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On a problem of P. Hall for Engel words

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Abstract. Let θ be a word in n variables and let G be a group; the marginal and verbal subgroups of G determined by θ are denoted by $\theta(G)$ and $\theta^*(G)$, respectively. The following problems are generally attributed to P. Hall:
(I) If π is a set of primes and $|G : \theta^*(G)|$ is a finite π-group, is $\theta(G)$ also a finite π-group?
(II) If $\theta(G)$ is finite and G satisfies maximal condition on its subgroups, is $|G : \theta^*(G)|$ finite?
(III) If the set $\{\theta(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G\}$ is finite, does it follow that $\theta(G)$ is finite?
We investigate the case in which θ is the n-Engel word $e_n = [x,_n y]$ for $n \in \{2, 3, 4\}$.

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1. Introduction. A classic result of Schur [12, p. 26] says that a group G whose center $Z(G)$ is of finite index has finite derived subgroup G'. The converse is due to P. Hall, under some natural restrictions on G. Another well-known result, belonging to the same line of investigation, shows that G' is finite whenever the set $\{x^{-1}y^{-1}xy \mid x, y \in G\}$ is finite. It is possible to generalize these three statements in terms of an arbitrary word, replacing in a suitable way the roles of $Z(G)$ and G'.

For a word θ in n variables, the *verbal subgroup* of G, determined by θ, is

$$\theta(G) = \langle \theta(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in G \rangle,$$

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and the *marginal subgroup*, determined by θ, is

$$\theta^*(G) = \{a \in G \mid \theta(x_1, \ldots, x_{i-1}, ax_i, x_{i+1}, \ldots, x_n) = \theta(x_1, \ldots, x_n), \forall i, \ \forall x_i \in G\}.$$

We also denote by $\{\theta\}(G)$ the set $\{\theta(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in G\}$.

The following problems are due to Hall (see [11, Chapter 4]):

(I) If π is a set of primes and $|G : \theta^*(G)|$ is a finite π-group, is $\theta(G)$ also a finite π-group?

(II) If $\theta(G)$ is finite and G satisfies maximal condition on its subgroups, is $|G : \theta^*(G)|$ finite?

(III) If the set $\{\theta(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G\}$ is finite, does it follow that $\theta(G)$ is finite?

If $\theta(x_1, x_2) = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ is the *commutator word* in 2 variables, then $\theta(G)$ becomes G', $\theta^*(G)$ becomes $Z(G)$ and all problems (I), (II) and (III) have positive solution by [12, p. 26]. Initial contributions were given by Merzljakov [9] and Turner-Smith [16], who generalized Schur's result to the so-called *outer commutator words*, in the context of linear groups and of finitely generated abelian-by-nilpotent groups. More recently, there have been further results. (II) has been proved in [10] for $[x_1, x_2]$ under the weaker assumption that G is finitely generated; (III) has been answered positively in [3] for $[x_1, x_2]$ and in [2] for the commutator word $[x_1, x_2, \ldots, x_n]$ in n-variables.

The choice of θ plays a fundamental role in the strategies of the proofs, otherwise counterexamples may occur. For instance, Ivanov [7] showed that there exist words θ for which $\{\theta\}(G)$ is finite and $\theta(G)$ is infinite. However $\theta(G)$ turns out to be not residually finite so the situation described by Ivanov is very special. The situation seems very complicated in general.

Here we will concentrate on $e_n(x, y) = [x,_n y]$, the *left normed n-Engel commutator word*, where $n \geq 1$. Consequently,

$$E_n(G) = \langle e_n(x, y) \mid x, y \in G\rangle \tag{1.1}$$

is the verbal subgroup, determined by $e_n(x, y)$, and

$$E_n^*(G) = \{a \in G \mid e_n(x, y) = e_n(ax, y) = e_n(x, ay) \ \forall x, y \in G\} \tag{1.2}$$

is the marginal subgroup, determined by $e_n(x, y)$. These subgroups are always characteristic in G, as noted in [14, Theorems 1.1], and are dual in the sense of [14, Theorems 1.1 and 1.2]. Furthermore, for left normed n-Engel words we have evidences from [1,3,5,15] that (I), (II) and (III) may have a positive solution. The following sections are devoted to prove this for small values of n.

2. Main results.

We begin with (I) for $e_2(x, y)$.

Proposition 2.1. *Let G be a group. If $G/E_2^*(G)$ is finite, then $E_2(G)$ is finite-by-cyclic.*

Proof. We invoke [2, Theorem B]: If the set

$$\{\gamma_n\}(G) = \{[x_1, x_2, \ldots, x_n] \mid x_1, x_2 \ldots, x_n \in G\}$$

is covered by finitely many cyclic groups, then the n-th term $\gamma_n(G)$ of the lower central series of G is finite-by-cyclic.

Let $g_1, \dots, g_n$ be a system of transversals of G in $E_2^*(G)$. We have

$$\{E_2\}(G) \subseteq \{\gamma_3\}(G) \subseteq G \subseteq \bigcup_{\substack{i=1\\ g_i \in G}}^{n} g_i E_2^*(G) \subseteq \bigcup_{\substack{i=1\\ g_i \in G}}^{n} \langle g_i E_2^*(G) \rangle.$$

Noting that subgroups of finite-by-cyclic groups remain finite-by-cyclic, the fact that $\gamma_3(G)$ is finite-by-cyclic, together with $E_2(G) \subseteq \gamma_3(G)$, implies that $E_2(G)$ is finite-by-cyclic. □

We continue with (II), generalizing indirectly [10, Main Theorem].

Proposition 2.2. *Let G be a finitely generated group. If $E_2(G)$ is finite, then $G/Z_2(G)$ is finite. In particular, $G/E_2^*(G)$ is finite.*

Proof. Since $G/E_2(G)$ is a 2-Engel group, it follows from [11, Corollary 3, vol. II, p. 45] that $G/E_2(G)$ is nilpotent of class at most 3 and $\gamma_3(G/E_2(G))$ is of exponent dividing 3. Thus $\gamma_4(G) \leq E_2(G)$ and $(\gamma_3(G))^3 \leq E_2(G)$. Since $\gamma_4(G)$ is finite, G is a finite-by-nilpotent group. Therefore G satisfies maximal condition on subgroups, since G is finitely generated. It follows that $\gamma_3(G)E_2(G)/E_2(G)$ is a finitely generated abelian group of exponent dividing 3. Thus $\gamma_3(G)$ is also finite and a result of Hall [4] implies that $G/Z_2(G)$ is finite. Since $Z_2(G) \leq E_2^*(G)$, the second part follows. □

Theorem 2.3. *Let G be a group such that $\{E_2\}(G)$ is finite. Then $E_2(G)$ is finite. If we assume further that G is finitely generated, then $G/E_2^*(G)$ is also finite.*

Proof. Let F_4 be the free group of rank 4 on the free generators f_1, f_2, f_3, f_4. Then $F_4/E_2(F_4)$ is nilpotent of class 3 (see [11, Corollary 3, vol. II, p. 45]). It follows that there are words x_i, y_i and z_i for $i = 1, \dots, t$ on f_1, f_2, f_3, f_4 and $\epsilon_i \in \{1, -1\}$ such that

$$[f_1, f_2, f_3, f_4] = [x_1, y_1, y_1]^{\epsilon_1 z_1} \ \dots \ [x_t, y_t, y_t]^{\epsilon_t z_t}.$$

Thus we have that $\{\gamma_4\}(G) \subseteq \{E_2\}(G)^{\epsilon_1} \ \dots \ \{E_2\}(G)^{\epsilon_t}$. Hence $\{\gamma_4\}(G)$ is finite and so $\gamma_4(G)$ is finite by [11, Vol. I, Corollary, p. 120]. There exists a finitely generated subgroup H such that $E_2(G) = E_2(H)$. Since H is finitely generated and $\gamma_4(H)$ is finite, H is also finite-by-nilpotent. Now it follows from [13, Theorem 1.4.2] that $E_2(H) = E_2(G)$ is finite. This completes the proof of the first part.
To prove the second part, note that since G is finitely generated and $\gamma_4(G)$ is finite, it follows from Hall's Theorem [4] that $G/Z_3(G)$ is finite. Now Proposition 2.2 completes the proof. □

Theorem 2.4. *Let $k \in \{3, 4\}$. If G is a group such that $\{E_k\}(G)$ is finite, then $E_k(G)$ is finite.*

Proof. Let F_d be the free group of rank d on the free generators $f_1, \ldots, f_d$ and $\overline{F_d} = F_d/E_k(F_d)$. Then $\overline{F_d}$ is a k-Engel group. In case of 3-Engel groups there are some classic results in [6,8] which allow us to conclude that every d-generated 3-Engel group is nilpotent of class at most $2d$. Similarly for 4-Engel groups, [5,15] allow us to conclude that every d-generated 4-Engel group is nilpotent of class at most $4d$. Therefore $\gamma_{2d+1}(F_d) \leq E_3(F_d)$ and $\gamma_{4d+1}(F_d) \leq E_4(F_d)$. By a similar argument as in the proof of Theorem 2.3, we find that

$$\{\gamma_{2d+1}\}(G) \subseteq \{E_3\}(G)^{\epsilon_1} \ \ldots \ \{E_3\}(G)^{\epsilon_{t_d}},$$
$$\{\gamma_{4d+1}\}(G) \subseteq \{E_4\}(G)^{\delta_1} \ \ldots \ \{E_4\}(G)^{\delta_{s_d}}$$

for some $\epsilon_i, \delta_i \in \{1, -1\}$ and some integers t_d, s_d. It follows from [11, Vol. I, Corollary, p. 120] that G is a finitely generated finite-by-nilpotent group. Now [13, Theorem 1.4.2] completes the proof. $\square$

3. Some applications. In this section we give two consequences, related to the minimal number of generators $d(G)$ of a finitely generated group G. The first generalizes [10, Main Theorem].

Corollary 3.1. *In a finitely generated group G, if $E_2(G)$ is finite, then*

$$d(G/E_2^*(G)) \leq d(C_G(E_2(G))/E_2^*(G)) \ |E_2(G)|^{d(E_2(G))}.$$

Proof. Since $[E_2^*(G), E_2(G)] = 1$, $E_2^*(G) \leq C_G(E_2(G))$. Then, we consider the (abelian) quotient group $C_G(E_2(G))/E_2^*(G)$ and deduce by Proposition 2.2 that $d(C_G(E_2(G))/E_2^*(G))$ is a positive integer.

The monomorphism $G/C_G(E_2(G)) \hookrightarrow \mathrm{Aut}(E_2(G))$ implies

$$|G : C_G(E_2(G))| \leq |\mathrm{Aut}(E_2(G))|$$

and then $G/C_G(E_2(G))$ is finite. It turns out that

$$|\mathrm{Aut}(E_2(G))| \leq |E_2(G)|^{d(E_2(G))}$$

and that $|G : E_2^*(G)| = |G : C_G(E_2(G))||C_G(E_2(G)) : E_2^*(G)|$. We conclude

$$d(G/E_2^*(G)) \leq d(C_G(E_2(G))/E_2^*(G)) \ |E_2(G)|^{d(E_2(G))}.$$

$\square$

The second is a specialization in case of finite p-groups (p prime).

Corollary 3.2. *Let G be a finite p-group such that $G' = HE_2(G)$ and $H \leq G' \cap Z_2(G)$. Then*

$$|G : E_2^*(G)| \leq \exp(Z_3(G)/E_2^*(G))^{d(Z_3(G)/E_2^*(G))} \cdot |E_2(G)|^{d(G/Z_3(G))}.$$

Proof. If $E_2^*(G)$ is trivial, then $Z(G)$ is trivial and this cannot be true, since G is a p-group. Then there is no loss of generality in assuming that $E_2^*(G)$ is non-trivial. We have

$$|G/Z_3(G)| = \left|\frac{G/Z_2(G)}{Z(G/Z_2(G))}\right| \leq |[G/Z_2(G), G/Z_2(G)]|^{d(G/Z_3(G))}$$
$$= |G'Z_2(G)/Z_2(G)|^{d(G/Z_3(G))} = |G' : G' \cap Z_2(G)|^{d(G/Z_3(G))}.$$

Since $Z_2(G) \leq E_2^*(G) \leq Z_3(G)$ by [14, Theorem 2.3 (i) and Corollary 2.8], $Z_3(G)/E_2^*(G)$ is abelian because a section of $Z_3(G)/Z_2(G)$. Therefore

$$\begin{aligned} |G/E_2^*(G)| &= |G/Z_3(G)| \cdot |Z_3(G)/E_2^*(G)| \\ &\leq |G' : G' \cap Z_2(G)|^{d(G/Z_3(G))} \cdot \exp(Z_3(G)/E_2^*(G))^{d(Z_3(G)/E_2^*(G))} \\ &\leq \exp(Z_3(G)/E_2^*(G))^{d(Z_3(G)/E_2^*(G))} \cdot |E_2(G)|^{d(G/Z_3(G))}. \end{aligned}$$

The result follows. □

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