Groups all of whose undirected Cayley graphs are integral

Alireza Abdollahi, Mojtaba Jazaeri

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

ARTICLE INFO

Article history:
Received 18 July 2013
Accepted 21 November 2013
Available online 13 December 2013

ABSTRACT

Let $G$ be a finite group, $S \subseteq G \setminus \{1\}$ be a set such that if $a \in S$, then $a^{-1} \in S$, where 1 denotes the identity element of $G$. The undirected Cayley graph $\text{Cay}(G, S)$ of $G$ over the set $S$ is the graph whose vertex set is $G$ and two vertices $a$ and $b$ are adjacent whenever $ab^{-1} \in S$. The adjacency spectrum of a graph is the multiset of all eigenvalues of the adjacency matrix of the graph. A graph is called integral whenever all adjacency spectrum elements are integers. Following Klotz and Sander, we call a group $G$ Cayley integral whenever all undirected Cayley graphs over $G$ are integral. Finite abelian Cayley integral groups are classified by Klotz and Sander as finite abelian groups of exponent dividing 4 or 6. Klotz and Sander have proposed the determination of all non-abelian Cayley integral groups. In this paper we complete the classification of finite Cayley integral groups by proving that finite non-abelian Cayley integral groups are the symmetric group $S_3$ of degree 3, $C_3 \times C_4$ and $Q_8 \times C_n^2$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Let $G$ be a finite group and $S$ be a subset of $G \setminus \{1\}$ such that $S = S^{-1}$, where 1 is the identity element of $G$. The undirected Cayley graph $\text{Cay}(G, S)$ is the graph whose vertex set is $G$ and two vertices $a, b \in G$ are adjacent whenever $ab^{-1} \in S$. The adjacency spectrum of a graph is the multiset of all eigenvalues of the adjacency matrix of the graph. A graph is called integral whenever all adjacency spectrum elements are integers. The question of “which graphs are integral?” was first proposed by...
Harary and Schwenk [9]. Many research papers were written on integral graphs e.g. [3, 5, 14]. Cayley graphs which are integral were studied by many people (e.g. [1, 2, 4, 7, 10, 11]). Following [10] we call a group $G$ Cayley integral whenever all undirected Cayley graphs over $G$ are integral. Finite abelian Cayley integral groups are classified in [10]; these are finite abelian groups of exponent dividing 4 or 6. In [10, p. 12, Problem 3] Klotz and Sander have proposed the problem of determination of all non-abelian Cayley integral groups. The non-abelian Cayley integral groups of order at most 12 found in [10, p. 12, Problem 3] are the symmetric group $S_3$ of degree 3, the quaternion group $Q_8$ of order 8, and the semidirect product $C_3$ by $C_4$ which is a non-abelian group of order 12. In this paper we complete the classification of finite Cayley integral groups by proving the following.

**Theorem 1.1.** A finite non-abelian group is Cayley integral if and only if it is isomorphic to one of the following groups:

1. the symmetric group $S_3$ of degree 3.
2. $C_3 \times C_4 = \langle x, y \mid x^3 = y^4 = 1, xy = x^{-1} \rangle$.
3. $Q_8 \times C_n^m$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8.

So combining the above mentioned result of Klotz and Sander and **Theorem 1.1**, the classification of finite Cayley integral groups completes as follows.

**Theorem 2.1.** A finite group is Cayley integral if and only if it is isomorphic to one of the following groups:

1. an abelian group of exponent dividing 4 or 6.
2. the symmetric group $S_3$ of degree 3.
3. $C_3 \times C_4 = \langle x, y \mid x^3 = y^4 = 1, xy = x^{-1} \rangle$.
4. $Q_8 \times C_n^m$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8.

Throughout we denote by $C_n$ the cyclic group of order $n$, the dihedral group of order $2n$ is denoted by $D_{2n}$, the alternating group of degree 4 is denoted by $A_4$; we denote by $S_3$ and $S_4$ the symmetric groups of degree 3 and 4, respectively, and $C_n^k$ denotes the direct product $C_n \times \cdots \times C_n$. The semidirect product of a group $N$ by a group $K$ is denoted by $N \rtimes K$ or $N \ltimes K$ which is a (not necessarily unique) group $G$ containing a normal subgroup $N_1$ isomorphic to $N$ and a subgroup $K_1$ isomorphic to $K$ such that $G = N_1 K_1$ and $N_1 \cap K_1 = 1$. For any two elements $x, y$ of a group $G$ we denote by $[x, y]$ the commutator $x^{-1}y^{-1}xy$. For a free group $F$ generated by free generators $x_1, \ldots, x_n$ and elements $r_1, \ldots, r_m$, the factor group $G = F / \langle r_1, \ldots, r_m \rangle$ is denoted by

$$\langle x_1, \ldots, x_n \mid r_1 = \cdots = r_m = 1 \rangle,$$

where $\langle r_1, \ldots, r_m \rangle$ is the normal closure of the subgroup $\langle r_1, \ldots, r_m \rangle$ in $F$. The notation $\langle \ast \rangle$ is called the presentation of the group $G$ by generators $x_1, \ldots, x_n$ and relations $r_1, \ldots, r_m$.

2. Preliminaries

In this section we state some facts which we need in the sequel. The following result is the classification of all undirected connected cubic Cayley integral graphs.

**Theorem 2.1** (Theorem 1.1 of [2]). There are exactly seven connected cubic integral Cayley graphs. In particular, for a finite group $G$ and a subset $T \subseteq H \setminus \{1\}$ with $T = T^{-1}$. Consider the Cayley graph $Cay(H, T)$. Note that $Cay(H, T)$ is isomorphic to a disjoint union of some $\Gamma = Cay(T, T)$. Thus $Cay(H, T)$ is integral if and only if $\Gamma$ is integral. Now, since $Cay(G, T)$ is also a disjoint union of some $\Gamma$, it follows that $\Gamma$ is integral. Hence $H$ is a Cayley integral graph. This completes the proof. □

**Lemma 2.2.** Let $G$ be a finite Cayley integral group. Then every subgroup of $G$ is also Cayley integral.

**Proof.** Let $H$ be an arbitrary subgroup of $G$ and let $T \subseteq H \setminus \{1\}$ with $T = T^{-1}$. Consider the Cayley graph $Cay(H, T)$. Note that $Cay(H, T)$ is isomorphic to a disjoint union of some $\Gamma = Cay(T, T)$. Thus $Cay(H, T)$ is integral if and only if $\Gamma$ is integral. Now, since $Cay(G, T)$ is also a disjoint union of some $\Gamma$, it follows that $\Gamma$ is integral. Hence $H$ is a Cayley integral group. This completes the proof. □
The above lemma will be frequently used in the sequel without further notice.

**Proposition 2.3** (See [8,12], Proposition 6.3.1 of [6]). Let \( G \) be a finite group and \( S \) a subset that is inverse closed and invariant under conjugation. The graph \( \text{Cay}(G, S) \) has eigenvalues \( \theta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s) \) with multiplicity \( \chi(1)^2 \), where \( \chi \) ranges over the irreducible characters of \( G \).

Note that for every finite group \( G \), since \( \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G| \cdot |\text{Irr}(G)| \) (\( \text{Irr}(G) \) is the set of all irreducible characters of \( G \)), the multiset \( \{\theta_\chi \mid \chi \in \text{Irr}(G)\} \) is the spectrum of \( \text{Cay}(G, S) \) for any \( S \) as in Proposition 2.3.

The following result has its own interest and we do not use it in the sequel but we would like to state it here!

**Proposition 2.4** (See Theorem 2 of [7]). Let \( G \) be a finite group and \( S \) be a member of the boolean algebra generated by the normal subgroups of \( G \). Then the Cayley graph \( \text{Cay}(G, S) \) is integral.

**Proof.** Since \( S \) can be obtained by arbitrary finite intersections, unions, or complements of normal subgroups of \( G \), \( S \) is closed under conjugation as well as inverse. Thus by Proposition 2.3 eigenvalues of \( \text{Cay}(G, S) \) are of the form \( \theta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s) \) for \( \chi \in \text{Irr}(G) \). Now, Corollary 4.2 of [4] shows that \( \sum_{s \in S} \chi(s) \) is an algebraic integer for each \( \chi \in \text{Irr}(G) \). Since each eigenvalue of any graph is an algebraic integer, it follows that \( \theta_\chi \) is integer. This completes the proof. \( \square \)

**Lemma 2.5** (Lemma 11 of [10]). If \( G \) is a Cayley integral group, then the order of every non-trivial element of \( G \) belongs to \( \{2, 3, 4, 6\} \).

**Lemma 2.6.** \( D_{2n} \) is not a Cayley integral group for all integers \( n \geq 4 \).

**Proof.** It is well-known that \( D_{2n} = \langle a, b \mid a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle \). If \( S = \{b, ba\} \), then \( \text{Cay}(D_{2n}, S) \) is a cycle on \( 2n \) vertices which is not an integral graph. Therefore \( D_{2n} \) is not an integral group. \( \square \)

**Lemma 2.7.** The following groups are not Cayley integral.

1. The alternating group \( A_4 \) of degree 4,
2. \( C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2 = 1 \rangle \),
3. \( S_3 \times C_3 \),
4. the special linear group \( SL(2, 3) \) of \( 2 \times 2 \) matrices over the field of order 3,
5. \( C_4 \times C_3 \cong \langle a, b \mid a^3 = b^4 = 1, a^b = a^{-1} \rangle \),
6. \( C_4 \times C_2 \cong \langle x, y \mid x^4 = y^4 = [x, y]^2 = [x^2, y] = [x, y^2] = 1 \rangle \),
7. the non-abelian group of exponent 3 and order 27: \( \langle x, y \mid x^3 = y^3 = (xy)^3 = (xy^{-1})^3 = 1 \rangle \),
8. \( (C_3 \times C_3) \times C_4 = \langle x, y, z \mid x^3 = y^3 = z^4 = [x, y] = 1, x^2 = x^{-1}, y^2 = y^{-1} \rangle \).

**Proof.** For each of groups (1)–(8) we have found an inverse closed subset \( S \) (\( 1 \not\in S \)) such that \( \text{Cay}(G, S) \) is not integral. We have used the following codes in GAP [16] to obtain the spectra of these graphs. The following code constructs the Cayley graph of \( A_4 \) on the set \( S \). We use GRAPE package of Soicher [15].

LoadPackage("grape");
F := FreeGroup(2);
x := F.1; y := F.2;
G := F/[x^3, y^2, (x*y)^3]; #G=A_4
a := G.1;
b := G.2;
S := [a, a^-1, b, a*b, b^-1*a^-1];
C := CayleyGraph(G, S);

By the function admat [2, p. 6] one can construct the adjacency matrix of a given graph by GRAPE [15].
A := admat(C, 12);
The following command computes the characteristic polynomial of the adjacency matrix of Cay(A₄, S) and the second command factorizes the polynomial into irreducible ones.

\[ P := \text{CharacteristicPolynomial}(A); \]
\[ \text{FP} := \text{Factors}(P); \]

It follows that the characteristic polynomial of Cay(A₄, S) is as follows:

\[ (x - 5)(x + 1)^5(x^2 - 5)^3, \]

and so Cay(A₄, S) is not integral showing that A₄ is not a Cayley integral group. For the other groups in (2)–(8) we give a presentation of G under which we introduce a subset S for which Cay(G, S) is not integral. Verifying that Cay(G, S) is not integral can be done as above by GAP. In each case, the factorized characteristic polynomial \( P(x) \) into irreducibles is exhibited.

(2) \( G = C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = 1, x^y = x^{-1} \rangle, S = \{ x, x^{-1}, y, y^{-1}, xy, y^{-1}x^{-1}, xy^2, y^2x^{-1} \} \) and \( P(x) = (x - 8)x^9(x + 4)^2(x^2 - 8)^2. \)

(3) \( G = S_3 \times C_3 = \langle x, y, z \mid x^2 = y^3 = z^3 = [x, z] = [y, z] = 1, y^x = y^{-1} \rangle, S = \{ x, y, y^{-1}, z, z^{-1}, zy^2, y^{-2}z^{-1} \} \) and \( P(x) = (x - 7)(x - 5)(x - 1)^4(x + 1)^4(x^2 + 3x - 1)^4. \)

(4) \( G = SL(2, 3) = \langle x, y \mid x^3 = y^3 = y^{-1}xyx^{-1}x = x^{-1}y^{-1}(x^{-1}y)^2 = (xy)^3 = 1, S = \{ x, x^{-1}, y, y^{-1} \} \) and \( P(x) = (x - 4)(x - 2)(x + 1)^8(x^2 - x - 4)^2. \)

(5) \( G = (C_4 \times C_3) \rtimes C_2 = \langle x, y, z \mid x^3 = y^3 = z^2 = [x, z] = [y, z] = 1, y^x = y^{-1} \rangle, S = \{ x, x^{-1}, y, y^{-1}, z, xy, y^{-1}x^{-1}, xz, z^{-1}x^{-1} \} \) and \( P(x) = (x - 9)(x - 3)^2(x - 1)^2x^6(x + 1)(x + 2)^4(x + 3)(x^2 + 12)^2. \)

(6) \( G = (C_4 \times C_2) \times C_4 = \langle x, y \mid x^4 = y^4 = [x^2, y] = [x, y^2] = 1, S = \{ x, x^{-1}, y, y^{-1} \} \) and \( P(x) = (x - 4)(x - 2)x^8(x + 4)(x^2 - 8)^2. \)

(7) \( G = \langle x, y \mid x^3 = (xy)^3 = (xy^{-1})^3 = 1, S = \{ x, x^{-1}, y, y^{-1} \} \) and \( P(x) = (x - 4)(x - 1)^4(x + 2)^{10}(x^2 - 2x - 2)^6. \)

(8) \( G = (C_3 \times C_3) \rtimes C_2 = \langle x, y, z \mid x^3 = y^3 = z^2 = [x, y] = 1, x^2 = x^{-1}, y^2 = y^{-1} \rangle, S = \{ x, x^{-1}, y, y^{-1}, z, z^{-1}, z^2, xy, (xy)^{-1}, xz, (xz)^{-1}, yz, (yz)^{-1} \} \) and \( P(x) = (x - 13)(x - 5)(x - 1)^2(x + 1)^{12}(x^2 - 2x - 11)^4(x^2 + 4x - 8)^2. \)

**Lemma 2.8.** Let \( G \) be a finite Cayley integral group and \( x \) be any element of order 2 of \( G \) and \( y \) be an arbitrary element of \( G \). Then \( (x, y) \) is isomorphic to one of the groups: \( C_2, C_2^2, C_4, C_6, S_3, C_2 \times C_4, C_2 \times C_6 \).

**Proof.** If \( y \in \langle x \rangle \), then \( (x, y) \cong C_2 \). Suppose that \( y \not\in \langle x \rangle \). If \( o(y) = 2 \), then \( (x, y) \) is a dihedral group of order at most 6 by Lemma 2.6. Thus in the latter case, \( (x, y) \cong C_2 \times C_2 \) or \( S_3 \). Now, assume that \( o(y) > 2 \). Let \( S = \langle x, y, y^{-1} \rangle \). Then Cay(S, S) is a cubic integral Cayley graph. It follows from Theorem 2.1 that \( S \) is isomorphic to one of the following groups: \( C_4, C_4, S_3, C_2 \times C_4, D_8, C_2 \times C_6, D_{12}, A_4, S_4, D_8 \times C_3, S_3 \times C_4, A_4 \times C_2 \). It follows from Lemma 2.5 that \( S \) is isomorphic to one of following groups: \( C_4, C_6, S_3, C_2 \times C_4, D_8, C_2 \times C_6, D_{12}, A_4, S_4, A_4 \times C_2 \). Since by Lemmas 2.6 and 2.7, \( G \) cannot have any subgroup isomorphic to \( D_{2n} \ (n \geq 4) \) or \( A_4, (S) \) is isomorphic to one of following groups: \( C_4, C_6, C_2 \times C_4, C_2 \times C_6, S_3 \). This completes the proof. □

**Lemma 2.9.** Let \( G \) be a finite Cayley integral group. Let \( x \) be any element of order 2 and \( y \in G \) is of order 4 or 6. Then \( xy = yx \).

**Proof.** It follows from Lemma 2.8 that the subgroup \( \langle x, y \rangle \) is abelian or isomorphic to \( S_3 \). Since \( S_3 \) has no element of order 4 or 6, we are done. □

**Lemma 2.10.** Let \( G \) be a finite non-abelian Cayley integral group generated by three distinct elements of order 2. Then \( G \cong S_3 \).

**Proof.** Suppose that \( G = \langle x, y, z \rangle \), where \( x, y, z \) are all distinct and \( o(x) = o(y) = o(z) = 2 \). Consider the cubic Cayley graph \( \Gamma = \text{Cay}(G, \{x, y, z\}) \). Since \( G \) is non-abelian and \( \Gamma \) is integral, it follows from Theorem 2.1 that \( G \) is isomorphic to one of the following groups: \( S_3, D_8, D_{12}, A_4, S_4, D_8 \times C_3, S_3 \times C_4, A_4 \times C_2 \).

The groups \( D_8, D_{12} \) and \( D_8 \times C_3 \) are ruled out by Lemma 2.6 and the groups \( A_4, S_4 \) and \( A_4 \times C_2 \) are not possible by Lemma 2.7. The group \( S_3 \times C_4 \) is not Cayley integral by Lemma 2.5. It follows that \( G \cong S_3 \). □
Lemma 2.11. Let $G$ be a finite 3-group. Then $G$ is Cayley integral if and only if $G$ is elementary abelian.

Proof. If $G$ is an elementary abelian 3-group, then $G \cong C_3^k$ for some integer $k \geq 0$. Now, it follows from [10] that $G$ is Cayley integral.

Now, assume that $G$ is a finite Cayley integral 3-group. By Lemma 2.5 the exponent of $G$ is 3. Suppose, for a contradiction, that $G$ is non-abelian. Then $G$ has two non-commuting elements $x$ and $y$. Thus $\langle x, y \rangle$ is the group of order 27 and exponent 3. This contradicts Lemma 2.7. Thus $G$ is abelian of exponent 3 which means that $G$ is an elementary abelian 3-group. □

Theorem 2.12. Let $G$ be a finite Cayley integral group. Then, there exist two non-commuting elements of order 2 in $G$ if and only if $G \cong S_3$.

Proof. Let $x, y \in G$ be two non-commuting elements of order 2. Then it follows from Lemma 2.8 that $\langle x, y \rangle \cong S_3$. Now, assume that $z$ is an element of order 2 such that $z \notin \{x, y\}$. It follows from Lemma 2.10 that $\langle x, y, z \rangle \cong S_3$. Therefore $z \in \langle x, y \rangle$. This means that all elements of order 2 of $\langle x, y \rangle$ are exactly all elements of order 2 of $G$. Thus $G$ has precisely three elements $x, y, z$ of order 2 and they are pairwise non-commuting. Now, assume that, if possible, $G$ has an element $t$ of order 4. Then by Lemma 2.9 $t$ commutes with all $x, y$ and $z$ and so $t^2g = gt^2$ for all $g \in \{x, y, z\}$; this is a contradiction since $t^2 \in \{x, y, z\}$. Therefore $G$ has no element of order 4. Now, if $G$ has a subgroup $K$ of order 4, it must be isomorphic to $C_2 \times C_2$, a contradiction as all elements of order 2 of $G$ are pairwise non-commuting. Hence 4 does not divide $|G|$ and so $|G| = 2m$ for some odd integer $m$. It follows from Lemma 2.5 that $m$ is a power of 3. Let $M$ be a Sylow 3-subgroup of $G$. By Lemma 2.11 $M$ is elementary abelian. Assume that $|M| \geq 9$. Thus $M$ has two elements $b_1$ and $b_2$ such that $\langle b_1, b_2 \rangle = \langle b_1 \rangle \times \langle b_2 \rangle$. Note that if $a \in G$ and $b \in G$ such that $o(a) = 2$ and $o(b) = 3$, it follows from Lemma 2.8 that $b^a = b$ or $b^a = b^{-1}$ since $(a, b)$ is either abelian or isomorphic to $S_3$. Since $G$ has a subgroup isomorphic to $S_3$ (say $\langle x, y \rangle$), we may assume that $b_1^2 = b_2^{-1}$. Thus $b_2^2 = b_1$ or $b_2^2 = b_1^{-1}$. If $b_2^2 = b_1$, then $\langle x, b_1, b_2 \rangle \cong S_3 \times C_3$ which is not possible by Lemma 2.7. If $b_2^2 = b_1^{-1}$, then $\langle x, b_1, b_2 \rangle$ has 9 elements of order 2 which is a contradiction, since $G$ has only 3 elements of order 2. Thus $|M| = 3$ and so $G \cong S_3$.

The converse is easy to verify. □

Theorem 2.13. Let $G$ be a finite non-abelian 2-group. Then $G$ is Cayley integral if and only if $G \cong Q_8 \times C_2^n$ for some integer $n \geq 0$.

Proof. Suppose that $G$ is a finite non-abelian Cayley integral 2-group. If we prove that every subgroup of $G$ is normal in $G$, it follows from a famous result of Dedekind–Baer (see Theorem 5.3.7 of [13]) that $G \cong Q_8 \times C_2^n$ for some integer $n \geq 0$.

To prove that every subgroup of $G$ is normal in $G$ it is enough to show that every cyclic subgroup of $G$ is normal. Since $G$ is a 2-group, every element of $G$ is of order 1, 2 or 4 by Lemma 2.5. Thus every cyclic subgroup of $G$ is either of order 1, 2 or 4. Every element of order 2 belongs to the center of $G$, this follows from Lemma 2.8 and so every (cyclic) subgroup of order 2 is normal in $G$. Therefore, it remains to prove that $\langle x \rangle \trianglelefteq G$ for all elements $x$ of order 4.

Suppose in contrary that there exists an element $a$ of order 4 such that $\langle a \rangle$ is not normal in $G$. Thus there exists an element $g$ of $G$ such that $g^{-1}ag \notin \langle a \rangle$. Consider the subgroup $H = \langle a, g \rangle$ of $G$. Clearly $H$ is non-abelian. The order of $g$ is not 1 or 2, otherwise $a^2 = a$ since elements of order 2 are central in $G$. Therefore the order of $g$ is 4. Now, we investigate group properties of $H$.

(1) $H = \langle a, g \rangle$ is of exponent 4, $o(a) = o(g) = 4$ and $g^{-1}ag \notin \langle a \rangle$.

(2) All elements of order 2 of $H$ are in the center $Z(H)$ of $H$. This follows from Lemma 2.8.

(3) $H/Z(H)$ is of exponent 2. If $x \in H$, then $x$ is of order 1, 2 or 4 by the property (1). Thus $o(x^2) \in \{1, 2\}$ and so $x^2 \in Z(H)$ by the property (2). This proves that $H/Z(H)$ is of exponent 2.

(4) $H$ is nilpotent of class 2. By the property (3), the derived subgroup $H'$ of $H$ is contained in $Z(H)$ and so $H$ is nilpotent of class 2.

(5) $H' = \langle [a, g] \rangle$ is of order 2. It is because that every commutator in $H$ is central by the property (4) and since $H$ is generated by two elements $a$ and $g$, we have $H' = \langle [a, g] \rangle$. Now, by the property (3), $[a, g]^2 = [a^2, g] = 1$ and so $|H'| = 2$. 

Author's personal copy
Lemma 2.7 implies that \( G \) elementsoforder 2 lie in the center of \( \langle \chi(o) \rangle \). This is because \( H/H' \) is an abelian group generated by two elements \( aH \) and \( gH \) which are both of orders dividing 4 and we note that \( H/H' \) cannot be cyclic otherwise \( H \) is also cyclic. It follows that the order of \( H \) is 8, 16 or 32.

If the order of \( H \) is 8, then \( H \) is isomorphic to \( D_8 \) or \( Q_8 \); since by Lemma 2.6, \( D_8 \) is not an integral group, \( H \not\cong D_8 \). Every subgroup of \( Q_8 \) is normal and so \( g^{-1}ag \in \langle a \rangle \) if \( H \cong Q_8 \). Therefore by the property (1), \( H \not\cong Q_8 \).

Thus \( |H| \in \{16, 32\} \). Now, suppose that the order of \( H \) is 16. By the following code written in GAP [16], one can see what are groups \( H \) of order 16 satisfying the properties (1)–(6).

```gap
a:=AllSmallGroups(16,IsAbelian,false);
b:=Filtered(a,i->Size(DerivedSubgroup(i))=2);
c:=Filtered(b,i->IdSmallGroup(FactorGroup(i,DerivedSubgroup(i)))=[8,2]);
d:=Filtered(c,i->IsSubgroup(Center(i),Subgroup(i,Filtered(i,j->Order(j)=2))))=true);
e:=Filtered(d,i->IsSubgroup(Center(i),Subgroup(i,Filtered(i,j->Order(j)=4))))=true);
```

The list \( e \) contains only one group isomorphic to \( C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = 1, x^y = x^{-1} \rangle \). Now, Lemma 2.7 states that \( H \) cannot be isomorphic to \( C_4 \times C_4 \) and so \( |H| \not\in \{16\} \).

Now, assume that \( |H| = 32 \). By a similar code as above one can see that there is only one group \( N \) satisfying the properties (1)–(6). The group \( N \) is isomorphic to \((C_4 \times C_2) \times C_4 = \langle x, y \mid x^4 = y^4 = [x,y]^2 = [x,y^2] = 1 \rangle \) which is not a Cayley integral group by Lemma 2.7. Thus \( |H| 
ot\in \{32\} \).

This completes the proof in this direction.

Now, let us prove that \( T_n = Q_8 \times C_{2}^{n} \) is a Cayley integral group for all integers \( n \geq 0 \). We first prove that the conjugacy class \( a^n = \{ a^{kn} \mid g \in T_n \} \) of any element \( a \in T_n \) is equal to \( \{ a \} \) or \( \{ a, a^{-1} \} \). For, if \( o(a) = 2 \), then \( a \) is central in \( T_n \) and so \( a^{kn} = \{ a \} \); and if \( o(a) = 4 \), then \( a = xt \), where \( x \in Q_8 \) and \( t \in C_{2}^{n} \). Let \( g = ys \) be any element of \( T_n \) such that \( y \in Q_8 \) and \( s \in C_{2}^{n} \). We have \( a^{k} = x^{k}t \) and since \( x^{k} \in \{ x, x^{-1} \} \) (it is easy to check that the latter is valid in \( Q_8 \)) it follows that \( a^{k} \in \{ a, a^{-1} \} \).

It follows that if \( S \) is an inverse closed subset of \( T_n \) (not containing 1), \( S \) is also closed under conjugation of elements of \( T_n \). Now, consider the Cayley graph \( Cay(T_n, S) \). Proposition 2.3 implies that for each irreducible character \( \chi \) of \( T_n \), \( \frac{1}{\chi(1)} \sum_{s \in S} \chi(s) \) is an eigenvalue \( \theta_{\chi} \) of \( Cay(T_n, S) \) of multiplicity \( \chi(1)^2 \). As we mentioned in the paragraph after the statement of Proposition 2.3 the multiset \( \{ \theta_{\chi} \mid \chi \in \text{Irr}(T_n) \} \) is the spectrum of \( Cay(T_n, S) \). Now, since \( \chi(g) \in \mathbb{Z} \) for all \( g \in T_n \) (because all irreducible characters of \( Q_8 \) or \( C_2 \) have integer values and an irreducible character of \( T_n \) is a tensor product of irreducible characters of \( Q_8 \) and \( C_2 \)'s), \( \sum_{s \in S} \chi(s) \in \mathbb{Z} \). Since \( \chi(1) \) is an integer, it follows that \( \theta_{\chi} \in \mathbb{Q} \) and so \( \theta_{\chi} \in \mathbb{Z} \) as each eigenvalue of the adjacency matrix of a graph is an algebraic integer. Hence \( Cay(T_n, S) \) is integral and so \( T_n \) is a Cayley integral group. \( \square \)

It should be mentioned that the subsets \( S \) of \( Q_8 \times C_{p}^{d} \) (\( p \) a prime) for which the Cayley graph \( Cay(Q_8 \times C_{p}^{d}, S) \) is integral are studied in the last section of [7]. We cannot derive the above result from discussions in [7]

3. Proof of the main theorem

In this section we prove the main theorem.

**Proof of Theorem 1.1.** Let \( G \) be a finite non-abelian Cayley integral group. By Lemma 2.11 and Theorem 2.13, we may assume that 6 divides the order of \( G \). By Theorem 2.12, we may assume that all elements of order 2 of \( G \) pairwise commute. If \( x \) is an arbitrary element of order 3, and \( y \) is any element of order 2, the subgroup \( \langle x, y \rangle \) must be abelian, otherwise it is isomorphic to \( S_3 \) by Lemma 2.8, and \( S_3 \) has non-commuting elements of order 2, a contradiction. Now, Lemmas 2.5 and 2.9 imply that elements of order 2 lie in the center of \( G \). If \( G \) has no elements of order 4, then \( G = P \times Q \), where \( P \) is the Sylow 2-subgroup of \( G \) and \( Q \) is the Sylow 3-subgroup of \( G \); this is because \( P \) is a central subgroup of \( G \). Now, Lemma 2.11 implies that \( G \cong C_{2}^{k} \times C_{3}^{\ell} \) for some positive integers \( k \) and \( \ell \) and so \( G \) is abelian, a contradiction. Thus we may assume that \( G \) has an element of order 4.
Now, let $a$ and $b$ be two elements of $G$ of order 4 and 3, respectively. By Lemma 2.5 $ab \neq ba$ and we claim that $b^a = b^{-1}$. To prove the latter, it is enough to show that $K = \langle a, b \rangle \cong C_4 \times C_3 = \langle x, y \mid x^4 = y^3 = 1, x^4 y = y = x^{-1} \rangle$. We need to note some group properties of $K$ as follows:

1. $K$ is a finite non-abelian Cayley integral group having two elements of orders 3 and 4.
2. The set of element orders of $K$ is contained in $\{1, 2, 3, 4, 6\}$.
3. All elements of order 2 of $K$ lie in the center of $K$.

By Von Dyck’s theorem, there exists an epimorphism from some $G_{ij}$ ($i, j \in \{2, 3, 4, 6\}$) onto $G$, where

$$G_{ij} = \langle x, y \mid x^4 = y^3 = [x^2, y] = (xy)^i = [x, y]^j = [(xy)^{\ell} x, x] = [(x, y)^{\ell} x, x] = 1 \rangle,$$

and $\epsilon(\ell) = \begin{cases} 1 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases}$ Therefore the group $K$ is isomorphic to a quotient of some $G_{ij}$. All groups $G_{ij}$ are finite and can be easily computed by GAP [16]. Hence we need to study quotients of $G_{ij}$. Using the following code in GAP [16], one can find all possible quotients (satisfying properties (1)–(3) above) of $G_{ij}$ which can be isomorphic to $K$. Note that the following code is for $G_{4,6}$.

```gap
f:=FreeGroup(2);
x:=f.1;;y:=f.2;;
G46:=f/[x^4,y^3,Comm(x^2,y),(x*y)^4,Comm(x,y)^6,Comm((x*y)^2,x),
Comm((x*y)^2,y),
Comm(Comm(x,y)^3,x),Comm(Comm(x,y)^3,y)];
N:=NormalSubgroups(G46);;
T:=List(N,i->FactorGroup(G46,i));;
Tn:=Filtered(T,i->IsAbelian(i)=false);
Tnl:=Filtered(Tn,i->exponent(i)=12);
Tn:=Filtered(Tn,i->IsAbelian(i));

These possible quotients are $SL(2, 3)$, $C_2 \times SL(2, 3)$ or $C_4 \times C_3$. Now, Lemma 2.7 implies that $K = C_4 \times C_3$ as we claimed.

Now, let $P$ be a Sylow 2-subgroup of $G$. Suppose for a contradiction that $P$ is non-abelian. Then Theorem 2.13 implies that $P$ has a subgroup $E$ isomorphic to the quaternion group $Q_8$ of order 8. Consider two elements $a$ and $a'$ of order 4 in $E$ such that $o(aa') = 4$ (take famous $i, j$ and $k$ in $Q_8$) and let $b$ be any element of order 3 in $G$. Since $b^a = b^{-1}$ and $b^{a'} = b^{-1}$, it follows that $b^{aa'} = (b^a)^{a'} = b$ and since $o(aa') = 4$, $b^{aa'} = b^{-1}$. Hence $b = b^{-1}$, a contradiction. Therefore Sylow 2-subgroups of $G$ are abelian.

Now, suppose for a contradiction that $P$ has an element $a$ of order 4 and an element $t$ of order 2 such that $(a, t) = \langle a \rangle \times \langle t \rangle$. Take any element $b$ of order 3 in $G$ and consider the group $M = \langle a, b, t \rangle$. The group $M$ is isomorphic to $(C_4 \times C_3) \times C_2$ which is listed in Lemma 2.7 as a group that is not Cayley integral. It follows that $P$ is cyclic of order 4. Now, let $Q$ be a Sylow 3-subgroup of $G$. Suppose for a contradiction that $|Q| \geq 9$. Then $Q$ has two elements $b$ and $b'$ of order 3 such that $\langle b, b' \rangle = \langle b \rangle \times \langle b' \rangle$. Since $b^a = b^{-1}$ and $b^{a'} = (b^b)^{a'} = b$ and since $o(aa') = 4$, $b^{aa'} = b^{-1}$. Hence $b = b^{-1}$, a contradiction. Therefore Sylow 2-subgroups of $G$ are abelian.

Acknowledgments

The authors thank the Graduate Studies of University of Isfahan. This research was partially supported by the Center of Excellence for Mathematics, University of Isfahan. This research was in part supported by a grant from IPM (No. 92050219).

References