Characterization of $\text{SL}(2, q)$ by its Non-commuting Graph*

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Abstract. Let $G$ be a non-abelian group and $Z(G)$ be its center. The non-commuting graph $\mathcal{A}_G$ of $G$ is the graph whose vertex set is $G \setminus Z(G)$ and two vertices are joined by an edge if they do not commute. Let $\text{SL}(2, q)$ be the special linear group of degree 2 over the finite field of order $q$. In this paper we prove that if $G$ is a group such that $\mathcal{A}_G \cong \mathcal{A}_{\text{SL}(2, q)}$ for some prime power $q \geq 2$, then $G \cong \text{SL}(2, q)$.

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1. Introduction and results

Let $G$ be a non-abelian group and $Z(G)$ be its center. One can associate with $G$ a graph whose vertex set is $G \setminus Z(G)$ and two vertices are joined by an edge whenever they do not commute. We call this graph the non-commuting graph of $G$ and it will be denoted by $\mathcal{A}_G$. The non-commuting graph $\mathcal{A}_G$ was first introduced by Paul Erdős [4] to formulate the following question: If every complete subgraph of $\mathcal{A}_G$ is finite, is there a finite bound on the cardinalities of complete subgraphs of $\mathcal{A}_G$?

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Neumann [4] answered positively Erdős question by proving that \(|G : Z(G)| = n\) is finite and \(n\) is obviously the requested finite bound.

The non-commuting graph has been studied by many people (see e.g., [1], [3] and [5]). It is proved in [7] (resp. in [8]) that if \(G\) is a finite group with \(A_G \cong A_{\text{PGL}(2,q)}\) (resp. \(A_G \cong A_{\text{SL}(2,q)}\)), then \(G \cong \text{PSL}(2,q)\) (resp., \(G \cong A_{10}\)). For any prime power \(q\), let \(\text{GL}(2,q)\) (resp. \(\text{SL}(2,q)\)) be the general (resp. special) linear group of degree 2 over the finite field of order \(q\). In this paper we study the groups whose non-commuting graphs are isomorphic to either \(\text{GL}(2,q)\) or \(\text{SL}(2,q)\). Our main results are the following.

**Theorem 1.1.** Let \(G\) be a group such that \(A_G \cong A_{\text{GL}(2,q)}\) for some prime power \(q > 3\). Then \(G/Z(G) \cong \text{PGL}(2,q)\), \(G' \cong \text{SL}(2,q)\) and \(Z(G)\) is of order \(q - 1\). In particular, if \(q\) is even, then \(G = G' \times Z(G)\).

**Theorem 1.2.** Let \(G\) be a group such that \(A_G \cong A_{\text{SL}(2,q)}\) for some prime power \(q \geq 2\). Then \(G \cong \text{SL}(2,q)\).

For any prime power \(q\), we denote by \(\text{PGL}(2,q)\) (resp. \(\text{PSL}(2,q)\)) the projective general (resp. special) linear group of degree 2 over the finite field of order \(q\).

2. Proofs

Here for convenience, we remind some of the properties of non-commuting graphs and common properties of groups with isomorphic non-commuting graphs.

Let \(G\) and \(H\) be two non-abelian groups such that \(A_G \cong A_H\). By Lemma 3.1 of [1], if one of \(G\) or \(H\) is finite, then so is the other. The order of \(A_G\) is \(|G| - |Z(G)|\) and so \(|G| - |Z(G)| = |H| - |Z(H)|\). The degree of a vertex \(x\) in \(A_G\) is equal to \(|G| - |C_G(x)|\). Thus the multisets of degrees of vertices of two graphs \(A_G\) and \(A_H\) are the same.

A non-abelian group \(G\) is called an AC-group, if the centralizer \(C_G(x)\) of every non-central element \(x\) of \(G\) is abelian.

Recall that a non-empty subset \(X\) of the vertices of a simple graph \(\Gamma\) is called independent if every two distinct vertices of \(X\) are not joint by an edge in \(\Gamma\). Thus an independent set \(S\) of the non-commuting graph of a group is a set of pairwise commuting non-central elements of the group.

**Lemma 2.1.** Let \(G\) and \(H\) be two finite non-abelian groups with \(A_G \cong A_H\).

1. If \(|G| = |H|\), then the multisets (sets with multiplicities) \(\{C_G(g) : g \in G \setminus Z(G)\}\) and \(\{C_H(h) : h \in H \setminus Z(H)\}\) are equal.
2. If \(G\) is an AC-group, then \(H\) is also an AC-group.

**Proof.** (1) It is straightforward, if we note that the set of non-adjacent vertices to a vertex \(x\) in the non-commuting graph \(H\) is \(C_H(x) \setminus Z(H)\), and note that from \(|G| = |H|\) we also have \(|Z(G)| = |Z(H)|\), since \(|H| - |Z(H)| = |G| - |Z(G)|\).

(2) Note that a subgroup \(S\) of a non-abelian group \(K\) is abelian if and only if either \(S \setminus Z(S)\) is empty or \(S \setminus Z(S)\) is an independent set in the non-commuting
graph $\mathcal{A}_G$. Let $\phi$ be a graph isomorphism from $\mathcal{A}_H$ onto $\mathcal{A}_G$. Then it is easy to see that for each $h \in H \setminus Z(H)$,

$$C_H(h) \setminus Z(H) = \phi^{-1}(C_G(\phi(h)) \setminus Z(G)).$$

(*)

Now since $G$ is an AC-group, $C_G(g)$ is abelian for all $g \in G \setminus Z(G)$ and so it follows from (*) and the remark above that $C_H(h)$ is abelian. Hence $H$ is also an AC-group.

Finite non-nilpotent AC-groups were completely characterized by Schmidt [6]. We use the following results in our proofs.

**Theorem 2.2.** ([6, Satz 5.9.]) Let $G$ be a finite non-solvable group. Then $G$ is an AC-group if and only if $G$ satisfies one of the following conditions:

1. $G/Z(G) \cong \text{PSL}(2, p^n)$ and $G' \cong \text{SL}(2, p^n)$, where $p$ is a prime and $p^n > 3$.
2. $G/Z(G) \cong \text{PGL}(2, p^n)$ and $G' \cong \text{SL}(2, p^n)$, where $p$ is a prime and $p^n > 3$.
3. $G/Z(G) \cong \text{PSL}(2, 9)$ and $G'$ is a covering group of $A_5$. In particular, $G'$ is isomorphic to

$$\mathcal{A} \cong \langle c_1, c_2, c_3, c_4, k | c_1^3 = c_2^2 = c_3^2 = c_4^2 = (c_1 c_2)^3 = (c_1 c_3)^2 = k^3, (c_1 c_4)^2 = k, c_2 c_4 = k^3 c_2 c_3, k c_4 = c_4 k \rangle.$$  

For a finite simple graph $I$, we denote by $\omega(I)$ the maximum size of a complete subgraph of $I$. So $\omega(\mathcal{A}_G)$ is the maximum number of pairwise non-commuting elements in a finite non-abelian group $G$.

**Theorem 2.3.** ([Satz 5.12 of [6]]) Let $G$ be a finite non-abelian solvable group. Then $G$ is an AC-group if and only if $G$ satisfies one of the following properties:

1. $G$ is non-nilpotent and it has an abelian normal subgroup $N$ of prime index and $\omega(\mathcal{A}_G) = |N : Z(G)| + 1$.
2. $G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, and $F$ and $K$ are abelian subgroups of $G$; and $\omega(\mathcal{A}_G) = |F : Z(G)| + 1$.
3. $G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, and $K$ is an abelian subgroup of $G$, $Z(F) = Z(G)$, and $F/Z(G)$ is of prime power order; and $\omega(\mathcal{A}_G) = |F : Z(G)| + \omega(\mathcal{A}_F)$.
4. $G/Z(G) \cong S_4$ and $V$ is a non-abelian subgroup of $G$ such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$; and $\omega(\mathcal{A}_G) = 13$.
5. $G = A \times P$, where $A$ is an abelian subgroup and $P$ is an AC-subgroup of prime power order.
Proof of Theorems 1.1 and 1.2. Let \( q_1 = p_1^{n_1} > 3 \) and \( q_2 = p_2^{n_2} \geq 2 \), where \( p_1 \) and \( p_2 \) are two prime numbers. Let \( M_1 = \text{GL}(2, q_1) \) and \( M_2 = \text{SL}(2, q_2) \) and suppose that \( G_1 \) and \( G_2 \) are two groups such that \( A_{G_i} \cong M_i \) for \( i = 1, 2 \).

If \( q_1 = 2 \), then \( M_2 \cong S_3 \) is the symmetric group of degree 3 and so by Proposition 3.2 of [1], \( G_2 \cong M_2 \). If \( q_2 = 3 \), then \( M_2 \) is a group of order 24 and its center has order 2. As there is some element \( g \) with \( |C_{G_2}(g)| = 6 \), we see that there is no normal Sylow 3-subgroup in \( G_2 \). Hence \( G_2/Z(G_2) \cong A_4 \). So either \( G_2 \cong M_2 \) or \( Z_2 \times A_4 \). But as there are elements \( h \in G_2 \) with \( |C_{G_2}(h)| = 4 \), we have \( G_2 \cong M_2 \).

Now let \( q_2 > 3 \). If \( q_2 \) is even, then \( \text{PSL}(2, q_2) \cong M_2 \) and so \( A_{G_2} \cong A_{\text{PSL}(2, q_2)} \). Then by Corollary 5.3 of [1], \( G_2 \cong \text{PSL}(2, q_2) \cong M_2 \). Therefore we may assume that \( q_2 \geq 5 \) is odd.

By Proposition 4.3 of [1], \( |G_i| = |M_i| \) for \( i = 1, 2 \). By Lemma 3.5 of [1], \( M_i \)'s are AC-groups and so by Lemma 2.1(2) \( G_i \)'s are also AC-groups. Now since \( A_{G_i} \cong M_i \) and \( |G_i| = |M_i| \), by Lemma 2.1 we have the following equality between multisets
\[
W_i = \{ |C_{G_i}(x)| \mid x \in G_i \setminus Z(G_i) \} = \{ |C_{M_i}(g)| \mid g \in M_i \setminus Z(M_i) \}, \quad i = 1, 2.
\]

Also, since the order of two graphs \( A_{G_i} \) and \( A_{M_i} \) are the same, we have that
\[
|G_i| - |Z(G_i)| = |M_i| - |Z(M_i)| \quad \text{and so} \quad |Z(G_i)| = |Z(M_i)| \quad (i = 1, 2).
\]

Therefore, it follows from Propositions 3.14 and 3.26 of [1] that the multiset \( W_1 \) (resp. \( W_2 \)) consists of three distinct integers \( q_1(q_1-1)^2 \) (resp. \( q_2(q_2-1)/2 \), \( q_2^2-1 \) (resp. \( q_2(q_2+1)/2 \)) and \( q_1(q_1-1) \) (resp. \( q_2 \)) with multiplicities \( \frac{q_1(q_1+1)}{2} \), \( \frac{q_2(q_2+1)}{2} \) and \( q_1+1 \), respectively.

We claim that both groups \( G_1 \) and \( G_2 \) are not nilpotent. Suppose, for a contradiction, that \( G_i \) is nilpotent, then so is \( G_i/Z(G_i) \). Therefore \( G_i/Z(G_i) \) has only one Sylow \( p_i \)-subgroup. Since \( W_1 \) (resp., \( W_2 \)) contains \( q_1+1 \) elements all equal to \( q_1(q_1-1) \) (resp., \( q_2(q_2+1)/2 \)), there exist two non-central elements \( x_1 \) and \( y_1 \) in \( G_1 \) (resp., \( x_2 \) and \( y_2 \) in \( G_2 \)) such that \( C_{G_1}(x_1) \neq C_{G_1}(y_1) \) and \( |C_{G_1}(x_1)| = |C_{G_1}(y_1)| = q_1(q_1-1) \) (resp., \( C_{G_2}(x_2) \neq C_{G_2}(y_2) \) and \( |C_{G_2}(x_2)| = |C_{G_2}(y_2)| = 2q_2 \)). Since \( C_{G_1}(x_1)/Z(G_1) \) and \( C_{G_2}(y_2)/Z(G_2) \) are of the same order \( q_1 \), they are Sylow \( p_1 \)-subgroups of \( G_1/Z(G_1) \) and \( G_2 \), respectively.

Now we prove that both \( G_1 \) and \( G_2 \) cannot be solvable. Suppose, for a contradiction, that \( G_i \)'s are solvable. Then since \( G_i \) are not nilpotent, it follows from Theorem 2.3 that \( G_i \)'s satisfy one of properties (1)-(4) in Theorem 2.3. Since \( q_1 > 3 \) is a prime power and \( q_2 \) is odd, both of \( |G_i/Z(G_i)| = q_1(q_1^2 - 1) \) and \( |G_2/Z(G_2)| = \frac{q_2(q_2^2 - 1)}{2} \) cannot equal to \( |S_4| = 24 \). Therefore \( G_i \)'s do not satisfy (4). If \( G_i \) satisfies either (1) or (2), then \( W_i \) contains only two distinct elements, since in the case (1), if \( x \in N \setminus Z(G_i) \), then \( C_{G_i}(x) = N \); and if \( x \in G \setminus N \) then \( C_N(x) = G \); and so \( |C_{G_1}(x)| \in \{ |G_1 : N| \} \). For every non-central element \( x \in G_i \), and in the case (3), \( |C_{G_i}(x)| \in \{ |K|, |F| \} \). This is not possible, since \( W_i \) has exactly three distinct elements.

Finally, suppose that \( G_i \) satisfies (3). Note that \( C_{G_i}(x) = C_F(x) \) for every non-central element \( x \in F \) and \( C_{G_i}(x) \) is equal to the conjugate of \( K \) which contains the non-central element \( x \). It follows that the three distinct elements of the multiset
$W'_i = \{ u/|Z(G)| \mid u \in W_i \}$ are $|K/Z(G_i)|$, $r^k$, $r^t$, where $|F/Z(G)| = r^m$ and $r$ is a prime number. This is impossible, since no two of the numbers $q_1$, $q_1 + 1$ or $q_1 - 1$ (resp., $q_2$, $(q_2 + 1)/2$ or $(q_2 - 1)/2$) can simultaneously be powers of the same prime.

Hence $G_i$'s are finite non-solvable AC-groups. By Theorem 2.2, $G_i$'s satisfy one of the conditions (1)–(4) stated in Theorem 2.2. If $G_i$ satisfies (3), then as $A_6$ has self-centralizing elements of order 4 and 5, $G_i$ contains two elements $x_i, y_i$ such that $|G_i(x_i)| = 4$ and $|G_i(y_i)| = 5$. This implies that $q_i \in \{4, 5\}$ and $q_2 = 9$. Therefore $|G_i/Z(G_i)| = 4 \cdot (4^2 - 1) = 4 \cdot (5^2 - 1)$, which is impossible, since $|G_i/Z(G_i)| = |\text{PSL}(2, 9)| = \frac{9 \cdot (9^2 - 1)}{2}$. Since $M_2 = \text{SL}(2, 9)$, $|Z(M_2)| = 2$. But $3$ divides $Z(G_i)$ by Theorem 2.2, a contradiction.

If $G_i$ satisfies (4), then as $\text{PGL}(2, 9)$ contains self-centralizing elements of order 8 and 10, $G_i$ contains two elements $x_i$ and $y_i$ such that $|C_{G_i}(x_i)| = 8$ and $|C_{G_i}(y_i)| = 10$.

It follows that $\{8, 10\} \subset \{q_1, \frac{q_2 - 1}{2}, \frac{q_2 + 1}{2}\}$, which is a contradiction as $q_1$ is a prime power; and for $i = 1$, it follows that $q_1 = 9$. Hence $|Z(M_1)| = 8$. But 3 divides $Z(G_i)$, a contradiction. Thus $G_i$ does not satisfy both (3) and (4).

Now suppose that $G_i$ satisfies either (1) or (2). The group $\text{PGL}(2, r^m)$ (resp. $\text{PSL}(2, r^m)$) has a partition $\mathcal{P}$ consisting of $r^m + 1$ Sylow $r$-subgroups, $\frac{(r^m + 1) \cdot r^m}{2}$ cyclic subgroups of order $r^m - 1$ (resp. $r^m - 1$, $r^m + 1$) Sylow $r$-subgroups of order $r^m + 1$ (resp. $r^m - 1$, $r^m + 1$) (see pp. 185–187 and p. 193 of [2]).

Now [5, 5.3.3] in p. 112 states that if $x \in G_i \setminus Z(G_i)$, then $C_{G_i}(x_i)/Z(G_i)$ belongs to $\mathcal{P}$. Suppose that $G_i/Z(G_i) \cong \text{PGL}(2, r^m)$ (resp. $\text{PSL}(2, r^m)$). Then there exist elements $x_1, x_2, x_3 \in G_i \setminus Z(G_i)$ such that $|C_{G_i}(x_1)/Z(G_i)| = r^m$, $|C_{G_i}(x_2)/Z(G_i)| = r^m + 1$ (resp. $\frac{(r^m + 1) \cdot r^m}{2}$, $\frac{(r^m + 1) \cdot r^m}{2}$).

Therefore, if $G_i/Z(G_i) \cong \text{PGL}(2, r^m)$ (resp. $\text{PSL}(2, r^m)$), then $\{q_1 - 1, q_1, q_1 + 1\} = \{r^m - 1, r^m, r^m + 1\}$ (resp. $\{r^m - 1, r^m, r^m + 1\}$) $\{r^m - 1, r^m, r^m + 1\}$ (resp. $\frac{(r^m + 1) \cdot r^m}{2}$, $\frac{(r^m + 1) \cdot r^m}{2}$).

It follows that, if $G_2/Z(G_2) \cong \text{PGL}(2, r^m)$ then $q_2 = r^m + 1$, $q_2 - 1 = r^m$ and $q_2 = r^m - 1$. Since $q_2 \geq 5$, we have a contradiction as $3 \leq q_2 - q_2 - 1 = r^m + 1 - r^m + 1 = 2$. Hence $G_2/Z(G_2) \cong \text{PGL}(2, r^m)$, $G_2' \cong \text{SL}(2, r^m)$ and $r^m = q_2$.

Now since $|G_2| = |G_2'| = |M_2|$, we have that $G_2 \cong M_2 = \text{SL}(2, q_2)$. This completes the proof of Theorem 1.2.

Now if $G_1/Z(G_1) \cong \text{PGL}(2, r^m)$ (resp. $\text{PSL}(2, r^m)$), it follows that $q_1 = r^m$ (resp. $q_1 = 2^m$). Since $\text{PSL}(2, 2^m) \cong \text{PGL}(2, 2^m)$, we have if $G_i$ satisfies either (1) or (2), then $G_1/Z(G_1) \cong \text{PGL}(2, q_1)$ and $G_1' \cong \text{SL}(2, q_1)$.

Therefore $G_1$ is a group satisfying the following conditions:

$G_1/Z(G_1) \cong \text{PGL}(2, q_1)$ (●), $G_1' \cong \text{SL}(2, q_1)$ and $|Z(G_1)| = q_1 - 1$.

If $q_1 = 2^m$ for some integer $m > 1$, then $\text{SL}(2, q_1) \cong \text{PGL}(2, q_1) \cong \text{PSL}(2, q_1)$. Thus as $\text{PSL}(2, q_1)$ is a non-abelian simple group, it follows from (●) that $G_1 =
and since $G_1'$ is also non-abelian simple, $G_1' \cap Z(G_1) = 1$. Therefore $G_1 = G_1' \times Z(G_1)$. This completes the proof of Theorem 1.1.

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References


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