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Cospectrality of graphs

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Richard Brualdi proposed in Stevanivić (2007) [6] the following problem:
(Problem AWGS.4) Let $G_n$ and $G'_n$ be two nonisomorphic graphs on $n$ vertices with spectra

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n$,

respectively. Define the distance between the spectra of $G_n$ and $G'_n$ as

$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2$ (or use $\sum_{i=1}^{n} |\lambda_i - \lambda'_i|$).

Define the cospectrality of $G_n$ by

$cs(G_n) = \min \{ \lambda(G_n, G'_n) : G'_n \text{ not isomorphic to } G_n \}$.

Let

$cs_n = \max \{ cs(G_n) : G_n \text{ a graph on } n \text{ vertices} \}$.

**Problem A.** Investigate $cs(G_n)$ for special classes of graphs.

**Problem B.** Find a good upper bound on $cs_n$.

In this paper we study **Problem A** and determine the cospectrality of certain graphs by the Euclidian distance.

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Let $K_n$ denote the complete graph on $n$ vertices, $nK_1$ denote the null graph on $n$ vertices and $K_2 + (n - 2)K_1$ denote the disjoint union of the $K_2$ with $n - 2$ isolated vertices, where $n \geq 2$. In this paper we find $cs(K_n)$, $cs(nK_1)$, $cs(K_2 + (n - 2)K_1)$ ($n \geq 2$) and $cs(K_{n,n})$.

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1. Introduction

Throughout the paper all graphs are simple, that is finite and undirected without loops and multiple edges. By the spectrum of a graph $G$, we mean the multiset of eigenvalues of adjacency matrix of $G$.

Richard Brualdi proposed in [6] the following problem:

(Problem AWGS.4) Let $G_n$ and $G'_n$ be two nonisomorphic graphs on $n$ vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

respectively. Define the distance between the spectra of $G_n$ and $G'_n$ as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \left( \text{or use} \sum_{i=1}^{n} |\lambda_i - \lambda'_i| \right).$$

Define the cospectrality of $G_n$ by

$$cs(G_n) = \min \{ \lambda(G_n, G'_n): G'_n \text{ not isomorphic to } G_n \}.$$

Thus $cs(G_n) = 0$ if and only if $G_n$ has a cospectral mate. Let

$$cs_n = \max \{ cs(G_n): G_n \text{ a graph on } n \text{ vertices} \}.$$

This function measures how far apart the spectrum of a graph with $n$ vertices can be from the spectrum of any other graph with $n$ vertices.

**Problem A.** Investigate $cs(G_n)$ for special classes of graphs.

**Problem B.** Find a good upper bound on $cs_n$.

In this paper we study Problem A and determine the cospectrality of certain graphs by the Euclidian distance, that is

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2.$$
For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively; $\overline{G}$ denotes the complement of $G$ and $A(G)$ denotes the adjacency matrix of $G$. For two graphs $G$ and $H$ with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, i.e. the disjoint union of two graphs $G$ and $H$. The complete product (join) $G \ast H$ of graphs $G$ and $H$ is the graph obtained from $G + H$ by joining every vertex of $G$ with every vertex of $H$. In particular, $nG$ denotes $G + \cdots + G$ and $\nabla_n G$ denotes $G \nabla G \nabla \cdots \nabla G$.

We denote by $\text{Spec}(G)$ the multiset of the eigenvalues of the graph $G$.

For positive integers $n_1, \ldots, n_\ell$, $K_{n_1, \ldots, n_\ell}$ denotes the complete multipartite graph with $\ell$ parts of sizes $n_1, \ldots, n_\ell$. Let $K_n$ denote the complete graph on $n$ vertices, $nK_1 = \overline{K}_n$ denote the null graph on $n$ vertices and $P_n$ denote the path with $n$ vertices. By the previous notation, for any integer $n \geq 2$, $K_2 + (n-2)K_1$ denotes the disjoint union of the $K_2$ with $n-2$ isolated vertices. In this paper we find $\text{cs}(K_n)$, $\text{cs}(nK_1)$, $\text{cs}(K_2 + (n-2)K_1)$ ($n \geq 2$) and $\text{cs}(K_{n,n})$. In particular, we find that there exists a unique graph $G_H$ such that $\lambda(H, G_H) = \text{cs}(H)$ if $H \in \{K_n, nK_1, K_2 + (n-2)K_1, K_{n,n}\}$. The main results of our paper are the following:

**Theorem 1.1.** For every integer $n \geq 2$, $\text{cs}(nK_1) = 2$. In particular, $\lambda(nK_1, G) = \text{cs}(nK_1)$ for some graph $G$ if and only if $G \cong K_2 + (n-2)K_1$.

**Theorem 1.2.** For every integer $n \geq 3$, $\text{cs}(K_2 + (n-2)K_1) = 2(\sqrt{2} - 1)^2$. Also, $\text{cs}(K_2) = \lambda(K_2, 2K_1) = 2$. In particular, $\lambda(K_2 + (n-2)K_1, G) = \text{cs}(K_2 + (n-2)K_1)$ for some graph $G$ if and only if $G \cong P_3 + (n-3)K_1$.

**Theorem 1.3.** For every integer $n \geq 2$, $\text{cs}(K_n) = n^2 + n - n\sqrt{n^2 + 2n + 7} - 2$. In particular, $\lambda(K_n, G) = \text{cs}(K_n)$ for some graph $G$ if and only if $G \cong K_n \setminus e$, where $K_n \setminus e$ is the graph obtaining from $K_n$ by deletion one edge $e$.

**Theorem 1.4.** Let $n \geq 2$ be an integer. Then $\text{cs}(K_{n,n}) = 2(n - \sqrt{n^2 - 1})^2$. In particular, $\lambda(K_{n,n}, G) = \text{cs}(K_{n,n})$ for some graph $G$ if and only if $G \cong K_{n-1,n+1}$.

2. Cospectrality of graphs with at most one edge

In this section we will determine the cospectrality of graphs with at most one edge. Let $G$ be a simple graph of order $n$ and size $m$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $G$. It is well known that $\lambda_1 + \cdots + \lambda_n = 0$ and $\lambda_1^2 + \cdots + \lambda_n^2 = 2m$. We now give the proof of Theorem 1.1 in which we determine the cospectrality of graphs with no edge.

**Proof of Theorem 1.1.** Let $G'_n$ be a simple graph of order $n$ and size $m'$ with eigenvalues $\lambda'_1 \geq \cdots \geq \lambda'_n$. Since the eigenvalues of $nK_1$ are $\lambda_1 = \cdots = \lambda_n = 0$, then

$$\lambda(nK_1, G'_n) = \lambda'_1^2 + \cdots + \lambda'_n^2 = 2m'.$$
Since $G'_n$ is not isomorphic to $nK_1$, $m' \geq 1$. So the minimum value of $\lambda(nK_1, G'_n)$ is 2 and it happens for $G'_n = K_2 + (n-2)K_1$. □

**Proof of Theorem 1.2.** Let $n \geq 3$ be an integer and let $G'_n$ be a simple graph of order $n$ and size $m'$ with eigenvalues $\lambda'_1 \geq \cdots \geq \lambda'_n$. Since the eigenvalues of $K_2 + (n-2)K_1$ are $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_{n-1} = 0$, $\lambda_n = -1$,

$$\lambda(K_2 + (n-2)K_1, G'_n) = (\lambda'_1 - 1)^2 + \lambda'_2^2 + \cdots + \lambda'_{n-1}^2 + (\lambda'_n + 1)^2 = 2m' + 2 - 2\lambda'_1 + 2\lambda'_n.$$  

Now, we want to find the minimum value of $m' - \lambda'_1 + \lambda'_n$ among all graphs of order $n$ and size $m'$. By Perron–Frobenius Theorem (see [2, Theorem 0.13]), $\lambda'_n \geq -\lambda'_1$. Thus

$$\lambda(K_2 + (n-2)K_1, G'_n) = 2m' + 2 - 2\lambda'_1 + 2\lambda'_n \geq 2m' + 4\lambda'_1 + 2.$$  

We claim that for every graph $G'_n \not\cong K_2 + (n-2)K_1, P_3 + (n-3)K_1$, the following holds

$$\lambda(K_2 + (n-2)K_1, G'_n) > 2(\sqrt{2} - 1)^2.$$  

Since

$$\lambda(K_2 + (n-2)K_1, P_3 + (n-3)K_1) = 2(\sqrt{2} - 1)^2,$$

the validity of our claim completes the proof. To prove the claim we consider the following cases:

**Case 1.** Let $\lambda'_1 \geq 1 + \sqrt{0.5}$. Then $(\lambda'_1 - 1)^2 \geq 0.5$. Thus

$$\lambda(K_2 + (n-2)K_1, G'_n) = (\lambda'_1 - 1)^2 + \lambda'_2^2 + \cdots + \lambda'_{n-1}^2 + (\lambda'_n + 1)^2 \geq 0.5 > 2(\sqrt{2} - 1)^2.$$  

**Case 2.** Let $\lambda'_1 < 1 + \sqrt{0.5}$. Then for $m' \geq 4$, $2m' \geq 4\lambda'_1$. Since

$$\lambda(K_2 + (n-2)K_1, G'_n) \geq 2m' - 4\lambda'_1 + 2,$$

it follows that $\lambda(K_2 + (n-2)K_1, G'_n) \geq 2$ and so we are done. To complete the proof of our claim it remains to compute $\lambda(K_2 + (n-2)K_1, G'_n)$ for all graphs $G'_n$ with at most 3 edges. These graphs are as follows, $nK_1$, $K_2 + (n-2)K_1$, $2K_2 + (n-4)K_1$, $P_3 + (n-3)K_1$, $3K_2 + (n-6)K_1$, $P_3 + K_2 + (n-5)K_1$, $K_3 + (n-3)K_1$, $P_4 + (n-4)K_1$ and $K_{1,3} + (n-4)K_1$. One can easily see that for the latter graphs the claim is valid. This completes the proof. □
3. Cospectrality of the complete graph

In this section we will determine the cospectrality of the complete graphs. Let $G$ be a simple graph of order $n$ and size $m$. In this section we show that for every integer $n \geq 2$, $cs(K_n) = \lambda(K_n, K_n \setminus e)$, where $e$ is an arbitrary edge of $K_n$. First we prove some lemmas.

**Theorem 3.1.** (Theorem 9.1.1 of [3]) Let $G$ be a graph of order $n$ and $H$ be an induced subgraph of $G$ with order $m$. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of $G$ and $H$, respectively. Then for every $i$, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.

Let $X$ be a graph. Recall that a partition $\pi$ of $V(X)$ with cells $C_1, \ldots, C_r$ is equitable if the number of neighbors in $C_j$ of a vertex $u$ in $C_i$ is a constant $b_{ij}$, independent of $u$.

**Theorem 3.2.** (Theorem 9.3.3 and Exercise 3 in page 213 of [3]) If $\pi$ is an equitable partition of a graph $X$, then the characteristic polynomial of $A(X/\pi)$ divides the characteristic polynomial of $A(X)$. Moreover, the spectral radius of $A(X/\pi)$ is equal to the spectral radius of $A(X)$.

Note that in Theorem 3.2, $X/\pi$ denotes the directed graph with the $r$ cells of $\pi$ as its vertices and $b_{ij}$ arcs from the $i$th to the $j$th cells of $\pi$ is called the quotient of $X$ over $\pi$, and denoted by $X/\pi$. The entries of the adjacency matrix of this quotient are given by $A(X/\pi)_{ij} = b_{ij}$.

Let $K_n^t$ be the graph obtained from $K_n$ by deleting $n - t - 1$ edges from one vertex of $K_n$. By the following lemma one can compute the spectrum of $K_n^t$.

**Lemma 3.3.** Let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. Let $1 \leq t \leq n-2$ and $K_n^t = K_n \setminus \{v_0v_{t+1}, v_0v_{t+2}, \ldots, v_0v_{n-1}\}$. Suppose that $f(\lambda) := \lambda^3 - (n-3)\lambda^2 - (n+t-2)\lambda + t(n-t-2)$. Then

$$ Spec(K_n^t) = \{-1, \ldots, -1, \lambda_1, \lambda_2, \lambda_3\}, $$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are the roots of the polynomial $f(\lambda)$ and $-1 \leq \max\{\lambda_1, \lambda_2, \lambda_3\}$.

**Proof.** Let $A = \{v_0\}$, $B = \{v_1, \ldots, v_t\}$ and $C = \{v_{t+1}, \ldots, v_{n-1}\}$. It is easy to see that the partition $\{A, B, C\}$ of vertices of $K_n^t$ is an equitable partition with the following matrix

$$ M = \begin{bmatrix}
0 & t & 0 \\
1 & t-1 & n-1-t \\
0 & t & n-t-2
\end{bmatrix}. $$
where the first (second and third) row and column corresponds to $A$ ($B$ and $C$, respectively). Since $\text{Spec}(K_{n-1}) = \{n-2, -1, \ldots, -1\}$ and $K_{n-1}$ is an induced subgraph of $K_n^t$,
by interlacing Theorem 3.1 we conclude that $K_n^t$ has at least $n - 3$ eigenvalues $-1$. By
Theorem 3.2, the three remaining eigenvalues of $K_n^t$ are the eigenvalues of the matrix $M$
and the largest eigenvalue of $K_n^t$ is among the latter three eigenvalues. On the other hand $f(\lambda) = \det(\lambda I - M)$ and $f(-1) \neq 0$. This completes the proof. $\square$

**Corollary 3.4.** For every integer $n \geq 2$ and every arbitrary edge $e$ of $K_n$,

$$\text{Spec}(K_n \setminus e) = \left\{ \frac{n-3+\sqrt{n^2+2n-7}}{2}, 0, -1, \ldots, -1, \frac{n-3-\sqrt{n^2+2n-7}}{2} \right\}.$$  

**Proof.** By the notation of Lemma 3.3, $K_n \setminus e = K_n^{n-2}$. This implies the result. $\square$

**Remark 3.5.** Let $G$ be a graph of order $n$ and size $m$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the
eigenvalues of $G$. Since $n-1 \geq -1 \geq \cdots \geq -1$ are the eigenvalues of $K_n$, one can obtain
that $\lambda(K_n, G) = 2m - 2n\lambda_1 + n^2 - n$. This equality shows that to obtain $\text{cs}(K_n)$ it is
sufficient to obtain a graph $G$ for which the parameter $m - n\lambda_1$ has the minimum value.
In sequel we show that among all graphs $G$ of order $n$ except $K_n$, the minimum value of
$m - n\lambda_1(G)$ is attained on the graph $K_n \setminus e$.

**Lemma 3.6.** For every integer $n \geq 3$ and every edge $e$ of $K_n$, $\lambda(K_n, K_n \setminus e) < 2$. If $n = 2$, then $\lambda(K_n, K_n \setminus e) = 2$.

**Proof.** For $n = 2$, there is nothing to prove. Let $n \geq 3$. Using Corollary 3.4 and
Remark 3.5 one can obtain the result. $\square$

**Lemma 3.7.** Let $n \geq 3$ be an integer. Let $K = K_n \setminus \{e, e'\}$, where $e$ and $e'$ are two
adjacent edges in $K_n$. Then $\lambda(K_n, K) > \lambda(K_n, K_n \setminus e)$.

**Proof.** Note that $K = K_n^{n-3}$. By Lemma 3.3 $\lambda_1(K)$ is a root of the polynomial $f(\lambda) = \lambda^3 - (n-3)\lambda^2 - (2n-5)\lambda + n - 3$. Since the roots of the derivation $f'(\lambda)$ (with respect
to $\lambda$) of $f(\lambda)$ are $\frac{n-3+\sqrt{n^2-6}}{3}$, $f$ is an increasing function on the interval $[\frac{n-3+\sqrt{n^2-6}}{3}, \infty)$. Using Corollary 3.4 one can see that $\lambda_1(K_n \setminus e) - \frac{1}{n} > n - 2 > \frac{n-3+\sqrt{n^2-6}}{3}$. It is
not hard to see that $f(\lambda_1(K_n \setminus e) - \frac{1}{n}) > 0$. On the other hand $f$ is increasing on
$[\frac{n-3+\sqrt{n^2-6}}{3}, \infty)$. Thus $f(\lambda) > 0$ for every $\lambda \geq \lambda_1(K_n \setminus e) - \frac{1}{n}$. Since $\lambda_1(K)$ is a root
of $f(\lambda)$, we conclude that $\lambda_1(K) < \lambda_1(K_n \setminus e) - \frac{1}{n}$. This shows that $\frac{n(n-1)}{2} - 2 - n\lambda_1(K) > \frac{n(n-1)}{2} - 1 - n\lambda_1(K_n \setminus e)$. Equivalently, $\lambda(K_n, K) > \lambda(K_n, K_n \setminus e)$ (see Remark 3.5). $\square$
We need the following theorems to prove the main result of this section.

**Theorem 3.8.** (See [5], and also Theorem 6.7 of [2].) A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.

**Lemma 3.9.** Let $G$ be a graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. The graph $G$ is isomorphic to one the following graphs if and only if $\lambda_1 > 0$, $\lambda_2 \leq 0$ and $\lambda_3 < 0$.

1. $G \cong K_n$,
2. $G \cong K_1 + K_{n-1}$,
3. $G \cong K_{2,1,\ldots,1} = K_n \setminus e$ for an edge $e$ of $K_n$.

**Proof.** Suppose that $\lambda_1 > 0$, $\lambda_2 \leq 0$ and $\lambda_3 < 0$. By Theorem 3.8, there exist some integers $t \geq 1$, $n_1, \ldots, n_t \geq 1$ and $r \geq 0$ such that $G \cong rK_1 + K_{n_1,\ldots,n_t}$. Since $\lambda_3 < 0$, $r \leq 1$. Thus it is enough to investigate the cases $G \cong K_{n_1,\ldots,n_t}$ and $G \cong K_1 + K_{n_1,\ldots,n_t}$. Note that $t \geq 2$, otherwise $G \cong \overline{K_n}$, a contradiction.

Suppose that $G \cong K_1 + K_{n_1,\ldots,n_t}$. If $n_i \geq 2$ for some $i$, then $G$ has a $3K_1$ as an induced subgraph and so it follows from Interlacing Theorem that $\lambda_3 \geq 0$, a contradiction. Therefore $G \cong K_1 + K_{n-1}$ in this case.

Now assume that $G \cong K_{n_1,\ldots,n_t}$. If $n_1 = \cdots = n_t = 1$, then $G \cong K_n$. Thus we may assume that $n_i \geq 2$ for some $i$. Suppose that $n_i, n_j \geq 2$ for some distinct $i$ and $j$. Thus the cycle $C_4$ of length 4 is an induced subgraph of $G$. Since the eigenvalues of $C_4$ are $2, 0, 0, -2$, it follows from Interlacing Theorem that $\lambda_3 \geq 0$, a contradiction. Thus we can assume that $n_1 \geq 2$ and $n_2 = \cdots = n_t = 1$. If $n_1 = 2$, then $G \cong K_{2,1,\ldots,1} = K_n \setminus e$ for some edge $e$. Therefore we may assume $n_1 \geq 3$. This shows that the star $K_{1,3}$ is an induced subgraph of $G$. Since the eigenvalues of $K_{1,3}$ are $\sqrt{3}, 0, 0, -\sqrt{3}$, it follows from Interlacing Theorem 3.1 that $\lambda_3 \geq 0$, a contradiction.

The converse follows from Corollary 3.4 and the fact that the eigenvalues of the complete graph $K_n$ are $n - 1, -1, \ldots, -1$. □

**Theorem 3.10.** (See [4].) Let $G$ be a graph without isolated vertices and let $\lambda_2(G)$ be the second largest eigenvalue of $G$. Then $0 < \lambda_2(G) \leq \sqrt{2} - 1$ if and only if one of the following holds:

1. $G \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$.
2. $G \cong (K_1 + K_{r,s})\nabla \overline{K_q}$.
3. $G \cong (K_1 + K_{r,s})\nabla K_{p,q}$.

Now, we are in a position to prove the main result of this section.

**Theorem 3.11.** Let $G$ be a graph of order $n$. If $G \not\cong K_n$ and $G \not\cong K_n \setminus e$, then $\lambda(K_n, G) \geq 2$ or $\lambda(K_n, G) > \lambda(K_n, K_n \setminus e)$. 

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Proof. Since $G \not\cong K_n, K_n \setminus e, n \geq 3$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $G$. Therefore

$$\lambda(K_n, G) = (\lambda_1 - (n - 1))^2 + (\lambda_2 + 1)^2 + \cdots + (\lambda_n + 1)^2.$$ 

It is easy to see that if one of the following cases holds, then $\lambda(K_n, G) \geq 2$.

Case 1. $\lambda_1 - (n - 1) \leq -\sqrt{2}$.

Case 2. $\lambda_2 \geq \lambda_3 \geq 0$.

Case 3. $\lambda_2 + 1 \geq \sqrt{2}$.

Now, suppose that none of the above cases occurs. Therefore we may assume that $\lambda_1 > n - 1 - \sqrt{2}, \lambda_2 < \sqrt{2} - 1$ and $\lambda_3 < 0$.

Suppose that $\lambda_2 \leq 0$. Thus it follows from Lemma 3.9 and the hypothesis, $G \cong K_1 + K_{n-1}$. Therefore $\lambda(K_n, G) = 2$.

Now suppose that $\lambda_2 > 0$. Thus $\lambda_1 > n - 1 - \sqrt{2}, 0 < \lambda_2 < \sqrt{2} - 1$ and $\lambda_3 < 0$. Hence Theorem 3.10 can be applied.

Case a. $G \cong (\nabla_1(K_1 + K_2))\nabla(K_{n_1, \ldots, n_m})$. If $t = 0$, then $G \cong K_{n_1, \ldots, n_m}$, which is not possible by Theorem 3.8. If $t \geq 2$, then $(K_1 + K_2)\nabla(K_1 + K_2)$ is an induced subgraph of $G$. Since

$$\text{Spec}((K_1 + K_2)\nabla(K_1 + K_2)) = \{3.73205, 4.1421, 26.795, -1, -1, -24.1421\},$$

by Interlacing Theorem $\lambda_3 \geq 0$, a contradiction. Now, suppose that $t = 1$. If there exists $i$ such that $n_i \geq 3$, $K_{1,3}$ is an induced subgraph of $G$. Now Interlacing Theorem implies that $\lambda_3 \geq 0$, a contradiction.

Now, we may assume that $n_i \leq 2$ for all $i$. If $m = 1$ and $n_1 = 1$, $G \cong (K_1 + K_2)\nabla K_1$, then

$$\text{Spec}(G) = \{2.17009, 3.1111, -1, -1, 48.119\}$$

and $\lambda(K_4, G) > 2$. If $m = 1$ and $n_1 = 2$, $G = (K_1 + K_2)\nabla K_2$, then

$$\text{Spec}(G) = \{2.85577, 3.2164, 0, -1, -2.17741\}.$$ 

Hence $\lambda_3 \geq 0$, a contradiction. Thus we may assume that $m \geq 2$. If there exist $i$ and $j$ such that $n_i, n_j \geq 2$, then $C_4$ is an induced subgraph of $G$, and since $\text{Spec}(C_4) = \{2, 0, 0, -2\}$, Interlacing Theorem implies that $\lambda_3 \geq 0$, a contradiction. Thus we can assume that $G \cong (K_1 + K_2)\nabla K_{n-3}$ or $G \cong (K_1 + K_2)\nabla K_{2,1,\ldots,1}$. We may also write $G$ as follows:

1. $G \cong (K_1 + K_2)\nabla K_{n-3} = K_n \setminus \{e, e'\}$
2. $G \cong (K_1 + K_2)\nabla(K_{n-3} \setminus e'') = K_n \setminus \{e, e', e''\}$
where $e, e', e'' \in E(K_n)$, $e$ and $e'$ have a common vertex and $e, e''$ and $e', e''$ are pairwise non-adjacent in $K_n$. If (1) happens then it follows from Lemma 3.7 that $\lambda(K_n, G) > \lambda(K_n, K_n \setminus e)$. Now suppose that (2) happens. Assume $e = \{v_1, v_2\}$, $e' = \{v_1, v_3\}$ and $e'' = \{v_4, v_5\}$, where $v_1, \ldots, v_5 \in V(K_n)$. The induced subgraph on vertices $v_1, \ldots, v_5$ is $(K_1 + K_2)\nabla K_2$. Since

$$\text{Spec}((K_1 + K_2)\nabla K_2) = \{2.85577, 32164, 0, -1, -2.17741\},$$

it follows from Interlacing Theorem that $\lambda_3 \geq 0$, a contradiction.

Now it remains to investigate the following cases:

**Case b.** $G \cong (K_1 + K_{r,s})\nabla K_q$.

**Case c.** $G \cong (K_1 + K_{r,s})\nabla K_{p,q}$.

In both cases b and c, if either of $r$ or $s$ is greater than 1, then an induced subgraph isomorphic to $K_1 + K_{1,2}$ in $G$ occurs. It now follows from Interlacing Theorem that $\lambda_3 \geq 0$ as $\text{Spec}(K_1 + K_{1,2}) = \{\sqrt{2}, 0, 0, -\sqrt{2}\}$. Now we may assume that $r = s = 1$. In both cases b and c, if either of $q$ or $p$ is greater than 1, then an induced subgraph isomorphic to $H = (K_1 + K_2)\nabla K_2$ in $G$ occurs. Since

$$\text{Spec}(H) = \{-2.17741, -1, 0, 32164, 2.85577\}$$

it follows from Interlacing Theorem that $\lambda_3 \geq 0$, a contradiction. Therefore $q = p = 1$ and so $G$ is isomorphic to one the following graphs:

(i) $G \cong G_1 = (K_1 + K_2)\nabla K_1$,

(ii) $G \cong G_2 = (K_1 + K_2)\nabla K_2$.

Since

$$\text{Spec}(G_1) = \{-1.48119, -1, 31111, 2.17009\}$$

and

$$\text{Spec}(G_2) = \{-1.68133, -1, -1, 35793, 3.32340\},$$

$\lambda(K_4, G_1) > 2$ and $\lambda(K_5, G_2) > 2$. This completes the proof. □

**Proof of Theorem 1.3.** By Lemmas 3.6 and 3.7 and Theorem 3.11, it follows that $cs(K_n) = \lambda(K_n, K_n \setminus e) = n^2 + n - n\sqrt{n^2 + 2n - 7} - 2$. This completes the proof. □

4. Cospectrality of complete bipartite graphs

Let $K_{m,n}$ be the complete bipartite graph with parts of sizes $m$ and $n$. It is known that if $G$ is a cospectral mate of $K_{m,n}$, then $G$ is bipartite and it follows from Theorem 3.8 that $G \cong K_{r,s} + tK_1$ for some positive integers $r$ and $s$ and an integer $t \geq 0$, where
r + s + t = n + m and rs = mn. Therefore if m and n are positive integers such that there exist integers r > 0 and s > 0 satisfying rs = mn and r + s < m + n, then cs(K_{m,n}) = 0.

**Proposition 4.1.** Let m and n be positive integers. Then cs(K_{m,n}) > 0 if and only if the minimum of x + y for all positive integers x, y such that xy = mn is attained on \{x, y\} = \{m, n\}.

If n > 0, then cs(K_{n,n}) > 0 by **Proposition 4.1.** We now compute the value of cs(K_{n,n}). We need the following result in the latter computation.

**Theorem 4.2.** (Theorem 2 of [1]) Let G be a graph of order n without isolated vertices. Then 0 < \lambda_2(G) < \frac{1}{3} if and only if G \cong (K_1 + K_2)\overline{K_{n-3}}, where \lambda_2(G) is the second largest eigenvalue of G.

**Proof of Theorem 1.4.** Since for every positive integer m and n

\[ \text{Spec}(K_{m,n}) = \{-\sqrt{mn}, 0, \ldots, 0, \sqrt{mn}\}, \]

it follows that \lambda(K_{n,n}, K_{n-1,n+1}) = 2(n - \sqrt{n^2 - 1})^2. If n = 2, then the result follows from Fig. 3, where the cospectrality of all graphs of order 4 is computed. Now assume that n \geq 3. Suppose, for a contradiction, that G is a graph non-isomorphic to K_{n,n} and K_{n-1,n+1} such that \lambda(K_{n,n}, G) < 2(n - \sqrt{n^2 - 1})^2. Assume \lambda_1 \geq \cdots \geq \lambda_2_n are the eigenvalues of G. Since for n \geq 3, 2(n - \sqrt{n^2 - 1})^2 < \frac{1}{9} we obtain \lambda_2 < \frac{1}{3}. Since for every graph except the complete graph, the second largest eigenvalue is non-negative (see [1, part 3 of Theorem 1]), the latter inequality shows that 0 \leq \lambda_2 < \frac{1}{3}. We can distinguish the following cases:

**Case 1.** Suppose that \lambda_2 = 0. Thus by Theorem 3.8, there exist some positive integers k and n_1, \ldots, n_k and an integer t \geq 0 such that G \cong tK_1 + K_{n_1,\ldots,n_k}. If k = 1, then G \cong \overline{K_{2n}} and so \lambda(K_{n,n}, G) = 2n^2 \geq 18, a contradiction. If k = 2, then G \cong tK_1 + K_{r,s} for some r and s such that r + s = 2n - t. In this case we have \lambda(K_{n,n}, G) = 2(n - \sqrt{rs})^2. It is not hard to see that if \{r, s\} \neq \{n, n\} and \{r, s\} \neq \{n-1, n+1\}, then 2(n - \sqrt{rs})^2 > 2(n - \sqrt{n^2 - 1})^2. Therefore k \geq 3. If n_1 = \cdots = n_k = 1, then G \cong tK_1 + K_{2n-t} and so \lambda(K_{n,n}, G) > 2(n - \sqrt{n^2 - 1})^2, a contradiction. Now we may assume that G has K_{1,1,2} as an induced subgraph. Since \lambda_3(K_{1,1,2}) = -1, then it follows from Interlacing Theorem that \lambda^2_{2n-1} \geq 1 and so \lambda(K_{n,n}, G) \geq 1, a contradiction.

**Case 2.** Assume that 0 < \lambda_2 < \frac{1}{3}. By Theorem 4.2, we conclude that there exists an integer t \geq 0 such that G \cong tK_1 + (K_1 + K_2)\overline{K_{2n-t-3}}. If 2n - t - 3 = 1, then it is easy to \lambda(K_{n,n}, G) > 2(n - \sqrt{n^2 - 1})^2, a contradiction. If 2n - t - 3 > 1, then G has K_{1,1,2} as an induced subgraph and the rest is similar to previous part.

This completes the proof. \(\Box\)
Fig. 1. Cospectrality of graphs with 2 vertices.

Fig. 2. Cospectrality of graphs with 3 vertices.

5. Cospectrality of graphs of order at most 4

In this section (see Figs. 1, 2 and 3), we find the cospectrality of all graphs of order at most 4.

Based on cospectrality of graphs with at most 4 vertices, one can observe the following facts:

1. It is not in general true that if \( cs(G) = \lambda(G, H) \) for some graph \( H \), then \( cs(H) = \lambda(G, H) \); for example \( cs(A_4) = \lambda(A_4, A_3) \) but \( cs(A_3) = \lambda(A_3, A_2) \) (see Fig. 2); also \( cs(B_3) = \lambda(B_3, B_4) \) but \( cs(B_4) = \lambda(B_4, B_9) \) (see Fig. 3).

2. It is not in general true that if \( G \) is a regular graph and \( cs(G) = \lambda(G, H) \) for some graph \( H \), then \( H \) is also regular; for example \( cs(A_4) = \lambda(A_4, A_3) \) (see Fig. 2); also \( cs(B_9) = \lambda(B_9, B_4) \) (see Fig. 3).

The following question is natural to ask.

**Question 5.1.** Let \( G \) and \( H \) be two graphs such that \( \lambda(G, H) = cs(G) = cs(H) \). For which graph theoretical property \( \rho \), if \( G \) has \( \rho \) then so does \( H \)?

It is well-known that if \( \lambda(G, H) = cs(G) = cs(H) = 0 \) (that is \( Spec(G) = Spec(H) \)), then the answer of **Question 5.1** for graph properties such as being bipartite and regularity is positive.

We propose the following question to finish the paper.

**Question 5.2.** Is there a constant \( c \) for which if \( cs(G) = \lambda(G, H) \), then \( |E(G)| - |E(H)| \leq c \)?
Remark 5.3. Concerning Question 5.2 we have the following remarks. By Figs. 2 and 3, $c$ can be chosen 1 for graphs with at most 4 vertices. However for graphs of order 5, it is easy to see that $\text{cs}(K_{2,3}) = 2(\sqrt{6} - 2)^2$ and $\lambda(K_{2,3}, G) = 2(\sqrt{6} - 2)^2$ for some graph $G$ if and only if $G \cong K_{1,4}$ or $G \cong K_{2,2} + K_1$. By similar arguments given in Section 4, it is not hard to see that for all $n \geq 3$, $\text{cs}(K_{n,n+1}) = \lambda(K_{n,n+1}, K_{n-1,n+2}) = 2(\sqrt{n^2 + n - \sqrt{n^2 + n - 2}})^2$ and the graph $K_{n-1,n+2}$ is the only graph $G$ satisfying the equality $\lambda(K_{n,n+1}, G) = \text{cs}(K_{n,n+1})$. Therefore for graphs with at least $2n + 1 \geq 7$ vertices if the constant $c$ exists it is at least 2.

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