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On noninner automorphisms of 2-generator finite $p$-groups

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ABSTRACT
A longstanding conjecture asserts that every finite nonabelian $p$-group admits a noninner automorphism of order $p$. In this paper we give some necessary conditions for a possible counterexample $G$ to this conjecture, in the case when $G$ is a 2-generator finite $p$-group. Then we show that every 2-generator finite $p$-group with abelian Frattini subgroup has a noninner automorphism of order $p$.

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1. Introduction

Throughout the paper $p$ denotes a prime number. For a finite $p$-group $G$, $Z(G)$ and $\Phi(G)$ denote the center and the Frattini subgroup of $G$, respectively. By $d(G)$ we denote the minimum number of generators of the group $G$ and $rk(G)$ stands for its rank (i.e. the minimum number $r$ such that every subgroup of $G$ can be generated by at most $r$ elements). The subgroup of $G$ generated by all elements of order $p$ is denoted by $\Omega_1(G)$ and for $i \geq 1$, $G^{p^i} = \langle g^{p^i} \mid g \in G \rangle$. We also use $Z_i(G)$ and $\gamma_i(G)$ to denote the $i$-th terms of the upper and lower central series of $G$, respectively.

One of the fundamental properties of finite $p$-groups is that, with the exception of the groups of order at most $p$, they always have a noninner automorphism of $p$-power order [9]. A well known related conjecture states that every finite nonabelian $p$-group admits a noninner automorphism of order $p$ (See [14, Problem 4.13]). This conjecture was established for various classes of $p$-groups, including; regular $p$-groups [8, 17], $p$-groups of nilpotency class 2 or 3 [1, 5, 13], $p$-groups of coclass 2 [6], $p$-groups of coclass 3 ($p \neq 3$) [16], $p$-groups with cyclic commutator subgroup [12], and $p$-groups $G$ in which $G/Z(G)$ is powerful [2], or $C_G(Z(\Phi(G))) \neq \Phi(G)$ [8]. For other results on the conjecture, see [3, 4, 10, 11, 18].

One can observe that in the most of the above mentioned cases, when the minimum number of generators of a group increases then the existence of a noninner automorphism of order $p$ becomes more apparent. This suggests that it is more appropriate to consider 2-generator $p$-groups for establishing the validity, or finding a minimal counterexample of the conjecture.

The object of this paper is to prove the following results.

Theorem 1. Let $G$ be a 2-generator finite $p$-group of odd order. If $G$ has no noninner automorphism of order $p$, then
(a) $Z(\Phi(G)) \leq C_G(G^p \gamma_3(G)) = Z(G^p \gamma_3(G))$,
(b) $d\left(\frac{Z_2(G) \cap Z(G^p \gamma_3(G))}{Z(G)}\right) \geq 4d(Z(G))$.

Theorem 2. Every 2-generator finite $p$-group with abelian Frattini subgroup has a noninner automorphism of order $p$. 

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2. Proofs of the main results

In the following remark we collect some existing necessary conditions for possible counterexamples to the conjecture. We will later often use them without further references. First, recall that a $p$-group $G$ is called powerful, if $\gamma_2(G) \leq G^p$, when $p$ is odd, and $\gamma_2(G) \leq G^4$, when $p = 2$.

**Remark 1.** Let $G$ be a finite nonabelian $p$-group with no noninner automorphism of order $p$. Then $G$ must have the following properties.

i. The nilpotency class of $G$ is at least 4 [1, 5, 13].

ii. $G/Z(G)$ is not powerful [2, Theorem 2.6]. Thus $G$ is not powerful too.

iii. The commutator subgroup $\gamma_2(G)$ of $G$ is not cyclic [12, Theorem 1.1].

iv. By the main result of [8], $G$ is strongly Frattinian that is

$$\quad C_G(Z(\Phi(G))) = \Phi(G).$$

v. It is observed in [11, Theorem 1.1] that

$$\quad Z_2^* G \leq C_G(Z_2^* G) \leq \Phi(G),$$

where $Z_2^* G = \{ g \in Z_2(G) \mid g^p \in Z(G) \}$. (In fact $p$-groups satisfying (2), form a subclass of strongly Frattinian $p$-groups).

vi. By [2, Lemma 2.2], the center of $G/Z(G)$ has the following limitation,

$$\quad d(Z_2(G)/Z(G)) = d(G)d(Z(G)).$$

**Remark 2.** It follows from (2) that $Z_2^* G$ is abelian and since $\Omega_1(Z_2(G)) = \Omega_1(Z_2^* G)$, we get $d(\Omega_1(Z_2(G))) \geq d(G)d(Z(G))$, by (3). Thus

$$\quad \Omega_1(Z_2(G)) = \Omega_1(Z(G)) \times V,$$

for some nontrivial subgroup $V$ of $\Omega_1(Z_2(G))$.

We also need the following known identities. First, if $G$ is a group and $x, y \in G$, then by Hall's compilation formula there exists elements $c_i = c_i(x, y) \in \gamma_1((x, y))$ such that

$$\quad x^n y^n = (xy)^n c_2^{(i)} c_3^{(i)} \cdots c_n^{(i)}.$$

for all $n \in \mathbb{N}$. Second, if $G$ is metabelian group and $x, y \in G$, then for each integer $n$ we have

$$\quad [x, y^n] = \Pi_{i=1}^n [x, y]^{(i)}.$$

Let $N(G)$ denote the normal subgroup $G^p \gamma_3(G)$, when $p > 2$ and $G^4 \gamma_3(G)$, when $p = 2$. We fix this notation throughout the paper.

**Lemma 1.** Let $G$ be a 2-generator finite $p$-group. Then

$$\quad [N(G), G] \leq \gamma_2(G)^p \gamma_4(G).$$

**Proof.** First, assume that $G$ is metabelian. Then by applying (5) and (6), one can see that (7) holds for $G$.

Then, note that every 2-generator $p$-group of nilpotency class at most 4 is metabelian. Now let $G$ be a 2-generator finite $p$-group. Then $\overline{G} = G/\gamma_5(G)$ is metabelian and therefore

$$\quad [N(\overline{G}), \overline{G}] \leq \gamma_2(\overline{G})^p \gamma_4(\overline{G}).$$

This implies that (7) holds for every 2-generator finite $p$-group $G$. \qed
Lemma 2. Let $G$ be a 2-generator finite $p$-group. If
\[
\gamma_3(G) \leq \gamma_2(G)^2 \gamma_4(G) \Omega_1(Z(G)),
\] (8)
then $G$ has a noninner automorphism of order $p$. In particular, if for some generating set $\{g, h\}$ of $G$ either of the following conditions holds, then $G$ has a noninner automorphism of order $p$.

(a) $p \geq 2$ and $[g, h] \in Z_2^*(G)N(G)$,
(b) $p = 2$ and $g^2[gh, h] \in Z_2^*(G)N(G)$ or
(c) $p = 2$ and $g^2, h^2 \in Z_2^*(G)N(G)$.

\textbf{Proof.} Let $G$ be a 2-generator finite $p$-group with no noninner automorphism of order $p$. Then we may assume that the nilpotency class of $G$ is greater than 3.

First, suppose that (8) holds. Thus we have
\[
\gamma_4(G) \leq \{\gamma_3(G), G\} \leq \{\gamma_2(G)^2 \gamma_4(G) \Omega_1(Z(G)), G\} \leq \gamma_2(G)^2 \gamma_5(G).
\]
Repeating this argument, we obtain $\gamma_4(G) \leq \gamma_2(G)^p$ and $\gamma_3(G) \leq \gamma_2(G)^p \Omega_1(Z(G))$. Then $\gamma_3(G/Z(G)) \leq \gamma_2(G/Z(G))^p \leq \Phi(\gamma_2(G/Z(G)))$ and since
\[
\gamma_2 \left( \frac{G}{Z(G)} \right) = \gamma_2(G)Z(G)/Z(G) = \left\{ [a, b]Z(G), \gamma_3 \left( \frac{G}{Z(G)} \right) \right\},
\]
we have $\gamma_2(G/Z(G))$ is cyclic. Therefore
\[
rk \left( \frac{G}{Z(G)} \right) \leq d \left( \frac{G}{\gamma_2(G)Z(G)} \right) + d \left( \frac{\gamma_2(G)Z(G)}{Z(G)} \right) = 3.
\]
Hence $Z(G)$ is cyclic, by Eq. (3). Since $\Omega_1(Z(G)) \leq \gamma_4(G)$, condition (8) reduces to $\gamma_3(G) \leq \gamma_2(G)^p$. Therefore $\gamma_3(G) \leq \Phi(\gamma_2(G))$ and it follows that $\gamma_2(G)$ is cyclic, which is a contradiction.

For the second assertion, it suffices to show that conditions (a), (b) and (c) imply (8). If condition (a) holds, then $\gamma_2(G) \leq Z_2^*(G)N(G)$ and by applying Lemma 1, one can see that (8) holds. If condition (b) holds, then $G = \langle g, h \rangle$. Thus by Lemma 1,
\[
[g, h, g] = [g^2[g, h], g] \in \gamma_2(G)^2 \gamma_4(G) \Omega_1(Z(G)).
\]
Moreover, since $[g^2[g, h], h] = [g, h, g][g, h, h]$ mod $\gamma_2(G)^2 \gamma_4(G)$, it follows from (8) and Lemma 1, that
\[
[g, h, h] \in \gamma_2(G)^2 \gamma_4(G) \Omega_1(Z(G)).
\]
Now, (8) and (**) imply that $G$ satisfies (8). A similar argument shows that (c) implies (8). This completes the proof.

\textbf{Proposition 1.} Let $G = \langle a, b \rangle$ be a 2-generator finite $p$-group of odd order which is not powerful. Suppose that $G$ has the presentation $G = F/R$, where $F$ is the free group on the set $\{a, b\}$ and $R$ is the normal subgroup of relations. Then $R \subseteq F^p \gamma_3(F)$.

\textbf{Proof.} Let $r \in R$. Then $r = d^b\{[a, b]^k, c\}$, for some integers $i, j, k$ and $c \in \gamma_3(F)$. If $gcd(i, p) = 1$, then $G = F/R = \langle a^i, b \rangle R/R = \langle b \rangle R/R = \langle b \rangle$ is cyclic, which contradicts the assumption. Hence we have $p|i$ and similarly $p|j$. If $gcd(k, p) = 1$, then $[a, b] \in F^p \gamma_3(F)R$. Thus $\gamma_2(F) \leq F^p \gamma_3(F)R$. So $\gamma_2(G) \leq G^p \gamma_3(G)$ and it implies that $G$ is powerful, a contradiction. Therefore $p|k$ and it follows that $r \in F^p \gamma_3(F)$.

\textbf{Proof of Theorem 1.} Let $G = \langle a, b \rangle$ be a 2-generator finite $p$-group of odd order and suppose that $G$ has no noninner automorphism of order $p$. Then we may assume that the nilpotency class of $G$ is at least 4. For the sake of clarity we divide the proof in several claims.
Claim 1. $Z^2_2(G) \subseteq N(G)$.

We know that $Z^2_2(G) \subseteq \Phi(G)$. If $Z^2_2(G) \not\subseteq N(G)$, then $\gamma_2(G) \subseteq \Phi(G) = Z^2_2(G)N(G)$. Now it follows from Lemma 2 that $G$ has a noninner automorphism of order $p$, a contradiction. So the claim holds.

Claim 2. Let $u, v \in \Omega_1(Z_2(G))$. Then the mapping $a \mapsto au$ and $b \mapsto bv$ is an automorphism of $G$ of order $p$ that fixes $N(G)$ elementwise.

Let $F$ be the free group on the set $\{a, b\}$ and let $R$ be a normal subgroup of $F$ such that $G = F/R$. Since $G$ is not powerful, it follows from Proposition 1 that $R \leq \Phi^p \gamma_2(F)$. Let $u, v \in \Omega_1(Z_2(G))$. Then for each $r(a, b) \in R$, we have $r(au, bv) = r(a, b)$. Thus by Von Dyck’s theorem there is an epimorphism $\alpha_{u, v} : G \rightarrow \langle au, bv \rangle \leq G$ with $a \mapsto au$ and $b \mapsto bv$. Since $u, v \in \Phi(G)$, $\alpha_{u, v}$ is an automorphism of $G$ which fixes $N(G)$ elementwise. Now Claim 1 implies that $\alpha_{u, v}$ has order $p$.

Claim 3. $Z(\Phi(G)) \leq C_{G}(N(G))$.

It follows from (1) that $C_G(\Phi(G)) = Z(\Phi(G))$. Thus $Z(\Phi(G)) \leq C_G(N(G))$. Assume that $Z(\Phi(G)) = C_G(N(G))$ and let $v$ be a nontrivial element of $V$. Suppose that $z = [a, v] \neq 1$ if $[a, v] = 1$, then $[v, b] \neq 1$ and similar argument holds). Then $\alpha = \alpha_{1, v}$ is an automorphism of order $p$ that fixes $N(G)$ elementwise. If $\alpha = \theta_g$ is inner, then $g \in Z(\Phi(G))$, by assumption. Thus $\langle a, b \rangle^g = [a, b] \neq [a, b]^g$, a contradiction. This implies the claim.

Claim 4. $C_G(N(G)) = Z(N(G))$.

By Claim 1, $Z^2_2(G) \leq N(G)$. Thus $C_G(N(G)) \leq C_G(Z^2_2(G)) = \Phi(G)$, by (2). Hence $C_G(N(G)) = C_{G\gamma_2}(N(G))$ and Claim 3, implies that $\Phi(G)$ is not abelian. Since $N(G)$ is a maximal subgroup of $\Phi(G) = \Phi^p \gamma_2(G)$, the claim follows from the following known fact: If $H$ is a finite nonabelian $p$-group and $M$ is a maximal subgroup of $H$ such that $Z(H) \leq M$, then $C_H(M) = Z(M)$ (See for example [15, (2.1) Lemma]).

Let $A = \{\alpha_{u, v} : u, v \in \Omega_1(Z_2(G))\}$ and let $W_{Z(G)} = \Omega_1(Z^2_2(G)/Z(N(G)))$.

Claim 5. $W_{Z(G)} \cong A$.

Define $\theta : W \rightarrow A$, by $w \mapsto \theta_w$, where $\theta_w$ is the inner automorphism of $G$ induced by $w$. Clearly $\theta$ is a homomorphism and $\ker(\theta) = Z(G)$. Now let $\alpha = \alpha_{u, v} \in A$. As we observed in Claim 2, $\alpha$ has order $p$ and fixes $N(G)$ elementwise. Hence $\alpha = \theta_g$ is inner, for some $g \in G$. Since $\alpha$ has order $p, \alpha|_{N(G)} = id$ and $\alpha|_{G\gamma_2(Z_2(G))} = id$, we get $g^p \in Z(G), g \in C_G(N(G)) = Z(N(G))$ and $g \in Z_3(G)$, respectively. Thus $g \in W$ and, therefore, $\theta$ is epimorphism. This proves the claim.

Clearly $A$ is an elementary abelian group of rank $2d(\Omega_1(Z_2(G))) \geq 4d(Z(G))$. Hence Theorem 1 follows from Claims 3–5.

Proof of Theorem 2. Let $G = \langle a, b \rangle$ be a 2-generator finite $p$-group with abelian Frattini subgroup. Suppose that $G$ has no noninner automorphism of order $p$. Then the nilpotency class of $G$ is at least 4, and by Theorem 1, we may assume that $p = 2$.

Let $G/\gamma_2(G) = \langle a \gamma_2(G) \rangle \times \langle b \gamma_2(G) \rangle$. Let $2^m = |a \gamma_2(G)|, 2^n = |b \gamma_2(G)|$ and $m \geq n \geq 1$. If $m = 1$, then by [7, Proposition 4.10], $G$ is of maximal class and it follows from [2, Corollary 2.4], that $G$ has a noninner automorphism of order 2, a contradiction. Thus we have $m \geq 2$.

We proceed by proving several claims.

Claim 1. $a^{2^m}, b^{2^n} \in \gamma_2(G)^2 \gamma_3(G) \leq N(G)$. 

If \( b^{2n} \not\in \gamma_2(G)^2 \), then \( b^{2n} = [a, b]c \), for some \( c \in \gamma_2(G)^2 \gamma_3(G) \). Now we have either \( n = 1 \), or \( n \geq 2 \). In the former case \( b^2[a, b] \in N(G) \), and in the latter case \( [a, b] \in N(G) \). Thus Lemma 2 gives a contradiction in both cases.

**Claim 2.** Let \( 1 \neq v \in V \) and \( z = [v, b] \). Set \( w = [a, b] \) and define \( \sigma : G \rightarrow G \), by

\[
(a^i b^i w^k c)^\sigma = (av)^i b^i w^k z^k c.
\]

Then \( \sigma \) is an automorphism of \( G \) that fixes \( N(G) \) elementwise.

Let \( a^i b^j w^k c = a^i b^j w^k c' \). Then \( a^i b^j = a^i b^j \mod \gamma_2(G) \). Thus \( i = r2^m + i' \) and \( j = s2^n + j' \), for some \( r, s \in Z \). Hence

\[
a^{2m}b^{i'}w^k = b^{j'}w^k c'.
\]

Now consider Eq. (9), modulo \( N(G) \). By Claim 1, and since \( G \) is not powerful we have \( k = 2t + k' \), for some integer \( t \). Thus

\[
(av)^i b^j w^k z^k c = (av)^{2m+i'} b^j w^{2t+k'} z^{2t+k'} c
\]

\[
= (av)^i a^{2m} b^j w^k c z^k
\]

\[
= (av)^i b^j w^k c' z^k.
\]

This shows \( \sigma \) is well defined. Next, note that

\[
[b^i, a^j] = [b, a]^i f(b^i, a^j) = w^{-i}f(b^i, a^j),
\]

for some \( f(b^i, a^j) \in \gamma_3(G) \). Therefore

\[
((a^i b^j w^k c)(a^i b^j w^k c'))^\sigma = (a^{i+i'} b^{j+i'} w^{k+k'} c f(b^j, a^i)[w^k c, a^j b^i] w^k c')^\sigma
\]

\[
= (av)^{i+i'} b^{j+i'} w^{k+k'} z^{k+k'} c f(b^j, a^i)[w^k c, a^j b^i] w^k c'
\]

\[
= (av)^{i+i'} b^{j+i'} w^{k+k'} z^{k+k'} c [b^j, a^i] [w^k c, a^j b^i] c'.
\]

On the other hand,

\[
(a^i b^j w^k c) (a^i b^j w^k c')^\sigma = (av)^i b^j w^k c [av]^i b^j w^k z^k c'
\]

\[
= (av)^{i+i'} b^{j+i'} w^{k+k'} c [b^j, a^i] z^{-i} [w^k c, a^j b^i] w^{k'} z^k c'.
\]

So \( \sigma \) is a homomorphism. Since \( \nu \in \Phi(G) \) and \( G \) is finite we have \( \sigma \) is an automorphism of \( G \). Clearly \( \sigma \) fixes \( N(G) \) elementwise.

**Claim 3.** \( Z_2^*(G) \not\subseteq N(G) \).

If \( Z_2^*(G) \subseteq N(G) \), then the automorphism \( \sigma \) defined in the previous claim has order 2. We have either \([a, v] \neq 1 \) or \([b, v] \neq 1 \). In the former case \( (a^2)^\sigma = (av)^2 \neq a^2 \). In the latter case we have \([a, b] \neq [a, b][v, b] \neq [a, b] \). Suppose that \( \sigma = \theta_x \) is inner. Since \( \sigma \) fixes \( Z_2^*(G) \) elementwise, we must have \( x \in C_G(Z_2^*(G)) = \Phi(G) \), which is impossible. Thus \( \sigma \) is noninner, contrary to our assumption. This proves the claim.

**Claim 4.** \( Z_2^*(G) \not\subseteq G^4 \gamma_2(G) \).

Suppose that \( Z_2^*(G) \subseteq G^4 \gamma_2(G) \). Since \(|G^4 \gamma_2(G) : N(G)| = 2\), Claim 3 implies that \( \gamma_2(G) \subseteq Z_2^*(G) \setminus N(G) \). Now Lemma 2 gives a contradiction.

From now on, let \( u \in Z_2^*(G) \setminus G^4 \gamma_2(G) \).
Claim 5.
- \( u = g^2c \), where \( g \in \{a, b, ab\} \) and \( c \in N(G) \), and
- if \( v \in Z^2(G) \), then either \( v \in N(G) \), or \( v = u \mod N(G) \).

Since \( u \in G^2 \), by Claim 4 we have \( u = g^2 \mod G^2_\gamma_2(G) \), where \( g \in \{a, b, ab\} \). Now \( G = \langle g, h \rangle \) for some \( h \in G \). Thus \( u = g^2[h, h]c \), for some \( c \in N(G) \). Since \( [g, h] \in N(G) \), we may assume that \( i \in \{0, 1\} \). But Lemma 2 implies that \( i = 0 \). This proves the first part of the claim.

Next, for the second part of the claim, suppose that \( v \in Z^2(G) \setminus N(G) \). Then as we saw in the first part of the claim, \( v = h^2c' \), for some \( h \in \{a, b, ab\} \) and \( c' \in N(G) \). If \( h \neq g \), then \( G = \langle g, h \rangle \), \( g^2, h^2 \in Z^2(G)N(G) \) and Lemma 2 gives a contradiction. Hence \( v = u \mod N(G) \).

Claim 6. \( V \cap N(G) \neq 1 \).

First, assume that \( Z(G) \) is cyclic. Since \( \gamma_2(G) \) is not cyclic, \( \frac{\Omega_1(\gamma_2(G))Z(G)}{Z(G)} \) is a nontrivial normal subgroup of \( G/Z(G) \). So there exists an element \( w \in \Omega_1(G') \cap Z_2(G) \). Thus \( w = zv \), for some \( z \in \Omega_1(Z(G)) \) and \( v \in V \). If \( v \notin \gamma_2(G)\gamma_2(G) \), then \( v = [a, b]c \), for some \( c \in \gamma_2(G)\gamma_2(G) \) which is impossible by Lemma 2. Thus the claim holds in this case.

Next, assume that \( Z(G) \) is not cyclic. Then it follows from Remark 4, that \( d(V) \geq 2 \). Let \( \langle v, w \rangle \leq V \) be a subgroup of order 4. If \( v, w \notin N(G) \), then by Claim 5, \( v = g^2c \) and \( w = g^2c' \), for some \( c, c' \in N(G) \). Thus \( vw \in N(G) \). This completes the proof of the claim.

To complete the proof of Theorem 2, first assume that \( g = a \). If \( b^2 \in \gamma_2(G) \), then by Claim 1, we have \( b^2 \in N(G) \) and since \( a^2 \in Z^2(G)N(G) \), it follows from Lemma 2 that \( G \) has a noninner automorphism of order 2, which is a contradiction. Hence \( n \geq 2 \). Now the argument of Claim 2, shows that the map \( \alpha \) given by \( a \mapsto a \) and \( b \mapsto bv \) is an automorphism of \( G \) of order 2 that fixes \( N(G) \) elementwise. It follows from Claim 5 that \( \alpha \) fixes \( Z^2(G) \) elementwise. If \( \alpha = \theta \), for some \( g \in G \), then we must have \( g \in C_G(Z^2(G)) = \Phi(G) \) and therefore \( \alpha \) fixes \( b^2 \) and \( [a, b] \).

Since \( (b^2)^\alpha = (bv)^2 = b^2[v, b] \) and \( [a, b]^\alpha = [a, b][a, v] \), we have \( [b, v] = 1 = [a, v] \). This implies that \( v \in Z(G) \), contrary to our choice of \( v \). Thus \( \alpha \) is noninner and this is a contradiction.

Next, if \( g = b \) (or \( g = ab \)), then the map \( a \mapsto av \) and \( b \mapsto b \) (or \( ab \mapsto abv \) and \( b \mapsto b \)) is a noninner automorphism of \( G \) of order 2, a contradiction. This proves the theorem. \( b \mapsto bv \).

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