Commuting graphs of full matrix rings over finite fields

Alireza Abdollahi*

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran

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Abstract

Let $R$ be a non-commutative ring and $Z(R)$ be its center. The commuting graph of $R$ is defined to be the graph $\Gamma(R)$ whose vertex set is $R \setminus Z(R)$ and two distinct vertices are joint by an edge whenever they commute. Let $F$ be a finite field, $n \geq 2$ an arbitrary integer and $R$ be a ring with identity such that $\Gamma(R) \cong \Gamma(M_n(F))$, where $M_n(F)$ is the ring of $n \times n$ matrices over $F$. Here we prove that $|R| = |M_n(F)|$.

We also show that if $|F|$ is prime and $n = 2$, then $R \cong M_2(F)$.

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1. Introduction and results

Let $R$ be a non-commutative ring and $Z(R)$ be its center. Following [2], the commuting graph of $R$, denoted by $\Gamma(R)$, is the graph whose vertex set is $R \setminus Z(R)$ and two vertices $a$ and $b$ are joint by an edge if $a \neq b$ and $ab = ba$. The commuting graphs of certain matrix rings have been studied in [2,3]. For any field $F$, $M_n(F)$ denotes the $n \times n$ full matrix ring with coefficients in $F$. Akbari et al. proposed the following interesting conjecture in [2] concerning the “uniqueness” of the commuting graph of $M_n(F)$.

* Tel.: +98 311 7932309; fax: +98 311 7932308.
E-mail address: a.abdollahi@math.ui.ac.ir

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AGHM conjecture. Let \( R \) be a ring, \( F \) be a finite field and \( n \geq 2 \). If \( \Gamma(R) \cong \Gamma(M_n(F)) \), then \( R \cong M_n(F) \).

In this paper, we investigate AGHM Conjecture. We first prove that the rings \( R \) with identity in AGHM Conjecture must have the same size \( |M_n(F)| \) (see Theorem 1.2, below). The property of having the same size for two rings with isomorphic commuting rings does not hold in general, as we shall show there are two rings \( R_1 \) and \( R_2 \) constructed from truncated skew-polynomial rings such that \( \Gamma(R_1) \cong \Gamma(R_2) \) and \( |R_1| \neq |R_2| \) (see Example 2.5, below).

Theorem 1.1. Let \( F \) be a finite field of characteristic \( p \) and let \( A \) be a finite commutative ring such that \( \text{gcd}(|A|, 2) = 1 \) whenever \( p \neq 2 \). Let \( S = M_2(F) \oplus A \) and let \( R \) be a ring with identity such that \( \Gamma(R) \cong \Gamma(S) \). Then \( R \) satisfies the following conditions:

1. \( |R| = |S| \).
2. \( |Z(R)| = |F||A| \).
3. The centralizer \( C_R(x) \) of any non-central element of \( R \) is a commutative ring of order \( |F|^2|A| \).
4. \( pR \subseteq Z(R) \).

Theorem 1.2. Let \( F \) be a finite field of characteristic \( p \), \( n \geq 2 \) an arbitrary integer and let \( A \) be a finite commutative ring such that \( \text{gcd}(|A|, 2) = 1 \) whenever \( p \neq 2 \). If \( S = M_n(F) \oplus A \) and \( R \) is a ring with identity such that \( \Gamma(R) \cong \Gamma(S) \), then \( |R| = |S| \).

We confirm AGHM Conjecture whenever \( F \) is a finite prime field and \( n = 2 \).

Theorem 1.3. Let \( F \) be an arbitrary finite prime field and \( R \) be a ring with identity such that \( \Gamma(R) \cong \Gamma(M_2(F)) \). Then \( R \cong M_2(F) \).

2. Proofs of Theorems 1.1 and 1.2

In the following lemma we give a characterization of rings whose commuting graphs are a disjoint union of some complete graphs.

Lemma 2.1. Let \( R \) be a non-commutative ring. Then \( \Gamma(R) \) is a disjoint union of some complete graphs if and only if the centralizer \( C_R(a) \) of each non-central element \( a \in R \) is commutative. In any case, \( \mathcal{C} = \{C_R(a) \setminus Z(R) | a \in R \setminus Z(R) \} \) is a partition of \( R \setminus Z(R) \) into sets of pairwise commuting elements and \( \Gamma(R) \) is a disjoint union of \( |\mathcal{C}| \) complete graphs whose sizes (counted with multiplicity) belong to the multiset \( \{|C||C| \in \mathcal{C}| \} \).

Proof. Suppose that \( \Gamma(R) \) is a disjoint union of some complete graphs. Let \( b \) and \( c \) be two distinct elements in \( C_R(a) \setminus Z(R) \). Since \( b \) and \( c \) both commute with \( a \), they are in the same connected component of \( \Gamma(R) \). As each connected component of \( \Gamma(R) \) is complete, \( b \) is adjacent to \( c \), that is \( bc = cb \). Thus \( C_R(a) \) is commutative.

Now, suppose that \( C_R(a) \) is commutative for any non-central element \( a \) of \( R \). We first prove that if \( C_R(a) \cap C_R(b) \neq Z(R) \), for non-central elements \( a, b \in R \), then \( C_R(a) = C_R(b) \). Let \( x \in C_R(a) \cap C_R(b) \setminus Z(R) \). Since \( x \) is non-central, \( A = C_R(x) \) is commutative. Now, as \( a, b \in A \) and \( A \) is commutative, this follows that \( A \subseteq C_R(a) \) and \( A \subseteq C_R(b) \). Since \( x \in C_R(a) \cap C_R(b) \).
and \( C_R(a) \) and \( C_R(b) \) are commutative, we have that both \( C_R(a) \) and \( C_R(b) \) are subsets of \( A \). Therefore \( A = C_R(a) = C_R(b) \).

Hence \( \mathcal{E} = \{ C_R(a) \setminus Z(R) | a \in R \setminus Z(R) \} \) is a partition of \( R \setminus Z(R) \) into sets of pairwise commuting elements. This now follows that \( \Gamma(R) \) is a disjoint union of \(|\mathcal{E}|\) complete graphs whose sizes (counted with multiplicity) belong to the multiset \([|C||C \in \mathcal{E}|]\).

In the proof of Theorem 1.1, we use the following result due to Isaacs [6] on equally partitioned groups.

**Theorem 2.2** [6]. Let \( A \) be a finite non-trivial group and let \( n > 1 \) be an integer such that \( \{ A_i | i = 1, \ldots, n \} \) is a set of subgroups of \( A \) with the property that \( A = \bigcup_{i=1}^{n} A_i \), \( |A_i| = |A_j| \) and \( A_i \cap A_j = \{1\} \) for any two distinct \( i, j \). Then \( A \) is a group of prime exponent.

In the proof of Theorem 1.1 we use a result of Le [7] on a solution of the Diophantine equation

\[
\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}
\]

for \( x, y, m, n \in \mathbb{N}, x > y > 1 \) and \( n > m > 2 \). It was conjectured by Goormaghtigh [5] that (GO) has only the solutions \((x, y, m, n) = (5, 2, 3, 5) \) and \((90, 2, 3, 13)\). This conjecture has not been solved so far.

**Theorem 2.3** [7, Theorem]. If \((x, y, 3, n) \not\in \{(5, 2, 3, 5), (90, 2, 3, 13)\}\) is a solution of the Eq. (GO) with \( m = 3 \), then we have \( \gcd(x, y) > 1 \) and \( y \nmid x \).

We need also the following lemma in the proof of Theorem 1.1. We postpone its proof to the last section.

**Lemma 2.4.** There is no finite ring \( R \) with identity satisfying the following conditions:

1. \( |R| = 2^7 \),
2. \( |Z(R)| = 4 \),
3. \( C_R(a) \) is a commutative ring of order 8 for all \( a \in R \setminus Z(R) \).

Before proving Theorem 1.1, as we promised in Section 1, we give an example of a pair of rings \( R_1 \) and \( R_2 \) such that \( \Gamma(R_1) \cong \Gamma(R_2) \) but \( |R_1| \neq |R_2| \).

**Example 2.5.** Let \( F = GF(p) \) (\( p \) prime) and \( E = GF(p^r) \) for some integer \( r \geq 2 \). We consider the truncated skew-polynomial ring

\[
R(p^r) = \{ \alpha_1 x + \alpha_2 x^2 | \alpha_1, \alpha_2 \in E \},
\]

where \( x^3 = 0 \) and \( x \alpha = \alpha p^r x \) for all \( \alpha \in E \). It is not hard to see with direct calculations that \( |R(p^r)| = p^{2r} \), \( |Z(R(p^r))| = p^r \), \( C_R(p^r)(\gamma) = (Z(R(p^r)), \gamma) \) is a commutative ring and \( |C_R(p^r)(\gamma)| = p^{r+1} \) for all non-central elements \( \gamma \) of \( R(p^r) \). This follows from Lemma 2.1 that \( \Gamma(R(p^r)) \) is the disjoint union of \( \frac{p^{r+1} - 1}{p-1} \) complete graphs each of which has the size \( p^{r+1} - p^r \).

Also it is easy to see that \( \Gamma(R(p^r) \oplus \mathbb{Z}_m) \), is the disjoint union of \( \frac{p^{r+1} - 1}{p-1} \) isomorphic complete graphs of size \( m(p^{r+1} - p^r) \).
Now, let $R_1 = R(25) \oplus \mathbb{Z}_{5^3}$ and $R_2 = R(5^3) \oplus \mathbb{Z}_{5^3}$. Then $\Gamma(R_1) \cong \Gamma(R_2)$ is the disjoint union of $\mathbb{Z}_{2950}$. Clearly, $|R_1| \neq |R_2|$.

We acknowledge that the idea of this example is given from an example constructed in [9] to refute Conjecture 1.1 of [1]. This is mentioned in [9] that the given example is due to Isaacs.

**Proof of Theorem 1.1.** Let $|F| = p^n$ for some integer $n$. First note that, by Remark 2 of [2], $R$ is a finite non-commutative ring. Since $C_S(x) = C_{M_2(F)}(x_1) \oplus A$, for all $x = (x_1, x_2) \in S$ where $x_1 \in M_2(F)$ and $x_2 \in A$, it follows from Theorem 2 of [2] that $\Gamma(R)$ is the disjoint union of $p^{2n} + p^n + 1$ copies of the complete graphs of the same size $(p^{2n} - p^n)|A|$. Now, consider the additive group $R/Z(R)$ and its subgroups $C_R(a)/Z(R)$, where $a$ ranges over the non-central elements of $R$. Since $\Gamma(R) \cong \Gamma(S)$ and $|C_S(x)| = |C_S(y)| = |F|^2|A|$ for any two non-central elements $x, y \in S$, we have that $|C_R(a)/Z(R)| = |C_R(b)/Z(R)|$ for any two non-central elements $a, b \in R$. Thus, it follows from Theorem 2.2, that the additive group $R/Z(R)$ is of prime exponent $q$, for some prime $q$. Thus $|R/Z(R)| = q^n$ and $|C_R(a)/Z(R)| = q^r$ for all non-central elements $a$ of $R$, where $m, r \in \mathbb{N}$, and $qR \subseteq Z(R)$. Note that, by Lemma 2.1 and [2, Theorem 2], each $C_R(a)/Z(R)$ is the set of vertices of a connected component of $\Gamma(R)$ and so the set $\mathcal{C} = \{C_R(a)/Z(R)|a \in R \setminus Z(R)\}$ has size $p^{2n} + p^n + 1$. Since $\{C_R(a)/Z(R) - Z(R)/Z(R)|a \in R \setminus Z(R)\}$ is a partition for $R/(R - Z(R))/Z(R)$, we have

$$q^m - 1 = (p^{2n} + p^n + 1)q^r - 1.$$

Then we find

$$\frac{q^m - 1}{q^r - 1} = p^{2n} + p^n + 1 = \frac{p^{3n} - 1}{p^n - 1}.$$

Since $\frac{q^m - 1}{q^r - 1} = p^{2n} + p^n + 1$ is an integer, $q^r - 1$ divides $q^m - 1$ and so $m = rs$ for some integer $s$. Therefore

$$\frac{(q^r)^s - 1}{q^r - 1} = \frac{(p^n)^s - 1}{p^n - 1}.$$

Now, it follows from Theorem 2.3, that either $(p, 3n, n) = (q, m, r)$ or $(p, n, q, m) = (5, 1, 2, 5)$. If $(p, 3n, n) = (q, m, r)$, then since $|R| - |Z(R)| = (p^{3n} - p^n)|A|$ and $|R/Z(R)| = q^m = p^{3n}$, we have $|R| = p^{3n}|A| = |S|$. This completes the proof, in this case.

Now, assume that $(p, n, q, m) = (5, 1, 2, 5)$. Thus $|R/Z(R)| = 32$ and $p = 5$. Now since $|C_R(a) - |Z(R)| = 20|A|$ for all $a \in R \setminus Z(R)$ and also $|R - |Z(R)| = (5^4 - 5)|A|$, it follows that $|R| = 27 \cdot 5|A|$, $|Z(R)| = 20|A|$, $|C_R(a)| = 40|A|$ for all $a \in R \setminus Z(R)$. Let $I_1 = \{x \in R|2x = 0\}$ and $I_2 = \{x \in R|5A|\}$. Then $I_1$ and $I_2$ are ideals of $R$. Since $p = 5 \neq 2$, we have gcd($5|A|, 2 = 1$ and this follows that $R = I_1 \oplus I_2$. This is now easy to see that $I_1$ is a ring with identity satisfying the conditions of Lemma 2.4. Thus this case cannot occur and $|R| = |S|$. □

We need the following easy lemma in the proof of Theorem 1.2.

**Lemma 2.6.** Let $F$ be any field and $E_{11} \in M_n(F)$ be the matrix only its $(1, 1)$ entry is $1$ and the others is equal to zero. Then $C_{M_n(F)}(E_{11}) \cong F \oplus M_{n-1}(F)$. 


Proof. By an easy calculation, we see that every matrix \( X \) in \( C_{M_n(F)}(E_{11}) \) is of the form \( X = \begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \), where \( a \in F \) and \( A \in M_{n-1} \). This completes the proof. \( \Box \)

Lemma 2.7. Let \( R \) and \( S \) be two non-commutative rings such that \( \phi : \Gamma(R) \to \Gamma(S) \) is a graph isomorphism. If \( C_R(x) \) is non-commutative for some non-central element \( x \in R \), then \( C_S(\phi(x)) \) is a non-commutative ring and \( \phi \) induces a graph isomorphism from \( \Gamma(C_R(x)) \) onto \( \Gamma(C_S(\phi(x))) \).

Proof. By an easy calculation, we see that every matrix \( X \) in \( C_{M_n(F)}(E_{11}) \) is of the form \( X = \begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \), where \( a \in F \) and \( A \in M_{n-1} \). This completes the proof. \( \Box \)

Proof of Theorem 1.3. It follows from Lemma 3.1 and Theorem 1.1. \( \Box \)
4. Proof of Lemma 2.4

For a ring $R$, we denote by $U(R)$ and $J(R)$ the set of units and the Jacobson radical of $R$, respectively. We use the following well-known result in the proof of Lemma 2.4.

**Lemma 4.1.** Let $R$ be a finite ring with identity and $J$ be its Jacobson radical. The element $a \in R$ is a unit if and only if the element $a + J$ is a unit. In particular, $|U(R)| = |U(R) / J| |J|$.

**Proof of Lemma 2.4.** Let $J(R)$ be the Jacobson radical of $R$. Then $R / J(R) \cong \bigoplus_{i=1}^{m} M_{n_i}(K_i)$, where $K_i$ is a finite field for $i = 1, \ldots, m$. Let $\overline{e_i}$ be the identity of $M_{n_i}(K_i)$. Then the set $\{\overline{e_i} | i = 1, \ldots, m\}$ may be lifted to a set of orthogonal idempotents $\{e_i | i = 1, \ldots, m\}$ in $R$ with $1 = e_1 + \cdots + e_m$ and $e_i \to \overline{e_i}$ under the natural epimorphism $R \to R / J(R)$. Set $S = \bigoplus_{i=1}^{m} e_i Re_i$ and $N = \bigoplus_{i \neq j} e_i Re_j$. Then by Theorem (VIII.27) of [8], $S \cap N = 0$, $R = S + N$, $N$ is an additive subgroup of $J(R)$, $S / J(S) \cong R / J(R)$ and $S \cong \bigoplus_{i=1}^{m} M_{n_i}(S_i)$ is a direct sum of $U_i$ by $n_i$ matrix rings over local rings $S_i$.

Since $e_i e_j = 0$ for all $i \neq j$ and $e_i$ is the identity element of $e_i Re_i$.

$$S + \left( \bigoplus_{j \neq i, j, k} e_j Re_k \right) \leq C_R(e_i). \quad (I)$$

Note that $Z(R) \setminus \{0\} \subseteq U(R)$, otherwise $Z(R)$ is the field of order 4 and so $R$ is a $Z(R)$-vector space. Thus $|R| = 4^t$ for some $t \in \mathbb{N}$, a contradiction. Hence $|Z(R) \cap U(R)| \in \{1, 2\}$.

We show that $m > 1$. Suppose, for a contradiction, that $m = 1$. Then it follows from Theorem (VIII.26) of [8], that $R$ is a semi-local ring and $R \cong M_n(L)$ for some local ring $L$. Thus $2^2 = |L|^n$ and so $|L| = 2^2$ and $n = 1$. Therefore $R$ is a local ring. Recall that in a finite ring with identity every non-unit element is a two-sided zero divisor (see page 199 of [10]). Thus it follows from Theorem (VI.1) of [8] that in any local ring with identity, the Jacobson radical is equal to the set of all zero-divisors. Therefore, if $m = 1$, $J(R)$ is the set of all zero divisors of $R$. Now Theorem 2 of [10] implies that $|J(R)| = 2^3$. It follows from Lemma 4.1 that $|U(R)| = 2^5$. The multiplicative group $G = U(R)$ is not abelian, since otherwise $G \subseteq C_R(x)$ for any $x \in G \setminus Z(R)$, which is impossible as $|C_R(x)| = 8$ for all $x \in R \setminus Z(R)$.

We now prove that $|C_R(g)| = 4$ for all $g \in G \setminus Z(G)$. If $g \in G$, then $g = 1 + x$ for some $x \in J(R)$. It is easy to see that $C_R(x) = C_R(1 + x)$ and $1 + (C_R(x) \cap J(R)) = C_R(x) \cap G = C_R(g)$. Since the unit $1 + x$ belongs to $C_R(x)$, $C_R(x) \subseteq J(R)$. Thus $R = C_R(x) + J(R)$ and so $|R| = |C_R(x)| |J(R)| / |C_R(x) \cap J(R)|$. This easily follows that $|C_R(g)| = 4$ for all $g \in G \setminus Z(G)$. Therefore $G$ is a non-abelian group of order 64 such that $|C_R(g)| = 4$ for all $g \in G \setminus Z(G)$. Now it is easy to see in the GAP library of groups of order 64 [11], that there is no such a group $G$. Therefore $m > 1$.

We now prove that $N \neq 0$. Suppose, for a contradiction, that $N = 0$ so that $R = S$. Therefore $|S_1|^{n_1} \cdots |S_m|^{n_m} = 2^7$. Since $|S_i| \geq 2$ for all $i \in \{1, \ldots, m\}$, at most one $n_i$ is equal to 2 and the others $n_j$'s are equal to 1. On the other hand $4 = |Z(R)| = |Z(S_1)| \cdots |Z(S_m)|$ which implies that $m = 2$ and $|Z(S_1)| = |Z(S_2)| = 2$. Since $R$ is non-commutative, either $M_{n_1}(S_1)$ or $M_{n_2}(S_2)$ so is. Assume that $S_1$ is non-commutative. Then for each $b \in S_1 \setminus Z(S_1)$, we have that $M_{n_1}(S_2) \leq C_R(b)$. Therefore $M_{n_2}(S_2)$ is commutative which implies $n_2 = 1$ and $|S_2| = 2$. This
follows that $|S_1|^2 = 2^6$. Then $|S_1| = 2^5$ and $n_1 = 1$. Therefore $S_1$ is a non-commutative local ring with identity of order $2^6$ such that $|Z(S_1)| = 2$. Thus the characteristic of $S_1$ is 2. By Theorem 2 of [10], we have that $J_1 = J(S_1)$ is of order $2^3$, $2^4$ or $2^5$. Now suppose that $n$ is a positive integer such that $J_1^{n-1} 
eq 0$ and $J_1^n = 0$, and let $x$ be a non-zero element of $J_1$. Since $|Z(S_1)| = 2$, $x \notin Z(S_1)$ and so $|C_R(x)| = 8$. Since $J_1 \subseteq C_R(x)$ and $|J_1| \geq 2^3$, we have that $J_1 = C_R(x)$ which is not possible, since the identity of $R$ belongs to $C_R(x)$.

Hence $N \neq 0$. Note that if $e_i \in Z(R)$, then $e_i Re_j = e_i e_j R = 0$ for every $j \neq i$. This implies that there exists $i \in \{1, \ldots, m\}$ such that $e_i \in R \setminus Z(R)$. Since $1 = e_1 + \cdots + e_m$, there exist at least two non-central $e_i$’s. We may assume without loss of generality that $e_1$ and $e_2$ are non-central elements of $R$.

Note that each summand $e_i Re_i$ of $S$ has size at least 2. Thus if $m > 3$, $|S| > 8$ and so by (I) we have $|C_R(e_1)| > 8$, a contradiction. Therefore $m \in \{2, 3\}$. If $m = 3$, then by (I) we have,

\[ e_1 Re_1 \oplus e_2 Re_2 \oplus e_3 Re_3 + e_2 Re_3 \oplus e_3 Re_2 \leq C_R(e_1), \]
\[ e_1 Re_1 \oplus e_2 Re_2 \oplus e_3 Re_3 + e_1 Re_3 \oplus e_3 Re_1 \leq C_R(e_2). \]

Since $e_1$ and $e_2$ are non-central, $|C_R(e_1)| = |C_R(e_2)| = 8$. It follows that $e_2 Re_3 = e_3 Re_2 = e_1 Re_3 = e_3 Re_1 = 0$. If $e_3$ is not central, then by a similar argument $e_1 Re_2 = e_2 Re_1 = 0$, which implies that $N = 0$, a contradiction. Therefore $e_3 \in Z(R)$, $|e_i Re_i| = 2$ for each $i \in \{1, 2, 3\}$. Hence $J(S) = 0$ and $N = J(R) = e_1 Re_2 \oplus e_2 Re_1$. Since $J(R)^2 \leq S \cap J(R) = 0$, $J(R)^2 = 0$. Now as $|Z(R)| = 4$ and $|J(R)| = 2^3$, there exists $x \in J(R) \setminus Z(R)$ so that $|C_R(x)| = 8$, but as $J(R)^2 = 0$, we have $J(R) \leq C_R(x)$, a contradiction.

Hence, so far, we have proved that $m = 2$ and $N \neq 0$, and both $e_1$ and $e_2$ are non-central.

Now we prove that $J(S) \neq 0$. If $J(S) = 0$ then $N = J(R)$. Since $N^2 \leq S$, we have that $J(R)^2 = 0$. Thus $J(R)$ is a commutative subring of $R$ of order $2^5$, a contradiction; since one may find a non-central element in $J(R)$ whose centralizer has at least $2^5$ element. Therefore $J(S) \neq 0$ and either $|e_i Re_i| > 2$ or $|e_2 Re_2| > 2$. This follows that $|S| \geq 8$ and as $S \leq C_R(e_1)$, we have that $S$ is a commutative rings of order 8. Thus $S \cong GF(2) \oplus L$, where $L$ is a local ring of order 4. Since $J(S) \neq 0$, this follows that $|J(S)| = 2$ and so $S/J(S) \cong GF(2) \oplus GF(2) \cong R/J(R)$. Now, it follows from Lemma 4.1 that $|U(R)| = 2^3$. Since $|Z(R) \cap U(R)| \in \{1, 2\}$, $G = U(R)$ is a non-abelian group such that $C_G(g)$ is abelian of order at most 8 for all $g \in G \setminus Z(G)$. Now, using the following program written in GAP [11], one can easily see that there is no such a group $G$ of order $2^5$.

```gap
a:=AllSmallGroups(2^5,IsAbelian,false);
b:=Filtered(a,x->Size(Center(x))<=8);; c:=Filtered(b,i->[true]=Set(List(Difference(i,Center(i)), j->IsAbelian(Centralizer(i,j))));
d:=Set(List(c,i->Set(List(Difference(i,Center(i)),
j->Size(Centralizer(i,j))))));
This completes the proof. □
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