem 2.6 (Macdonald [33])** For any prime number $p$ and each multiple $n > 2$ of $p$, there is a finite metabelian $p$-group $G$ containing an element $a \in R_n(G)$ such that $a \notin L_n(G)$ and $a^{-1} \notin L_n(G)$.

Macdonald’s result was sharpened by L.-C. Kappe [26]. Newman and Nickel [36] showed that the situation may be more bad: no non-trivial power of a right $n$-Engel element can be a left $n$-Engel element.

### 3 Four Engel Subsets and Corresponding Subgroups

In the most of groups $G$ for which we know that parts 1 to 4 of Question 1.1 are all true, the corresponding Engel subsets are equal to the following subgroups, respectively: the Hirsch–Plotkin radical, the hypercenter, the Baer radical and the $\omega$-center of $G$. In this section, we first shortly recall definitions of these subgroups and then in the next section we collects known relations with the corresponding Engel subsets. There are also two other less famous subgroups of an arbitrary group $G$ denoted by $\varphi(G)$ and $\overline{\varphi}(G)$ which are related to the right Engel elements.

Let $G$ be any group. We denote by $Fitt(G)$ the Fitting subgroup of $G$ which is the subgroup generated by all normal nilpotent subgroups of $G$. By [10, p. 100] the normal closure of each element of $Fitt(G)$ in $G$ is nilpotent.

Let $HP(G)$ (called the Hirsch–Plotkin radical of $G$) be the subgroup generated by all normal locally nilpotent subgroups of $G$. By [23] or [40] $HP(G)$ is locally nilpotent.
We denote by $B(G)$ the set of elements $x \in G$ such that $\langle x \rangle$ is a subnormal subgroup in $G$. Then by [5, §3, Satz 3] $B(G)$ (called the Baer radical of $G$) is a normal locally nilpotent subgroup of $G$ such that every cyclic subgroup of $B(G)$ is subnormal in $G$.

We define inductively $\zeta_\alpha(G)$ (called $\alpha$-center of $G$) for each ordinal number $\alpha$. For $\alpha = 0, 1$, we have $\zeta_0(G) = 1$ and $\zeta_1(G) = Z(G)$ the center of $G$. Now suppose $\zeta_\beta(G)$ has been defined for any ordinal $\beta < \alpha$. If $\alpha$ is not a limit ordinal (i.e., $\alpha = \alpha' + 1$ for some ordinal $\alpha' < \alpha$), we define $\zeta_\alpha(G)$ to be such that

$$Z(G/\zeta_\alpha(G)) = \zeta_\alpha(G)/\zeta_{\alpha'}(G),$$

and if $\alpha$ is a limit ordinal, we define $\zeta_\alpha(G) = \bigcup_{\beta < \alpha} \zeta_\beta(G)$.

We denote by $\omega$ the ordinal of natural numbers $\mathbb{N}$ with the usual order $\lt$. The ordinal $\omega$ is the first infinite ordinal and it is a limit one. It follows that $\zeta_\omega(G) = \bigcup_{\beta < \omega} \zeta_\beta(G)$.

Since every ordinal $\beta < \omega$ is a finite one, every such $\beta$ can be considered as a non-negative integer. Thus we have

$$Z(G/\zeta_i(G)) = \zeta_{i+1}(G)/\zeta_i(G) \text{ for each integer } i \geq 0.$$

Since the cardinal of a group $G$ cannot be exceeded, there is an ordinal $\beta$ such that $\zeta_\beta(G) = \zeta_\beta(G)$ for all ordinal $\lambda \geq \beta$. For such an ordinal $\beta$, we call $\zeta_\beta(G)$ the hypercenter of $G$ and it will be denoted by $\tilde{\zeta}(G)$.

We denote by $Gr(G)$ the set of elements $x \in G$ such that $\langle x \rangle$ is an ascendant subgroup in $G$. Then by [14, Theorem 2] $Gr(G)$ (called the Grunenberg radical of $G$) is a normal locally nilpotent subgroup of $G$ such that every cyclic subgroup of $Gr(G)$ is ascendant in $G$. A group $G$ is called a Fitting, hypercentral, Baer or Grunenberg group if $Fitt(G) = G$, $\tilde{\zeta}(G) = G$, $B(G) = G$ and $Gr(G) = G$, respectively. Note that

$$Fitt(G) \leq B(G) \leq Gr(G) \leq HP(G)$$

for any group $G$.

For a group $G$, following Grunenberg [14, p. 159] we define $\varrho(G)$ to be the set of all elements $a$ of $G$ such that $\langle x \rangle$ is an ascendant subgroup of $\langle x \rangle \langle a \rangle^G$ for every $x \in G$. Similarly, $\overline{\varrho}(G)$ is defined to be the set of all elements $a \in G$ for which there is a positive integer $n = n(a)$ such that $\langle x \rangle$ is a subnormal subgroup in $\langle x \rangle \langle a \rangle^G$ of defect at most $n$ for every $x \in G$.

By [14, Theorem 3] $\varrho(G)$ and $\overline{\varrho}(G)$ are characteristic subgroups of $G$ satisfying

$$\zeta_\omega(G) \leq \overline{\varrho}(G) \text{ and } \tilde{\zeta}(G) \leq \varrho(G).$$

In addition, $\varrho(G) \leq Gr(G)$ and $\overline{\varrho}(G) \leq B(G)$.
3.1 Left Engel Elements

In this section we will deal with left Engel elements.

**Proposition 3.1** For any group \( G \), \( \text{HP}(G) \subseteq L(G) \).

**Proof** Let \( g \in \text{HP}(G) \) and \( x \in G \). Then \( \langle g^{-x}, g \rangle \leq \text{HP}(G) \), as \( \text{HP}(G) \) is normal in \( G \). Thus \( \langle g^{-x}, g \rangle \) is nilpotent of class at most \( n \geq 1 \), say. By Theorem 2.2, we have \( [x_{n+1} g] = [g^{-x}, g]^{g} \). As \( [g^{-x}, g] = 1 \), we have \( [x_{n+1} g] = 1 \). This implies that \( g \in L(G) \).

Let \( p \) be a prime number or zero and let \( d \geq 2 \) be any integer. Golod [12] has constructed a non-nilpotent (infinite) \( d \)-generated, residually finite, \( p \)-group (torsion-free group whenever \( p = 0 \)) \( G_{d}(p) \) in which every \( d-1 \) generated subgroup is a nilpotent group. Hence for any \( d \geq 3 \) every two generated subgroup of \( G_{d}(p) \) is nilpotent so that the Golod group \( G_{d}(p) \) is an Engel group, that is, \( G_{d}(p) = R(G_{d}(p)) = L(G_{d}(p)) \). Therefore for an arbitrary group \( G \), it is not necessary to have \( \text{HP}(G) = L(G) \) as \( L(G_{d}(p)) \neq \text{HP}(G_{d}(p)) \).

**Proposition 3.2** For any group \( G \), \( B(G) \subseteq \overline{L}(G) \).

**Proof** Let \( g \in B(G) \). Thus \( \langle g \rangle \) is subnormal in \( G \) of defect \( k \), say. Therefore \( \langle g \rangle \langle G_{k} \langle g \rangle \rangle = \langle g \rangle \). It follows that \( [G_{k} \langle g \rangle] \leq \langle g \rangle \) and so \( [G_{k+1} \langle g \rangle] = 1 \). Hence \( [x_{k+1} g] = 1 \) for all \( x \in G \) and \( g \in \overline{L}(G) \).

So, elements of the Hirsch–Plotkin radical (the Baer radical, resp.) of a group are potential examples of (bounded, resp.) left Engel elements of a group. An element of order 2 (if exists) in a group, under some extra conditions, can also belong to the set of (bounded) left Engel elements.

**Proposition 3.3** Let \( x \) be an element of any group \( G \) such that \( x^{2} = 1 \).

1. \([g_{n} x] = [g, x]^{(-2)^{n-1}} \) for all \( g \in G \) and all integers \( n \geq 1 \).

2. If every commutator of weight 2 containing \( x \) is a 2-element, then \( x \in L(G) \).

   In particular, if \( G' \) is a 2-group, then \( x \in L(G) \).

3. If every commutator of weight 2 containing \( x \) is of order dividing \( 2^{n} \), then \( x \in L_{n+1}(G) \). In particular, if \( G' \) is of exponent dividing \( 2^{n} \), then \( x \in L_{n+1}(G) \).

**Proof** It is enough to show part 1. We argue by induction on \( n \). It is clear, if \( n = 1 \). We have

\[
[g_{n+1} x] = [g_{n} x]^{-1} [g_{n} x]^{x} \\
= ([g, x]^{(-2)^{n-1}})^{-1} ([g, x]^{(-2)^{n-1}})^{x} \\
= [g, x]^{-1} [g, x]^{(-1)(-2)^{n-1}} \\
= [g, x]^{-1} [g, x]^{(-1)(-2)^{n-1}} \\
= [g, x]^{(-2)^{n}}.
\]

This completes the proof.
The above phenomenon, something like part 2 of Proposition 3.3, may not be true for elements of other prime orders. For, if \( p \geq 4381 \) is a prime, then the free 2-generated Burnside group \( B = B(2, p) \) of exponent \( p \), by a deep result of Adjan and Novikov [4] is infinite and every abelian subgroup of \( B \) is finite. Now, if a non-trivial element \( a \in B \) were in \( L(B) \), then \( A = \langle a \rangle \) would be nilpotent by Theorem 3.14. If \( A \) is infinite, then as it is nilpotent, it contains an infinite abelian subgroup which is not possible and if \( A \) is finite, then \( a \) has finitely many conjugates in \( B \) and in particular the centralizer \( C_B(a) \) is infinite, which is again impossible as the centralizer of every non-trivial element in \( B \) is finite by [4].

The following result was announced by Bludov in [7].

**Theorem 3.4 (Bludov [7])** There exist groups in which a product of left Engel elements is not necessarily a left Engel element.

This refutes part 1 of Question 1.1 for an arbitrary group, i.e., the set of left Engel elements is not in general a subgroup. He constructed a non Engel group generated by left Engel elements. In particular, he shows a non left Engel element which is a product of two left Engel elements. His example is based on infinite 2-groups constructed by Grigorchuk [13]. Note that part 3 of Question 1.1 is still open, i.e., we do not know whether the set of bounded left Engel elements is a subgroup or not.

To end this section we prove the following result.

**Proposition 3.5** At least one of the following happens.

1. The free 2-generated Burnside group of exponent \( 2^{48} \) is a \( k \)-Engel group for some integer \( k \).
2. There exists a group \( G \) of exponent \( 2^{48} \) generated by four involutions which is not an Engel group.

**Proof** Let \( n = 2^{48} \) and \( B(X, n) \) be the free Burnside group on the set \( X = \{ x_i \mid i \in \mathbb{N} \} \) of the Burnside variety of exponent \( n \) defined by the law \( x^n = 1 \). Lemma 6 of [25] states that the subgroup \( \langle x^{n/2}_{2k-1} x^{n/2}_{2k} \mid k = 1, 2, \ldots \rangle \) of \( B(X, n) \) is isomorphic to \( B(X, n) \) under the map \( x^{n/2}_{2k-1} x^{n/2}_{2k} \to x_k, k = 1, 2, \ldots \). Therefore the subgroup \( \mathcal{G} := \langle x_1^{n/2}, x_2^{n/2}, x_3^{n/2}, x_4^{n/2} \rangle \) is generated by four elements of order 2, contains the subgroup \( \mathcal{H} = \langle x_1^{n/2}, x_2^{n/2}, x_3^{n/2}, x_4^{n/2} \rangle \) isomorphic to the free 2-generator Burnside group \( B(2, n) \) of exponent \( n \). It follows from Proposition 3.3 that the group \( \mathcal{G} \) can be generated by four left 49-Engel elements of \( \mathcal{G} \). Thus

\[
\mathcal{G} = \langle L_{49}(\mathcal{G}) \rangle = \langle L(\mathcal{G}) \rangle = \langle \mathcal{L}(\mathcal{G}) \rangle.
\]

Suppose, if possible, \( \mathcal{G} \) is an Engel group. Then \( \mathcal{H} \) is also an Engel group. Let \( Z \) and \( Y \) be two free generators of \( \mathcal{H} \). Thus \( [Z, k Y] = 1 \) for some integer \( k \geq 1 \). Since \( \mathcal{H} \) is the free 2-generator Burnside group of exponent \( n \), we have that every group of exponent \( n \) is a \( k \)-Engel group. Therefore, \( \mathcal{G} \) is an infinite finitely generated \( k \)-Engel group of exponent \( n \), as \( \mathcal{H} \) is infinite by a celebrated result of Ivanov [24]. This completes the proof.

We believe that the group \( \mathcal{H} \) cannot be an Engel group, but we are unable to prove it.
3.1.1 Left Engel Elements in Generalized Soluble and Linear Groups

In this subsection we collect main results on Engel structure left Engel elements in generalized soluble and linear groups.

The papers [14] and [15] by Gruenberg are essential to anyone who wants to know the Engel structure of soluble groups. In particular, all four Engel subsets are subgroups in any soluble group.

**Theorem 3.6 (Gruenberg [14, Proposition 3, Theorem 4])** Let $G$ be a soluble group. Then $L(G) = H_\text{P}(G)$ and $\mathcal{L}(G) = B(G)$.

A group is called radical if it has an ascending series with locally nilpotent factors. Define the upper Hirsch–Plotkin series of a group to be the ascending series $1 = R_0 \leq R_1 \leq \cdots$ in which $R_{\alpha+1}/R_\alpha = H_\text{P}(G/R_\alpha)$ and $R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$ for limit ordinals $\lambda$. It can be proved that radical groups are precisely those groups which coincide with a term of their upper Hirsch–Plotkin series.

**Theorem 3.7 (Plotkin [41, Theorem 9])** Let $G$ be a radical group. Then $L(G) = H_\text{P}(G)$.

**Question 3.8 (Robinson [43, p. 63 of Part II])** Let $G$ be a radical group. Is it true that $L(G) = B(G)$?

**Theorem 3.9 (Gruenberg [16])** Let $R$ be a commutative Noetherian ring with identity and $G$ be a group of $R$-automorphisms of a finitely generated $R$-module. Then $L(G) = H_\text{P}(G)$ and $\mathcal{L}(G) = B(G)$.

To state some results on certain soluble groups we need the following definitions. Let $\mathcal{A}_0$ be the class of abelian groups of finite torsion-free rank and finite $p$-rank for every prime $p$; $\mathcal{A}_1$ be the class of abelian groups $A$ of finite Prüfer rank such that $A$ contains only a finite number of elements of prime order; $\mathcal{A}_2$ be the class of abelian groups which have a series of finite length each of whose factors satisfies either the maximal or the minimal condition for subgroups; and let $\mathcal{S}_i$ be the class of all poly $\mathcal{A}_i$-groups. The class of all $\mathcal{S}_0$-groups in which the product of all periodic normal subgroups is finite will be denoted by $\mathcal{S}_n$.

**Theorem 3.10 (Gruenberg [16, Theorem 4] and [15, Theorem 1.3, Proposition 1.1, Lemma 2.2, Proposition 6.1])** If $G$ is an $\mathcal{S}_0$-group, then

1. $H_\text{P}(G) = \text{Gr}(G) = L(G)$ is hypercentral;
2. $\bar{\zeta}(G) = \bar{\varphi}(G) = R(G) = \zeta_n(G)$, where $\alpha \leq n\omega$ for some positive integer $n$;
3. $\text{Fitt}(G)$ need not be nilpotent.

If $G$ is an $\mathcal{S}_1$-group, then

1. $\text{Fitt}(G) = B(G) = L(G)$ is nilpotent;
2. $\zeta_n(G) = \bar{\pi}(G) = R(G)$;
3. $H_\text{P}(G)$ need not be nilpotent and $\bar{\zeta}(G)$ may not equal $\zeta_n(G)$ for some positive integer $n$, even if $G$ satisfies the minimal condition and therefore is an $\mathcal{S}_2$-group.
If $G$ is an $S_t$-group, then $HP(G) = Fitt(G)$ and $\zeta(G) = \zeta_k(G)$ for some positive integer $k$.

**Theorem 3.11 (Wehrfritz [51, Theorem E2])** Let $G$ be a group of automorphisms of the finitely generated $S_0$-group $A$. Then

(i) $HP(G) = Gr(G) = L(G)$ is hypercentral;

(ii) $\overline{L}(G) = \varrho(G) = R(G) = \zeta_\alpha(G)$, where $\alpha \leq n\omega$ for some positive integer $n$;

(iii) $HP(G) = B(G) = \overline{L}(G)$ is nilpotent;

(iv) $\zeta_\omega(G) = \overline{\varrho}(G) = \overline{R}(G)$.

If in addition $A$ is an $S_t$-group, then

(v) $HP(G) = Fitt(G)$ and $\zeta(G) = \zeta_k(G)$ for some positive integer $k$.

**Theorem 3.12 (Gruenberg [16])** Let $R$ be a commutative Noetherian ring with identity and $G$ be a group of $R$-automorphisms of a finitely generated $R$-module. Then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$.

Let us finish this section with some results on the structure of generalized linear groups given by Wehrfritz. Let $R$ denote a commutative ring with identity and $M$ an $R$-module. Let $G$ be a group of finitary automorphisms of $M$ over $R$; that is,

$$G \leq FAut_R M = \{g \in Aut_R M : M(g - 1) \text{ is } R\text{-Noetherian}\} \leq Aut_R M.$$

**Theorem 3.13 (Wehrfritz [53, 4.4])** Let $G$ be a group of finitary automorphisms of a module over a commutative ring with identity. Then $L(G) = HP(G) = Gr(G)$ and $\overline{L}(G) = B(G)$.

Wehrfritz has also studied the Engel structure of finitary skew linear groups [52].

### 3.1.2 Left Engel Elements in Groups Satisfying Certain Min or Max Conditions

The famous structure result for left Engel elements is due to Baer [6, p.257], where he proved that a left Engel element defined by right-normed commutators of a group satisfying maximal condition on subgroups belongs to the Hirsch–Plotkin radical. Therefore in groups satisfying maximal conditions, the set of left Engel elements defined by right-normed commutators is a subgroup and so it coincides with the one defined by left-normed commutators. Hence, Baer’s result is also valid for left Engel element defined by left-normed commutators.

**Theorem 3.14 (Plotkin [42])** Let $G$ be a group which satisfies the maximal condition on its abelian subgroups. Then $L(G) = \overline{L}(G) = HP(G)$ which is nilpotent.

Theorem 3.14 follows from the following key result due to Plotkin [42].
Theorem 3.15 (Plotkin [42, Lemma 2]) Let $G$ be an arbitrary group and $g \in L(G)$. Then there exists a sequence of subgroups

$$H_1 \leq H_2 \leq \cdots \leq H_n \leq \cdots$$

in $G$ satisfying the following conditions:
1. $H_i$ is nilpotent for all integers $i \geq 1$,
2. $H_1 = \langle g \rangle$ and for each $i \geq 2$, $H_i = \langle H_{i-1}, g^{h_i} \rangle$ for some $h_i \in G$,
3. $H_i$ is normal in $H_{i+1}$ for all $i \geq 1$.
4. there is an integer $n \geq 1$ such that $H_{n+1} = H_n$ if and only if $H_n$ is a normal subgroup of $G$.

Theorem 3.15 follows from the following important result of [42, Lemma 1]. Note that if a group satisfies maximal condition on its abelian subgroups, then by [43, Theorem 3.31] it also satisfies maximal condition on its nilpotent subgroups.

Theorem 3.16 (Plotkin [42, Lemma 1]) Let $H$ be a nilpotent subgroup of any group $G$ such that $H = \langle H \cap L(G) \rangle$. If $H$ is not normal in $G$, then there is an element $x \in H \cap L(G)$ which is conjugate to some element of $H \cap L(G)$.

To taste a little of the proof of Theorem 3.16, let us treat the case in which $H$ is finite cyclic generated by $g \in L(G)$. Since $H$ is not normal, there is an element $x \in G$ such that $g^x \notin H$. Thus $[x, g] \notin H$ and since $g \in L(G)$, there exists an integer $n \geq 2$ such that $[x_n g, x_k g] = 1$. It follows that there is a positive integer $k$ such that $[x_k g] \notin H$ but $[x_{k+1} g] \in H$. Since $[x_{k+1} g] = g^{-[x_k g]} g \in H$ and $g \in H$, we have that $g^{[x_k g]} \in H$. This implies that $H^{[x_k g]} \leq H$ and since $H$ is finite, $[x_k g] \in N_G(H)$.

Corollary 3.17 Let $G$ be a locally finite group. Then $L(G) = HP(G)$.

Proof Let $x \in L(G)$ and $g_1, \ldots, g_n \in G$. Then $H = \langle x^{g_1}, \ldots, x^{g_n} \rangle$ is a finite group generated by left Engel elements and so by Theorem 3.14, $H$ is nilpotent. This implies that $\langle x \rangle^G$ is locally nilpotent and so $x \in HP(G)$, as required.

A group is said to satisfy Max locally whenever every finitely generated subgroup satisfies the maximal condition on its subgroups.

Theorem 3.18 (Plotkin [42]) Let $G$ be a group having an ascending series whose factors satisfy Max locally. Then $L(G) = HP(G)$.

Theorem 3.19 (Held [21]) Let $G$ be a group satisfying minimal condition on its abelian subgroups. Then $L(G) = Fitt(G)$.

As far as we know the following result is still unpublished.

Theorem 3.20 (Martin, [43, p. 56 of Part II]) Let $G$ be a group satisfying minimal condition on its abelian subgroups. Then $L(G) = HP(G)$.
A group $G$ is called an $M_c$-group or said to satisfy $M_c$ (the minimal condition on centralizers) whenever for the centralizer $C_G(X)$ of any set of elements $X$ of $G$, there is a finite subset $X_0$ of $X$ such that $C_G(X) = C_G(X_0)$.

**Theorem 3.21 (Wagner [50, Corollary 2.5])** Let $G$ be an $M_c$-group. Then $\mathcal{L}(G) = \text{Fitt}(G)$.

**Theorem 3.22** Let $G$ be an $M_c$-group. Then every left Engel element of prime power order of $G$ lies in the Hirsch–Plotkin radical of $G$.

**Proof** Let $x \in \mathcal{L}(G)$ be a $p$-element for some prime $p$. Then the set of all conjugates of $x$ in $G$ is a $G$-invariant subset of $p$-elements in which every pair of elements satisfies some Engel identity (in the sense of [50, Definition 1.2]). Now [50, Corollary 2.2] implies that $\langle x \rangle^G$ is a locally finite $p$-group so that $x \in HP(G)$. This completes the proof.

**Question 3.23** Let $G$ be an $M_c$-group. Is it true that $\mathcal{L}(G) = HP(G)$?

A group $G$ is said to have finite (Prüfer) rank if there is an integer $r > 0$ such that every finitely generated subgroup of $G$ can be generated by $r$ elements. A group $G$ is said to have finite abelian subgroup rank if every abelian subgroup of $G$ has finite rank.

**Question 3.24** Is $\mathcal{L}(G)$ a subgroup for groups $G$ with finite rank? Is $\mathcal{L}(G)$ a subgroup for groups $G$ with finite abelian subgroup rank?

### 3.1.3 Left $k$-Engel Elements

In this subsection, we deal with left $k$-Engel elements, specially for small values of $k$.

Left 1-Engel elements are precisely elements of the center. Left 2-Engel elements can easily be characterized:

**Proposition 3.25** 1. For any group $G$, $L_2(G) = \{ x \in G \mid \langle x \rangle^G \text{ is abelian} \}$. In particular, $L_2(G) \subseteq \text{Fitt}(G)$

2. There is a group $K$ in which $L_2(K)$ is not a subgroup.

**Proof** 1. The proof follows from the fact that for any elements $a, b, x \in G$, we have:

$$[ab^{-1}x, x] = 1 \iff [x^{-ab^{-1}}, x] = 1 \iff [x^{ab^{-1}}, x] = 1 \iff [x^{a^2}, x^b] = 1.$$  

2. Take $K$ to be the standard wreath product of a group of order 2 and with an elementary abelian group of order 4. The group $K$ is generated by left 2-Engel elements but $K \neq L_2(K)$. This completes the proof of part 2.
Proposition 3.26 Let $A$ be any group of exponent $2^k$ for some integer $k \geq 1$ and $\langle x \rangle$ and $\langle y \rangle$ be cyclic groups of order 2. Let $G$ be the standard wreath product $A \wr (\langle x \rangle \times \langle y \rangle)$. Then

1. $x, y, xy \in L_{k+1}(G) \setminus L_k(G)$.
2. $ax \notin L_{k+1}(G)$ for all $1 \neq a \in A$.

In particular, for any integer $n \geq 2$, there exists a group $G$ containing two elements $a, b \in L_n(G)$ such that $ab \notin L_n(G)$.

Let $x$ be a bounded left (right, resp.) Engel element of a group $G$. The left (right, resp.) Engel length of $x$ is defined to be the least non-negative integer $n$ such that $x \in L_n(G)$ ($x \in R_n(G)$, resp.) and it is denoted by $\ell^l_G(x)$ ($\ell^r_G(x)$, resp.). Roman’kov [30, Question 11.88] asked whether for any group $G$ there exists a linear (polynomial) function $\phi(x, y)$ such that $\ell^l_G(uy) \leq \phi(\ell^l_G(u), \ell^l_G(v))$ for elements $u, v \in \mathcal{T}(G)$. Dolbak [9] answered negatively the question of Roman’kov [30, Question 11.88]. We propose the following problem.

Problem 3.27 Let $G$ be an arbitrary group. Find all pairs $(n, m)$ of positive integers such that, $xy \in L_n(G)$ whenever $x \in L_n(G)$ and $y \in L_m(G)$.

We now shortly show that for every integer $m > 0$, all pairs $(2, m)$ are of the solutions of part 1 of Problem 3.27.

Proposition 3.28 Let $G$ be any group and $a \in L_2(G)$ and $b \in L_n(G)$ for some $n \geq 1$. Then both $ab$ and $ba$ are in $L_2(G)$.

Proof Let $g \in G$ and $X = \langle a \rangle^G$. Then

$$[g, ab]X = [gX, abX] = [g, b]X = X,$$

where the last equality holds as $b \in L_n(G)$. Therefore $[g, ab] \in X$. So we have

$$[g, 2n, ab] = [[[g, ab], b]_{2n}, ab]$$

$$= [[[g, ab], b], \, Since \, [g, ab], \, a \in X \, and \, X \, is \, abelian \, normal \, in \, G$$

$$= 1 \, Since \, b \in L_n(G).$$

This proves that $ab \in L_{2n}(G)$. Since $L_{2n}(G)$ is closed under conjugation, $(ab)^n = ba$ is also in $L_{2n}(G)$. This completes the proof.

Let us ask the following question that we suspect it to be true.

Question 3.29 Is it true that the product of every two left 3-Engel element is a left Engel element?

The following question is arisen by part 1 of Proposition 3.25. Is it true that every bounded left Engel element is in the Hirsch–Plotkin radical? In general for an arbitrary group $K$ it is not necessary that $L_n(K) \subseteq \text{HP}(K)$. Suppose, for a contradiction, that $L_n(K) \subseteq \text{HP}(K)$ for all $n$ and all groups $K$. By a deep result of Ivanov [24], there is a finitely generated infinite group $M$ of exponent $2^k$ for some
positive integer $k$. Suppose that $k$ is the least integer with this property, so every finitely generated group of exponent dividing $2^{k-1}$ is finite. By Proposition 3.3 every element of order 2 in $M$ belongs to $L_{k+1}(M)$. So by hypothesis, $M = \text{HP}(M)$ is of exponent dividing $2^{k-1}$ and so it is finite. Since $M$ is finitely generated, $\text{HP}(M)$ so is. But this yields that $\text{HP}(M)$ is a periodic finitely generated nilpotent group and so it is finite. It follows that $M$ is finite, a contradiction. This argument can be found in [1].

Hence the following question naturally arises.

**Question 3.30** What is the least positive integer $n$ for which there is a group $G$ with $L_n(G) \not\subseteq \text{HP}(G)$?

If one uses Lysenkov’s result [32] instead of Ivanov’s one [24] in the above argument, we find that the requested integer $n$ in Question 3.30 is less than or equal to 13. To investigate Question 3.30 one should first study the case $n = 3$ which was already started in [1].

**Proposition 3.31** (Abdollahi [1, Corollary 2.2]) For an arbitrary group $G$,

$$L_3(G) = \{ x \in G \mid \langle x, x^y \rangle \text{ is nilpotent of class at most 2 for all } y \in G \}.$$ 

In particular, every power of a left 3-Engel element is also a left 3-Engel element.

**Theorem 3.32** (Abdollahi [1, Theorem 1.1]) Let $p$ be any prime number and $G$ be a group. If $x \in L_3(G)$ and $x^{p^n} = 1$ for some integer $n > 1$, then $\langle x^{p^n} \rangle^G$ is soluble of derived length at most $n - 1$ and $x^p \in B(G)$. In particular, $\langle x^{p^n} \rangle^G$ is locally nilpotent.

Theorem 3.32 reduces the verification of the question whether any left 3-Engel element of prime power order lies in the Hirsch–Plotkin radical to the following.

**Question 3.33** Let $G$ be a group and $x \in L_3(G)$ of prime order $p$. Is it true that $x \in \text{HP}(G)$?

The positive answer of Question 3.33 for the case $p = 2$ gives a new proof for the local finiteness of groups of exponent 4.

Two left 3-Engel elements generate a nilpotent group of class at most 4.

**Theorem 3.34** (Abdollahi [1, Theorem 1.2]) Let $G$ be any group and $a, b \in L_3(G)$. Then $\langle a, b \rangle$ is nilpotent of class at most 4.

NQ package [37] can show 4 is the best bound in Theorem 3.34.

**Question 3.35** (Abdollahi [1, Question]) Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every nilpotent group generated by $d$ left 3-Engel elements is nilpotent of class at most $f(d)$?
In particular, whether the number \( n \) of Question 3.30 is greater than 4 or not is of special interest. Indeed, if it were greater than 4, then every group of exponent 8 would be locally finite. Here are some thought on the left 4-Engel elements of a group.

**Theorem 3.36 (Abdollahi & Khosravi [3, Theorem 1.5])** Let \( G \) be an arbitrary group and both \( a \) and \( a^{-1} \) belong to \( L_4(G) \). Then \( \langle a, a^b \rangle \) is nilpotent of class at most 4 for all \( b \in G \).

**Theorem 3.37 (Abdollahi & Khosravi [3, Theorem 1.6])** Let \( G \) be a group.

1. If \( p = 2 \) then \( a^4 \in B(G) \).
2. If \( p \) is an odd prime, then \( a^p \in B(G) \).

### 3.2 Right Engel Elements

In this section we discuss on right Engel elements of groups.

**Proposition 3.38** Let \( G \) be any group and \( g \in \zeta_\omega(G) \). Then there is an integer \( n > 0 \) such that \( \langle g, x \rangle \) is nilpotent of class at most \( n \) for all \( x \in G \). In particular, \( \zeta_\omega(G) \subseteq \overline{R}(G) \).

**Proof** Since \( g \in \zeta_\omega(G) \), there exists an integer \( n > 0 \) such that \( g \in \zeta_n(G) \). Now consider the factor group \( \langle g, x \rangle \zeta_n(G)/\zeta_n(G) \). As \( g \in \zeta_n(G) \), \( \langle g, x \rangle / (g, x) \cap \zeta_n(G) \) is cyclic. Since \( \langle g, x \rangle \cap \zeta_n(G) \) is contained in \( \zeta_n(\langle g, x \rangle) \), we have \( \langle g, x \rangle / \zeta_n(\langle g, x \rangle) \) is cyclic. It follows that \( \langle g, x \rangle = \zeta_n(\langle g, x \rangle) \) and so \( \langle g, x \rangle \) is nilpotent of class at most \( n \). This easily implies that \( [g, n] = 1 \) and so \( g \in \overline{R}(G) \).

**Proposition 3.39 (Gruenberg [14, Theorem 3])** Let \( G \) be any group. Then \( \varrho(G) \subseteq R(G) \) and \( \overline{\varrho}(G) \subseteq \overline{R}(G) \).

**Proof** Let \( a \in \varrho(G) \) and \( x \in G \). Then \( \langle x \rangle \) is ascendant in \( \langle x \rangle \langle a \rangle \). Therefore \( [a, n] \leq 1 \) for some integer \( n \); for otherwise, by examining those terms of ascending series between \( \langle x \rangle \) and \( \langle x \rangle \langle a \rangle \) that contain \( [a, n] \), we should be able to find a set of ordinals without a first element. Hence \( \varrho(G) \subseteq R(G) \). It is equally easy to see that \( \overline{\varrho}(G) \subseteq \overline{R}(G) \).

### 3.2.1 Right Engel Elements in Generalized Soluble and Linear Groups

**Theorem 3.40 (Gruenberg [14, Theorem 4])** Let \( G \) be a soluble group. Then \( R(G) = \varrho(G) \) and \( \overline{R}(G) = \overline{\varrho}(G) \).

The standard wreath product of a cyclic group of prime order \( p \) by an elementary abelian \( p \)-group of infinite rank is an example of a soluble group with trivial hypercenter, but in which all elements are right \( (p + 1) \)-Engel. Replacing the elementary abelian \( p \)-group in the wreath product by a quasicyclic \( p \)-group gives a soluble group \( G \) with \( \overline{\zeta}(G) = \overline{R}(G) = 1 \) but satisfying \( G = R(G) \). Gruenberg [14,
Theorem 1.7] showed that $\overline{R}(G) = \zeta_\omega(G)$ when $G$ is a finitely generated soluble group. Brookes [8] proved that all four subsets are the same in a finitely generated soluble group.

**Theorem 3.41 (Brookes [8, Theorem A])** Let $G$ be a finitely generated soluble group. Then

$$R(G) = \zeta(G) = \overline{R}(G) = \zeta_\omega(G).$$

The proof of Theorem 3.41 follows from a result on constrained modules over integer group rings: For a group $G$ and a commutative Noetherian ring $R$, an $RG$-module $M$ is said to be constrained if for all $m \in M$ and $g \in G$ the $R$-module $mR(g)$ is finitely generated as an $R$-module. Let us explain how such modules are appeared in the study of right Engel elements. Let $G$ be any group, $H$ a normal subgroup of $G$ and $K$ a normal subgroup of $H$ such that $M = H/K$ is abelian and suppose further that $H = (H \cap R(G))$. Then $G$ acts by conjugation on $M$ as a group and so $M$ can be considered as a $\mathbb{Z}G$-module. Now we prove that $M$ is a constrained $\mathbb{Z}G$-module. We need the following technical and useful result to show our claim as well as in the sequel.

**Lemma 3.42** Let $x, y$ be elements of a group $G$.

1. For each integer $k \geq 0$, there exist elements $g_k(x, y), f_k(x, y), h_k(x, y) \in G$ such that

$$[x, y^k] = g_k(x, y)x^{-(k-1)} h_k(x, y) = f_k(x, y)x^{y^k},$$

and

$$g_k(x, y) \in \langle x^{y}, \ldots, x^{y^{k-1}} \rangle, \quad h_k(x, y) \in \langle x^{y^k}, \ldots, x^{y^{k-1}}, x^{y^k} \rangle$$

and $f_k(x, y) \in \langle x, x^y, \ldots, x^{y^{k-1}} \rangle$.

2. If $[x, y] = 1$, then

$$\langle x \rangle^{(y)} = \langle x, [x, y], \ldots, [x, n-1, y] \rangle = \langle x, x^{y}, \ldots, x^{y^{n-1}} \rangle.$$  

**Proof** 1. Using $[x, k+1, y] = [x, k, y]^{-1} [k, y]^y$, the proof follows from an easy induction on $k$.

2. Let $H = \langle x, [x, y], \ldots, [x, n-1, y] \rangle$ and $K = \langle x, x^{y}, \ldots, x^{y^{n-1}} \rangle$. Since $x$ belongs to both $H$ and $K$ and they are contained in $\langle x \rangle^{(y)}$, it is enough to show that both of $H$ and $K$ are normal subgroups of $\langle x, y \rangle$. To prove the latter, it is sufficient to show that $[x, k, y]^{y^k}, [x, k, y]^{y^{k-1}} \in H$ for all $k \in \{0, 1, \ldots, n-1\}$ and $x^{y^k}, x^{y^{k-1}} \in K$.

We first show the former. Since $[x, k, y]^{y^k} = [x, k, y][x, k+1, y]$ and $[x, n, y] = 1$, $[x, k, y]^y \in H$ for all $k \in \{0, \ldots, n-1\}$. Now $[x, n-1, y]^{y^k} = [x, n-1, y]^{y^{n-1}} \in H$ and assume by a backward induction on $K$ that $[x, k, y]^{y^{n-1}} \in H$ for $k < n-1$. Then, as $[x, k-1, y]^{y^{n-1}} = [x, k-1, y][x, k, y]^{-y^{-1}}$, we have $[x, k-1, y]^{y^{-1}} \in H$. This shows $H$ is also invariant under conjugation of $y^{-1}$.

Now we prove $x^{y^k}, x^{y^{k-1}} \in K$. From part (1) and $[x, n, y] = 1$, it follows that $x^{y^n} = f_n(x, y)^{-1} \in K$ and $x = (g_n(x, y)^{-1} h_n(x, y)^{-1})^{-y^{n-1}} \in \langle x^y, \ldots, x^{y^n} \rangle$. By conjugation of $y^{-1}$, it now follows from the latter that $x^{y^{-1}} \in K$. This completes the proof. \[\square\]
This lemma is very useful to study Engel groups. We do not know where is its origin and who first noted, however part 2 of Lemma 3.42 has already appeared as Exercise 12.3.6 of the first edition of [44] published in 1982 and Rhemtulla and Kim [29] groups $G$ having the property that $\langle x^{(y)} \rangle$ is finitely generated for all $x, y \in G$ called restrained groups and if there is a bound on the number of generators of such subgroups, they called $G$ strongly restrained.

Now we can prove our claim.

**Proposition 3.43** Let $G$ be any group, $H$ a normal subgroup of $G$ and $K$ a normal subgroup of $H$ such that $M = H/K$ is abelian and suppose further that $H = \langle H \cap R(G) \rangle$. Then $M$ is a restrained $\mathbb{Z}G$-module.

**Proof** We have to prove that

$$S = \frac{\langle x_1^{k_1} \cdots x_n^{k_n} \rangle K}{K}$$

is a finitely generated abelian group for any $x_1, \ldots, x_n \in R(G) \cap H$, $k_1, \ldots, k_n \in \mathbb{Z}$ and any $g \in G$. Clearly $S$ is a subgroup of

$$L = \frac{\langle x_1, \ldots, x_n \rangle^{(g)} K}{K}.$$

Now since $x_i \in R(G)$, part 2 of Lemma 3.42 implies that $\langle x_i^{(g)} \rangle$ is finitely generated for each $i$ and so $L$ is a finitely generated abelian group. Thus $S$ is also finitely generated. This completes the proof. \qed

**Theorem 3.44 (Robinson [43, Theorem 7.34])** Let $G$ be a radical group. Then $R(G)$ is a locally nilpotent subgroup of $G$. Furthermore, $R(G) = \varrho(G)$ if and only if $R(G)$ is a Gruenberg group.

A group $G$ is called an $SN^*$-group, if $G$ admits an ascending series whose factors are abelian.

**Corollary 3.45 (Robinson [43, Corollary 1, p. 60 of Part II])** If $G$ is an $SN^*$-group, then $R(G) = \varrho(G)$.

**Corollary 3.46 (Robinson [43, Corollary 2, p. 60 of Part II])** In an arbitrary group the right Engel elements that lie in the final term of the upper Hirsch–Plotkin series from a subgroup.

**Question 3.47 (Robinson [43, p. 63 of Part II])** Let $G$ be an $SN^*$-group. Is it true that $\overline{R}(G) = \overline{\varrho}(G)$?

**Theorem 3.48 (Gruenberg [16])** Let $R$ be a commutative Noetherian ring with identity and $G$ be a group of $R$-automorphisms of a finitely generated $R$-module. Then $R(G) = \zeta(G)$ and $\overline{R}(G) = \zeta_\omega(G)$. 
Theorem 3.49 (Wehrfritz [53, 4.4]) Let $G$ be a group of finitary automorphisms of a module over a commutative ring with identity. Then $R(G) = \varrho(G)$ and $\overline{R}(G) = \overline{\varrho}(G)$.

Wehrfritz [52] has also studied the Engel structure of certain linear groups over skew fields.

3.2.2 Right Engel Elements in Groups Satisfying Certain Min or Max Conditions

Theorem 3.50 (Peng [39]) Let $G$ be a group which satisfies the maximal condition on its abelian subgroups. Then $R(G) = \overline{R}(G) = \zeta(G)$.

Corollary 3.51 Let $G$ be a locally finite group. Then $R(G)$ is a subgroup of $\text{HP}(G)$.

Proof Let $a, b \in R(G)$ and $x \in G$. Then $H = \langle a, b, x \rangle$ is a finite group and so by Theorem 3.50, $ab^{-1} \in R(H)$. Hence $[ab^{-1}, x] = 1$ for some integer $k \geq 0$. Now Corollary 3.17 and Theorem 2.2 complete the proof. \hfill \Box

Theorem 3.52 (Plotkin [42]) Let $G$ be a group having an ascending series whose factors satisfy Max locally. Then $R(G)$ is a subgroup of $G$.

Theorem 3.53 (Held [21]) Let $G$ be a group satisfying minimal condition on its abelian subgroups. Then $\overline{R}(G) = \zeta_\omega(G)$.

Theorem 3.54 (Martin & Pamphilon [34, Theorem (iii), (iv)]) Let $G$ be a group satisfying minimal condition on those subgroups which can be generated by their left Engel elements. Then $R(G) = \overline{\zeta}(G)$ and $\overline{R}(G) = \zeta_\omega(G)$.

As far as we know the following result is still unpublished.

Theorem 3.55 (Martin, [43, p. 56 of Part II]) Let $G$ be a group satisfying minimal condition on its abelian subgroups. Then $R(G) = \zeta(G)$.

Question 3.56 Let $G$ be an $\mathfrak{M}_\omega$-group.
1. Is it true that $R(G) = \varrho(G)$?
2. Is it true that $\overline{R}(G) = \overline{\varrho}(G)$?

Question 3.57 Is $R(G)$ a subgroup for groups $G$ of finite rank? Is $R(G)$ a subgroup for groups $G$ with finite abelian subgroup rank?
3.2.3 Right $k$-Engel elements

For any group $G$, $R_1(G) = Z(G)$.

**Theorem 3.58 (Levi & W. P. Kappe [31], [28])** Let $G$ be a group, $a \in R_2(G)$ and $x, y, z \in G$.

1. $a \in L_2(G)$ so that $R_2(G) \subseteq L_2(G)$ and $\langle a \rangle^G$ is an abelian group.
2. $\langle a \rangle^G \subseteq R_2(G)$.
3. $[a, x, y] = [a, y, x]^{-1}$.
4. $[a, [x, y]] = [a, x, y]^2$.
5. $[a, [x, y, z]] = [a, y, z]^2 = 1$ so that $a^2 \in \zeta_3(G)$.
6. $[a, [x, y, z]] = 1$.

W. P. Kappe proved explicitly in [28] that $R_2(G)$ is a characteristic subgroup for any group $G$.

**Theorem 3.59 (W. P. Kappe [28])** Let $G$ be a group. Then $R_2(G)$ is a characteristic subgroup of $G$.

**Proof** As $R_2(G)$ is invariant under automorphisms of $G$, it is enough to show that $R_2(G)$ is a subgroup. Let $a, b \in R_2(G)$ and $x \in G$. Then

$$[ab^{-1}, x] = [[a, x]^{b^{-1}}, [b, x]^{-b^{-1}}, x]$$

$$= [[a, x][b, x]^{-1}, x[x, b]]^{b^{-1}}$$

$$= 1$$

by parts 3 and 4 of Theorem 3.58. Hence $ab^{-1} \in R_2(G)$.

**Theorem 3.60 (Newell [35])** Let $G$ be any group and $x \in R_3(G)$. Then $\langle x \rangle^G$ is a nilpotent group of class at most 3.

An essential ingredient to proving Theorem 3.60 was to show $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_3(G)$ and $x \in G$.

The following asks of a similar property mentioned in Theorem 3.58 of right 2-Engel elements for right 3-Engel ones.

**Question 3.61** Let $G$ be an arbitrary group and $a \in R_3(G)$. Are there positive integers $n$ and $m$ such that $a^m \in \zeta_n(G)$?

**Theorem 3.62 (Macdonald [33])** There is a finite 2-group $G$ containing an element $a \in R_3(G)$ such that $a^{-1} \notin R_3(G)$ and $a^2 \notin R_3(G)$.

On the positive side, we have the following results.

**Theorem 3.63 (Heineken [20])** If $A$ is the subset of a group $G$ consisting of all elements $a$ such that both $a$ and $a^{-1}$ belong to $R_3(G)$, then $A$ is a subgroup if either $G$ has no element of order 2 or $A$ consists only of elements having finite odd order.
Theorem 3.64 (Abdollahi & Khosravi [2]) Let $G$ be a group such that $\gamma_5(G)$ has no element of order 2. Then $R_3(G)$ is a subgroup of $G$.

Proof It follows from detail information of the subgroup $\langle a, b, x \rangle$ where $a, b \in R_3(G)$ and $x \in G$.

L.-C. Kappe and Ratchford [27] have shown that $R_n(G)$ is a subgroup of $G$ whenever $G$ is metabelian or center-by-metabelian with certain extra properties.

Nickel [38] generalized Macdonald’s example (Theorem 3.62) to all right $n$-Engel elements for any $n \geq 3$ by proving that there is a nilpotent group of class $n + 2$ containing a right $n$-Engel element $a$ and an element $b$ such that $[a^{-1,n}, b] = [a^{2,n}, b]$ is non-trivial.

Using the group constructed by Newman and Nickel [36], it is shown in [2] that there is a group containing a right $n$-Engel element $x$ such that $x^k$ and $x^{-1}$ are not in $R_n(G)$ for all $k \geq 2$.

By Theorem 3.60 of Newell, we know that $R_3(G) \subseteq \text{Fitt}(G)$ for any group $G$ and on the other hand Gupta and Levin [17] have shown that the normal closure of an element in a 5-Engel group need not be nilpotent (see also [49, p. 342]).

Theorem 3.65 (Gupta & Levin [17]) For each prime $p \geq 3$, let $G$ be the free nilpotent of class 2 group of exponent $p$ and of countably infinite rank. Let $M_p$ be the the multiplicative group of $2 \times 2$ matrices over the group ring $\mathbb{Z}_pG$ of the form $\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}$, where $g \in G$ and $r \in \mathbb{Z}_pG$. Then the group $M_p$ has the following properties:

1. $M_p$ has exponent $p^2$ and $\gamma_3(M_p)$ has exponent $p$;
2. $M_p$ is abelian-by-(nilpotent of class 2);
3. $M_p$ is a $(p + 2)$-Engel group;
4. $M_p$ has an element whose normal closure in $M_p$ is not nilpotent.

This result of Gupta and Newman raises naturally the following question.

Question 3.66 Let $n$ be a positive integer. For which primes $p$, there exists a soluble $n$-Engel $p$-group $M(n, p)$ which is not a Fitting group?

By Gupta–Levin’s Theorem 3.65, for all $p \geq 3$ and for all $n \geq p + 2$, $M(n, p)$ exists. We observed that $M(6, 2)$ also exists. In fact, a similar construction of Gupta and Levin gives $M(6, 2)$.

Proposition 3.67 Let $G$ be the free nilpotent of class 2 group of exponent 4 and of countably infinite rank. Let $M$ be the the multiplicative group of $2 \times 2$ matrices over the group ring $\mathbb{Z}_2G$ of the form $\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}$, where $g \in G$ and $r \in \mathbb{Z}_2G$. Then the group $M$ has the following properties:

1. $M$ has exponent 8 and $\gamma_3(M)$ has exponent 2;
2. $M$ is abelian-by-(nilpotent of class 2);
3. $M$ is a 6-Engel group;
4. $M$ has an element whose normal closure in $M$ is not nilpotent.

**Proof** We first observed that the elements $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ of $M$ constitute an elementary abelian 2-subgroup $K$. Since $\exp(G) = 4$ and $\cl(G) = 2$, it follows that $\gamma_5(M) \leq K$ and $M^4 \leq K$. Thus we have proved parts 1 and 2. For the proof of part 3, we first note that if $A = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and $B = \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} g^{-1} & 0 \\ -rg^{-1} & 1 \end{pmatrix}$. Thus the commutator $[A, B] = \begin{pmatrix} 1 & 0 \\ s(g-1) & 1 \end{pmatrix}$, and by iteration $[A, A, B] = \begin{pmatrix} 1 & 0 \\ s(g-1)^4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since every element of $\gamma_5(M)$ is of the form $A$, it follows that $M$ is a 6-Engel group. Finally, for the proof of part 4, we first note that for $X_i = \begin{pmatrix} x_i & 0 \\ 1 & 1 \end{pmatrix}$, $i \geq 0$, the commutator $[X_i, X_j]$ is of the form $\begin{pmatrix} [x_i, x_j] & 0 \\ r & 1 \end{pmatrix}$. Thus if $Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $[Y, [X_0, X_1], \ldots, [X_0, X_m]] = \begin{pmatrix} 1 & 0 \\ u_m & 1 \end{pmatrix}$, where $u_m = ([x_0, x_1] - 1) \cdots ([x_0, x_m] - 1)$ and it is a non-zero element of $\mathbb{Z}_2 G$ for all $m \geq 1$. It now follows that the normed closure of $X_0$ in $M$ is not nilpotent. This completes the proof. 

**Question 3.68** Does there exist a soluble 5-Engel 2-group which is not a Fitting group?

It follows that $R_n(G) \not\subseteq \text{Fitt}(G)$ for $n \geq 5$. The following question naturally arises.

**Question 3.69** What are the least positive integers $n$, $m$ and $\ell$ such that
1. $R_n(G_1) \not\subseteq \text{Fitt}(G_1)$ for some group $G_1$?
2. $R_m(G_2) \not\subseteq B(G_2)$ for some group $G_2$?
3. $R_\ell(G_3) \not\subseteq \text{HP}(G_3)$ for some group $G_3$?

Therefore, to find integer $n$ in Question 3.69 we have to answer the following.

**Question 3.70** Let $G$ be an arbitrary group. Is it true that $R_4(G) \subseteq \text{Fitt}(G)$?

For right 4-Engel elements there are some results.

**Theorem 3.71 (Abdollahi & Khosravi [3, Theorem 1.3])** Let $G$ be any group. If $a \in G$ and both $b, b^{-1} \in R_4(G)$, then $\langle a, a^b \rangle$ is nilpotent of class at most 4.
Theorem 3.72 (Abdollahi & Khosravi [2]) Let $G$ be a $\{2, 3, 5\}'$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and any $x \in G$. Then $R_4(G)$ is a subgroup of $G$.

An important tool in the proof of Theorem 3.72 is the nilpotent quotient algorithm as implemented in the NQ package [37] of GAP [11]. Indeed we need to know the structure of the largest nilpotent quotient of a nilpotent $\{2, 3, 5\}'$-group generated by two right 4-Engel elements and an arbitrary element. It is a byproduct of the proof of Theorem 3.72 that

Corollary 3.73 Let $G$ be a $\{3, 5\}'$-group such that $\langle a, x \rangle$ is nilpotent for all $a \in R_4(G)$ and any $x \in G$. Then $R_4(G)$ is inverse closed.

Corollary 3.74 (Abdollahi & Khosravi [2]) Let $G$ be a $\{2, 3, 5\}'$-group such that $\langle a, b, x \rangle$ is nilpotent for all $a, b \in R_4(G)$ and for any $x \in G$. Then $R_4(G)$ is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of $G$ is nilpotent of class at most 7.

Proof By Theorem 3.72, $R_4(G)$ is a subgroup of $G$ and so it is a 4-Engel group. Now it follows from a result of Havas and Vaughan-Lee [18] that 4-Engel groups are locally nilpotent, $R_4(G)$ is locally nilpotent. By [47], we know that every locally nilpotent 4-Engel $\{2, 3, 5\}'$-group is nilpotent of class at most 7. Therefore $R_4(G)$ is nilpotent of class at most 7. Since $R_4(G)$ is a normal set, the second part follows easily. 

Question 3.75 Let

$$\mathcal{C}_4 = \{ \text{cl}(\langle x \rangle^G) \mid G \text{ is a group such that } x \in R_4(G) \text{ and } \langle x \rangle^G \text{ is nilpotent} \},$$

where cl$(X)$ denotes the nilpotent class of a nilpotent group $X$.

1. Is the set $\mathcal{C}_4$ bounded?
2. If the part 1 has positive answer, what is the maximum of $\mathcal{C}_4$? Is it 4?

Theorem 3.76 (Abdollahi & Khosravi [2]) In any $\{2, 3, 5\}'$-group, the normal closure of any right 4-Engel element is nilpotent if and only if every 3-generator subgroup in which two of the generators can be chosen to be right 4-Engel, is nilpotent.

Proof By Corollary 3.74, it is enough to show that a $\{2, 3, 5\}'$-group $H = \langle a, b, x \rangle$ is nilpotent whenever $a, b \in R_4(H)$, $x \in H$ and both $\langle a \rangle^H$ and $\langle b \rangle^H$ are nilpotent. Consider the subgroup $K = \langle a \rangle^H \langle b \rangle^H$ which is nilpotent by Fitting’s theorem. We have $K = \langle a, b \rangle^{(2)}$ and since $a$ and $b$ are both right Engel, we have (see e.g., [44, Exercise 12.3.6, p. 376] that both $\langle a \rangle^{(2)}$ and $\langle b \rangle^{(2)}$ are finitely generated. Thus $K$ is also finitely generated. Hence $H$ satisfies maximal condition on its subgroups. Now Theorem 3.50 completes the proof.

It may be interesting to know that in a nilpotent $\{2, 3, 5\}$-free group every two right 4-Engel element with an arbitrary element always generate a 4-Engel group.
Theorem 3.77 Let \( G \) be any group, \( a, b \in R_4(G) \) and \( x \in G \). If \( \langle a, b, x \rangle \) is a nilpotent \( \{2, 3, 5\} \)-free group, then it is a 4-Engel group of class at most 7.

Proof A proof is similar to one of [2, Theorem 3.4].

The following problem is the right analog of Problem 3.27.

Problem 3.78 Let \( G \) be an arbitrary group. Find all pairs \((n, m)\) of positive integers such that, \( xy \in R(G) \) whenever \( x \in R_n(G) \) and \( y \in R_m(G) \).

Since the set of right 2-Engel elements is a subgroup, \((2, 2)\) belongs to the solutions of Problem 3.78. We show that \((2, 3)\) and \((3, 3)\) also belong to the solutions. In fact we prove more.

Proposition 3.79 Let \( G \) be an arbitrary group, \( a \in R_2(G) \) and \( b, c \in R_3(G) \). Then \( ab \in R_3(G) \) and \( bc \in R_4(G) \).

Proof By [35] \( K = \langle b, c, x \rangle \) is nilpotent for all \( x \in G \). In particular, \( H = \langle a, b, x \rangle \) is also nilpotent for all \( x \in G \). Now, thanks to the NQ package [37] of GAP [11], one can easily construct the freest nilpotent groups with the same defining relations as \( K \) and \( H \). Then the conclusion can be easily checked through two line commands in GAP [11].

By using the positive solution of restricted Burnside’s problem due to Zel’manov [54, 55], Shalev has proved that:

Theorem 3.80 (Shalev [46, Proposition D]) There is a function \( c : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that for any \( d \)-generated nilpotent group \( G \) and any normal subgroup \( H \) of \( G \) with \( H \subseteq R_n(G) \), we have \( H \subseteq \zeta_{c(n,r)}(G) \).

We finish by the following question.

Question 3.81 Are there functions \( c, e : \mathbb{N} \to \mathbb{N} \) such that for any nilpotent group \( G \) and any normal subgroup \( H \) of \( G \) with \( H \subseteq R_n(G) \), we have \( H^{c(n)} \subseteq \zeta_{c(n)}(G) \)?

Acknowledgments. This survey was completed during the author’s visit to University of Bath in 2009. The author is very grateful to Department of Mathematical Sciences of University of Bath and specially he wishes to thank Gunnar Traustason for their kind hospitality. The author gratefully acknowledges financial support of University of Isfahan for his sabbatical leave study. This work is also financially supported by the Center of Excellence for Mathematics, University of Isfahan.

References

[27] L. C. Kappe and P. M. Ratchford, On centralizer-like subgroups associated with the n-Engel word, Algebra Colloq. 6 (1999), 1–8.
[48] G. Traustason, Locally nilpotent 4-Engel groups are Fitting groups, *J. Algebra* 270 (2003), 7–27.