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Non-triviality of Tate cohomology for certain classes of finite $p$-groups

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**ABSTRACT**

We prove that the Tate cohomology groups $\hat{H}^n(G/\Phi(G), Z(\Phi(G)))$ are non-trivial, whenever $G$ is a finite $p$-group of class 3, or the $p$th term of the upper central series of $G$ contains $Z(\Phi(G))$. This confirms a conjecture of Schmid for these groups.

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**1. Introduction**

Let $Q$ be a finite group and $A$ be a $Q$-module. We say that $A$ is **cohomologically trivial over $Q$** if the Tate cohomology groups $\hat{H}^n(S, A)$ are trivial for all integers $n$, and all subgroups $S \leq Q$.

The interest in modules of trivial cohomology over finite groups was first influenced by class field theory. Remarkable work in this direction was done by Tate [12], and Nakayama (see [8] and [9, 10]). Similar results, probably with different motivations, were obtained by Gaschütz [5] (the same main result was obtained independently by Uchida [14]). Gaschütz–Uchida result (which we shall use freely in the sequel) states that if $Q$ and $A$ are finite $p$-groups, then $A$ is cohomologically trivial over $Q$ if and only if there exists an integer $n$ such that $\hat{H}^n(Q, A) = 0$. This result was used by Gaschütz to prove his famous result on the existence of non-inner $p$-automorphisms of finite $p$-groups. In several situations, regarding the non-inner automorphisms of finite $p$-groups (see e.g. [3] for explaining the main problem), one encounters cohomology groups arising as follows:

Suppose that $G$ is a finite $p$-group and $N \triangleleft G$, then $Z(N)$ can be viewed as a $G/N$-module with the action $a^xN = x^{-1}ax, a \in Z(N)$ and $x \in G$. Thus we can consider the cohomology groups $\hat{H}^n(G/N, Z(N))$. Schmid has conjectured in [11] that

**Conjecture 1.1 ([11, p. 3]).** For every finite non-abelian $p$-group $G$, $Z(\Phi(G))$ does not have trivial cohomology over $G/\Phi(G)$.

Some investigations of Conjecture 1.1 and related problems can be found in [1, 2]. A finite non-abelian $p$-group which satisfies Conjecture 1.1 will be termed a **Schmid group** and an NS-group otherwise.

Schmid has shown that regular $p$-groups satisfy Conjecture 1.1 (see [11]). More generally, every semi-abelian $p$-group is a Schmid group by [4, Theorem 1]. However, it is shown in [2] that there are NS-groups...
of order $2^8$. These counter examples were found via the following GAP code ([13]):

```gap
f:=function(G)
local H0,ZP,P,i,j,L1,L2;
P:=FrattiniSubgroup(G);
ZP:=Center(P);
L1:=List(RightCosets(G,P),i->Representative(i));
L2:=List(ZP,i->Product(L1,j->iˆj));
H0:=FactorGroup(Centralizer(ZP,G),Group(L2));
return Size(H0);
end;
```

This code computes the size of the cohomology group $\hat{H}^0(G/\Phi(G),Z(\Phi(G)))$, for every given group $G$. So, by Gaschütz–Uchida, a finite non-abelian $p$-group $G$ is an NS-group if, and only if, the corresponding output is zero.

With the above code, one can see that there are exactly 10 NS-groups among the 56092 groups of order $2^8$ which are all of nilpotency class 4. More precisely, these are the groups of order $2^8$ with IdSmallGroup 298, . . . , 307 in GAP library of small groups [13]. There are 10494213 groups of order $2^9$, and surprisingly when applying the above code to them, one finds no NS-groups. So still the conjecture deserves some interest.

A question that remained open since the publication of [2] is whether there exist NS-groups of class 3. The main aim of our paper is to answer the latter question.

**Theorem 1.2.** Every finite non-abelian $p$-group of class at most 3 is a Schmid group.

The above result can be strengthened for $p \geq 3$ (though, the proof is simpler!).

**Theorem 1.3.** Every finite non-abelian $p$-group $G$ satisfying $Z(\Phi(G)) \subseteq Z_p(G)$ is a Schmid group.

Our notations are standard and some of them are as follows. We use $A^Q$ to denote the submodule of $A$ formed by the elements fixed by $Q$. For a subset $B$ of $A$, $C_Q(B)$ denotes the subgroup of $Q$ formed by the $x \in Q$ satisfying $b^x = b$, for all $b \in B$. The map $\tau : A \to A$ defined by $\tau (a) = \prod_{x \in Q} a^x$ will be called the trace map (induced by $Q$). The terms of the lower and the upper central series of $Q$ are denoted by $\gamma_i(Q)$ and $Z_i(Q)$, respectively. $\mathbb{Z}_n$ denotes the cyclic group of order $n$, and any further unexplained notation may produce no ambiguity.

**2. Some consequences of the triviality of the cohomology**

In this section, we assume that $Q$ and $A$ are finite $p$-groups, and $A$ is a $Q$-module.

**Proposition 2.1.** If $A$ is a cohomologically trivial $Q$-module, then

$$[A, Q]/[A, Q, Q] \cong Q/Q' \otimes A^Q.$$  

**Proof.** Let $K$ be the kernel of the trace map $\tau : A \to A$. The condition on $A$ implies that $K = [A, Q]$, and $A^\tau = A^Q$. Hence, we have a short exact sequence of $Q$-modules

$$0 \to K \to A \to A^Q \to 0.$$  

This sequence induces a long exact sequence of (Tate) cohomology groups

$$\cdots \to \hat{H}^n(Q,A) \to \hat{H}^n(Q,A^Q) \to \hat{H}^{n+1}(Q,K) \to \hat{H}^{n+1}(Q,A) \to \cdots$$
As $\hat{H}^{n}(Q, A) = 0$, for all $n$, it follows that
$$
\hat{H}^{n}(Q, A^{Q}) \cong \hat{H}^{n+1}(Q, K).
$$

For $n = -2$, we have $\hat{H}^{-1}(Q, K) = \ker \tau'/[K, Q]$; where $\tau'$ is the restriction of $\tau$ on $K$. Therefore, by definition of $K$, $\ker \tau' = K$, hence $\hat{H}^{-1}(Q, K) = [A, Q]/[A, Q, Q]$. On the other hand, $\hat{H}^{-2}(Q, A^{Q})$ is nothing but the first homology group $H_{1}(Q, A^{Q})$; and since $A^{Q}$ is a trivial $Q$-module, we have $\hat{H}^{-2}(Q, A^{Q}) = Q/Q' \otimes A^{Q}$. Thus $[A, Q]/[A, Q, Q] \cong Q/Q' \otimes A^{Q}$.

**Corollary 2.2.** If $A$ is a cohomologically trivial $Q$-module, then
$$
|A| = |A^{Q}||Q/Q' \otimes A^{Q}||[A, Q, Q]|.
$$

**Proof.** Note that $A/[A, Q] \cong A^{T} = A^{Q}$. The claim now follows from Proposition 2.1.

The following result is due to Schmid (see [11, Proposition 1]).

**Proposition 2.3.** If $A \neq 0$ is a cohomologically trivial over $Q$, then for every subgroup $H$ of $Q$, we have $C_{Q}(A^{H}) = H$.

Assume that we have an NS-group $G$. For every $x \in G - \Phi(G)$, we can consider $A = Z(\Phi(G))$ as a $(\bar{x})$-module, where $\bar{x} = x\Phi(G)$. It follows that $A$ has trivial cohomology over $(\bar{x})$; hence by Proposition 2.3, $C_{(\bar{x})}(A) = 1$. In other words, $x$ does not commute with $A$. This proves the following.

**Lemma 2.4.** For every NS-group $G$, we have $C_{Q}(Z(\Phi(G))) = \Phi(G)$.

### 3. Proof of Theorem 1.3

First, we need the following result (see [7, Corollary 1.2]).

**Lemma 3.1.** If $H$ is a finite $p$-group of class at most $p$, then $\gamma_{2}(H)$ and $H/Z(H)$ have the same exponent.

**Lemma 3.2.** Let $G$ be a finite $p$-group, and let $A = Z(\Phi(G))$. If $A \subseteq Z_{p}(G)$, then for every $x \in G$, the group $(A, x)$ has nilpotency class at most $p$, and $\gamma_{2}((A, x))$ has exponent at most $p$.

**Proof.** We have $\gamma_{2}((A, x)) = \langle [a, x^2] | a \in A, i \in \mathbb{N} \rangle \subseteq Z_{p-1}(G)$. Hence, by induction, $\gamma_{n}((A, x)) = Z_{p-1}(G)$, for $2 \leq n \leq p + 1$. In particular, $\gamma_{p+1}((A, x)) = 1$, as desired.

Let $\bar{x} = x\Phi(G)$, $K = (\bar{x})$, and consider $A$ as a $K$-module. For $n \geq 2$, let $\gamma_{n}(A, K)$ be the subgroup generated by all the commutators $[t_{1}, t_{2}, \ldots, t_{i}]$, with $i \geq n$, $t_{1} \in A$, and the other $t_{j}$’s lie in $A \cup K$ and at least $n - 1$ of them lie in $K$. As $A$ is abelian, it follows that $\gamma_{n}(A, K) = [A, A^{n-1}K]$ which coincides with $\gamma_{n}((A, x))$. As we have seen above, $\gamma_{p}((A, x)) \subseteq Z_{p}(G)$; thus $\gamma_{p}(A, K) \subseteq Z_{p}(G)$. Therefore $K$ acts $p$-centrally on $\gamma_{p}(A, K)$, that is, $K$ fixes every element of order dividing $p$ (if $p = 2$) in $\gamma_{p}(A, K)$. By [7, Theorem 1.1 (iii)], $K$ and $[A, K]$ have the same exponent, so $\exp([A, K]) \leq p$. But, $\gamma_{2}((A, x)) = [A, K]$, the result follows.

**Proof of Theorem 1.3.** Set $A = Z(\Phi(G))$, and assume for a contradiction that $A$ has trivial cohomology over $Q = G/\Phi(G)$. It follows that for every $\bar{x} \in Q$, $A$ is a cohomologically trivial $(\bar{x})$-module. Therefore, $0 = \hat{H}^{0}(\bar{x}, A) = A^{\bar{x}}/A^{1}$, where $\tau$ is the trace map $\tau(a) = a^{1+x+\cdots+x^{p-1}}$, $a \in A$. We claim that $A^{T} \leq Z(G)$. It is straightforward to see that $\tau(a) = (ax^{-1})^{p}x^{p}$, for all $a \in A$. By the Hall-Petrescu formula,

$$
(ax^{-1})^{p} = a^{p}x^{-p}c_{2}^{\binom{p}{2}} \cdots c_{p}^{\binom{p}{p}}
$$
with \( c_i \in \gamma_i((a, x)) \). By Lemma 3.2, we have 
\[
(ax^{-1})^p = a^p x^{-p} c_p,
\]
with \( c_p \in Z(G) \). Thus \( \tau(a) = a^p x^{-p} c_p x^p = a^p c_p \). By Lemmas 3.2 and 3.1, we have \( a^p \in Z((a, x)) \), for all \( x \in G \). Hence, \( a^p \in Z(G) \). Thus \( \tau(a) \in Z(G) \), for all \( a \in A \); this proves the claim. Now, as \( A^{(x)} = A^\tau \), we have \( C_Q(A^{(x)}) = Q \); however, by Proposition 2.3, \( C_Q(A^{(x)}) = \langle x \rangle \), so \( Q = \langle x \rangle \). This holds if and only if \( G \) is cyclic, a contradiction. \( \square \)

4. Proof of Theorem 1.2

By the proof of Theorem 1.3 (or alternatively by [1, Theorem 3.6]), we may assume here that \( p = 2 \) and that our group has class exactly 3.

First, we need the following important reduction from [2].

Lemma 4.1 ([2, Theorem 3.7]). Let \( G \) be a finite non-abelian 2-group of class 3, and let \( A = Z(\Phi(G)) \). Then \( G \) is an NS-group if and only if the following conditions hold:

1. \( C_G(A) = \Phi(G) \).
2. \( A^4 = Z(G) \).
3. \( d(A) = 2 \).

Lemma 4.2. If \( G \) is an NS-group of class 3, and \( A \) denotes \( Z(\Phi(G)) \), then
\[
A/ Z(G) \cong Z_4 \oplus Z_2.
\]

Proof. Let \( Q = G/\Phi(G) \). First, we claim that \( A^Q = Z(G) \). Indeed, if \( Q = \langle x_1 \Phi(G), x_2 \Phi(G) \rangle \), then \( G = \langle x_1, x_2 \rangle \); hence \( A^Q = A \cap Z(G) \). But by Lemma 2.4, \( Z(G) \) lies in \( A \), the claim follows. By [2, Lemma 3.9], \( Z(G) \) is cyclic, and by Lemma 4.1(3), \( Q = Z_2 \oplus Z_2 \); therefore \( Q/ Q \otimes A^Q \cong Z_2 \oplus Z_2 \). It follows from Corollary 2.2 that \( |A| = 4|Z(G)||[A, Q, Q]| \). Let \( a \in A \) and \( x \in Q \). As \( [A, Q, Q] \subseteq Z(G) \), the map \( t \mapsto [a, x, t] \) is a group homomorphism from \( Q \) to \( [A, Q, Q] \). Hence, \( [a, x, t]^2 = [a, x, t^2] = 1 \), for all \( t \in Q \). This shows that \( [A, Q, Q] \) has exponent 2 (\( [A, Q, Q] \) is not trivial by Theorem 1.3). Since \( Z(G) \) is cyclic, we have \( |[A, Q, Q]| = 2 \). This proves that \( A/ Z(G) \) has order 8. Among the abelian groups of order 8, only \( Z_4 \oplus Z_2 \) has exponent 4, but by Lemma 4.1(2), \( A/ Z(G) \) has exponent 4, the result follows. \( \square \)

Lemma 4.3. With the assumptions of Lemma 4.2, there exist \( a, b \in A \) such that
\[
A = \langle a \rangle \oplus \langle b \rangle
\]
with \( b \) of order 2, and \( a \) of order \( 2^{r+2} \), where \( 2^r = |Z(G)| \).

Proof. We can write \( A = \langle a \rangle \oplus B \), where \( a \) is an element of \( A \) of maximal order. As \( A^4 = \langle a^4 \rangle \oplus B^4 \), and \( A^4 = Z(G) \) is cyclic, we have \( B^4 = 1 \). Now we have \( Z(G) = A^4 = \langle a^4 \rangle \), so \( a^4 \) has order \( 2^r \) and \( a \) has order \( 2^{r+2} \). By Lemma 4.2,
\[
(\langle a \rangle/\langle a^4 \rangle) \oplus B \cong Z_4 \oplus Z_2,
\]
hence by the unique factorization theorem of finite abelian groups, we have \( B \cong Z_2 \). If we set \( B = \langle b \rangle \), then \( A = \langle a \rangle \oplus \langle b \rangle \), and \( a, b \) are as desired. \( \square \)

Lemma 4.4. With the assumptions and notation of Lemma 4.3, there exists \( x, y \in G \) such that \( G = \langle x, y \rangle \), and \( x \) centralizes \( b \).

Proof. Let \( A_1 = \Omega_1(A) = \langle a^{r+1} \rangle \oplus \langle b \rangle \). Then \( A_1 \) is a normal elementary abelian subgroup of \( G \). Thus \( G/C_G(A_1) \) can be embedded in \( \text{Aut}(A_1) = GL(2, 2) \). The 2-part of \( |GL(2, 2)| \) is 2, so \( |G: C_G(A_1)| \leq 2 \).
As $b$ is not central, $|G : C_G(A_1)| = 2$; that is, $C_G(A_1)$ is a maximal subgroup of $G$. Now any $y \in G - C_G(A_1)$ and $x \in C_G(A_1) - \Phi(G)$ do the claim.

In conclusion, if $G$ is an NS-group of class 3, then $G$ is a 2-group which satisfies:
- $G = \langle x, y \rangle$;
- $A := Z(\Phi(G)) = \langle a \rangle \oplus \langle b \rangle$;
- $b^2 = [a, x] = 1$, and $(a^4) = Z(G)$.

Now, consider $A$ as a $(\bar{x})$-module. It follows that $\hat{H}^0((\bar{x}), A) = 0$. We have $A^{(\bar{x})}$ contains $(a^4) \oplus \langle b \rangle$, so it is not cyclic; but $A^{(\bar{x})}$ coincides with the image of the trace map $\tau' : A \to A$, $\tau'(t) = tt^x$. As $\tau'(b) = 1$, $A^{\tau} = A^{(\bar{x})}$ is cyclic; a contradiction. This completes the proof of Theorem 1.2.

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