STRONG INSERTION OF A CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE PRECONTINUOUS (SEMI-CONTINUOUS) FUNCTIONS

Majid Mirmiran
Department of Mathematics, University of Isfahan
Isfahan 81746-73441, Iran
mirmir@sci.ui.ac.ir

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Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a continuous function between two comparable real-valued functions.

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1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [3]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$. A set $A$ is called preclosed if its complement (denoted by $A^c$) is a preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [11], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [3].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [10]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open [10] if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semi-closed if its complement is a semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$.

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$–continuous [13] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$–continuity. However, for unknown concepts the reader may refer to [4, 5].
Hence, a real-valued function \( f \) defined on a topological space \( X \) is called precontinuous (resp. semi-continuous) if the preimage of every open subset of \( \mathbb{R} \) is a preopen (resp. semi-open) subset of \( X \).

Precontinuity was called by V. Ptik nearly continuity [14]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [6]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [1].

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give necessary and sufficient conditions for the strong insertion of a continuous function between two comparable real-valued functions.

If \( g \) and \( f \) are real-valued functions defined on a space \( X \), we write \( g \leq f \) in case \( g(x) \leq f(x) \) for all \( x \) in \( X \).

The following definitions are modifications of conditions considered in [9].

A property \( P \) defined relative to a real-valued function on a topological space is a \( c \)-property provided that any constant function has property \( P \) and provided that the sum of a function with property \( P \) and any continuous function also has property \( P \). If \( P_1 \) and \( P_2 \) are \( c \)-properties, the following terminology is used:(i) A space \( X \) has the weak \( c \)-insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f \), \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a continuous function \( h \) such that \( g \leq h \leq f \). (ii) A space \( X \) has the strong \( c \)-insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f \), \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a continuous function \( h \) such that \( g \leq h \leq f \) and if \( g(x) < f(x) \) for any \( x \) in \( X \), then \( g(x) < h(x) < f(x) \).

In this paper, a sufficient condition for the weak \( c \)-insertion property is given. Also for a space with the weak \( c \)-insertion property, we give necessary and sufficient conditions for the space to have the strong \( c \)-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

The abbreviations \( pc \), \( sc \) and \( c \) are used for precontinuous, semicontinuous and continuous, respectively.

Let \( (X, \tau) \) be a topological space, the family of all semi-open, semi-closed, pre-open and preclosed will be denoted by \( sO(X, \tau) \), \( sC(X, \tau) \), \( pO(X, \tau) \) and \( pC(X, \tau) \), respectively.

**Definition 2.1.** Let \( A \) be a subset of a topological space \( (X, \tau) \). Respectively, we define the \( s \)-closure, \( s \)-interior, \( p \)-closure and \( p \)-interior of a set \( A \), denoted by \( s\text{Cl}(A) \), \( s\text{Int}(A) \), \( p\text{Cl}(A) \) and \( p\text{Int}(A) \) as follows:
Definition 2.2. If $\rho$ is a binary relation in a set $S$ then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any $u$ and $v$ in $S$.

Definition 2.3. A binary relation $\rho$ in the power set $P(X)$ of a topological space $X$ is called a strong binary relation in $P(X)$ in case $\rho$ satisfies each of the following conditions:

1. If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2. If $A \subseteq B$, then $A \bar{\rho} B$.

3. If $A \rho B$, then $Cl(A) \subseteq B$ and $A \subseteq Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a lower indefinite cut set in the domain of $f$ at the level $\ell$.

We now give the following main result:

Theorem 2.1. Let $g$ and $f$ be real-valued functions on a topological space $X$ with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a continuous function $h$ defined on $X$ such that $g \leq h \leq f$.

Proof. Let $g$ and $f$ be real-valued functions defined on $X$ such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions $F$ and $G$ mapping the rational numbers $\mathbb{Q}$ into the power set of $X$ by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [8] it
follows that there exists a function $H$ mapping $\mathbb{Q}$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2), H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any $x$ in $X$, let $h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' < t$; since $x$ is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = Int(H(t_2)) \setminus Cl(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is an open subset of $X$, i.e., $h$ is a continuous function on $X$.

The above proof used the technique of proof of Theorem 1 of [7].

If a space has the strong $c$–insertion property for $(P_1, P_2)$, then it has the weak $c$–insertion property for $(P_1, P_2)$. The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $c$–insertion property to satisfy the strong $c$–insertion property.

**Theorem 2.2.** Let $P_1$ and $P_2$ be $c$–properties and $X$ be a space that satisfies the weak $c$–insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g \leq f, g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the strong $c$–insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of $X$ such that (i) for each $n, F_n$ and $A(f - g, 2^{-n})$ are completely separated by continuous functions, and (ii) $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

**Proof.** Theorem 3.1, of [12].

**Theorem 2.3.** Let $P_1$ and $P_2$ be $c$–properties and assume that the space $X$ satisfies the weak $c$–insertion property for $(P_1, P_2)$. The space $X$ satisfies the strong $c$–insertion property for $(P_1, P_2)$ if and only if $X$ satisfies the strong $c$–insertion property for $(P_1, c)$ and for $(c, P_2)$.

**Proof.** Theorem 3.2, of [12].

3. Applications

**Corollary 3.1.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$ of $X$, there exist open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the weak $c$–insertion property for $(pc, pc)$ (resp. $(sc, sc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis $\rho$ is a strong
binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);$$

since $\{ x \in X : f(x) \leq t_1 \}$ is a preclosed (resp. semi-closed) set and since $\{ x \in X : g(x) < t_2 \}$ is a preopen (resp. semi-open) set, it follows that $\rho A(f, t_1) \subseteq p\text{Int}(A(g, t_2))$ (resp. $s\text{Cl}(A(f, t_1)) \subseteq s\text{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1.

\[\square\]

**Corollary 3.2.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$, there exist open sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every precontinuous (resp. semi-continuous) function is continuous.

**Proof.** Let $f$ be a real-valued precontinuous (resp. semi-continuous) function defined on $X$. Set $g = f$, then by Corollary 3.1, there exists a continuous function $h$ such that $g = h = f$.

\[\square\]

**Corollary 3.3.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$ of $X$, there exist open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the strong $c-$insertion property for $(pc, pc)$ (resp. $(sc, sc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$). Also, by Corollary 3.2, since $g$ and $f$ are continuous functions hence $h$ is a continuous function.

\[\square\]

**Corollary 3.4.** If for each pair of disjoint subsets $F_1, F_2$ of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ have the weak $c-$insertion property for $(pc, sc)$ and $(sc, pc)$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$, such that $g$ is $pc$ (resp. $sc$) and $f$ is $sc$ (resp. $pc$), with $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $s\text{Cl}(A) \subseteq p\text{Int}(B)$ (resp. $p\text{Cl}(A) \subseteq s\text{Int}(B)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);$$

since $\{ x \in X : f(x) \leq t_1 \}$ is a semi-closed (resp. preclosed) set and since $\{ x \in X : g(x) < t_2 \}$ is a preopen (resp. semi-open) set, it follows that $s\text{Cl}(A(f, t_1)) \subseteq p\text{Int}(A(g, t_2))$ (resp. $p\text{Cl}(A(f, t_1)) \subseteq s\text{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1.

\[\square\]

Before stating consequences of Theorems 2.2, and 2.3, we state and prove some necessary lemmas.
Lemma 3.1. The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $F_1, F_2$ of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist open subsets $G_1, G_2$ of $X$ such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.

(ii) If $F$ is a semi-closed (resp. preclosed) subset of $X$ which is contained in a preopen (resp. semi-open) subset $G$ of $X$, then there exists an open subset $H$ of $X$ such that $F \subseteq H \subseteq Cl(H) \subseteq G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where $F$ and $G$ are semi-closed (resp. preclosed) and preopen (resp. semi-open) subsets of $X$, respectively. Hence, $G^c$ is a preclosed (resp. semi-closed) and $F \cap G^c = \emptyset$.

By (i) there exist two disjoint open subsets $G_1, G_2$ of $X$ such that, $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since $G_2^c$ is a closed set containing $G_1$ we conclude that $Cl(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $F_1, F_2$ are two disjoint subsets of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and $F_1^c$ is a preopen subset of $X$. Hence by (ii) there exists an open set $H$ such that, $F_2 \subseteq H \subseteq Cl(H) \subseteq F_1^c$.

But

$$H \subseteq Cl(H) \Rightarrow H \cap (Cl(H))^c = \emptyset$$

and

$$Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (Cl(H))^c.$$ 

Furthermore, $(Cl(H))^c$ is an open set of $X$. Hence $F_2 \subseteq H, F_1 \subseteq (Cl(H))^c$ and $H \cap (Cl(H))^c = \emptyset$. This means that condition (i) holds.

Lemma 3.2. Suppose that $X$ is a topological space. If each pair of disjoint subsets $F_1, F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, can be separated by open subsets of $X$ then there exists a continuous function $h : X \rightarrow [0, 1]$ such that, $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$. 

**Proof.** Suppose $F_1$ and $F_2$ are two disjoint subsets of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. In particular, since $F_1^c$ is a preopen subset of $X$ containing the semi-closed subset $F_2$ of $X$, by Lemma 3.1, there exists an open subset $H_{1/2}$ of $X$ such that,

$$F_2 \subseteq H_{1/2} \subseteq Cl(H_{1/2}) \subseteq F_1^c.$$  

Note that $H_{1/2}$ is also a preopen subset of $X$ and contains $F_2$, and $F_1^c$ is a preopen subset of $X$ and contains the semi-closed subset $Cl(H_{1/2})$ of $X$. Hence, by Lemma 3.1, there exists open subsets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_2 \subseteq H_{1/4} \subseteq Cl(H_{1/4}) \subseteq H_{1/2} \subseteq Cl(H_{1/2}) \subseteq H_{3/4} \subseteq Cl(H_{3/4}) \subseteq F_1^c.$$  

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain open subsets $H_t$ of $X$ with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function $h$ on $X$ by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., $h$ maps $X$ into $[0,1]$. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that $h$ is a continuous function on $X$. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \emptyset$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \{t \in H_t : t < \beta\}$, hence, they are open subsets of $X$. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \{t \in H_t : t > \beta\}$ hence, each of them is an open subset of $X$. Consequently $h$ is a continuous function. 

**Lemma 3.3.** Suppose that $X$ is a topological space. If each pair of disjoint subsets $F_1, F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, can separate by open subsets of $X$, and $F_1$ (resp. $F_2$) is a countable intersection of open subsets of $X$, then there exists a continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$).

**Proof.** Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), where $G_n$ is an open subset of $X$. We can suppose that $G_n \cap F_2 = \emptyset$ (resp. $G_n \cap F_1 = \emptyset$), otherwise we can substitute $G_n$ by $G_n \setminus F_2$ (resp. $G_n \setminus F_1$). By Lemma 3.2, for every $n \in \mathbb{N}$, there exists a continuous function $h_n : X \to [0,1]$ such that, $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$) and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that $h$ is a continuous function from $X$ to $[0,1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$ (resp. $F_1 \subseteq X \setminus G_n$), therefore $h_n(F_2) = \{1\}$ (resp. $h_n(F_1) = \{1\}$) and consequently $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$). Since $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$), hence $h(F_1) = \{0\}$ (resp. $h(F_2) = \{0\}$). It suffices to show that if $x \notin F_1$ (resp. $x \notin F_2$), then $h(x) \neq 0$.

Now if $x \notin F_1$ (resp. $x \notin F_2$), since $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), therefore there exists $n_0 \in \mathbb{N}$ such that, $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$).
Lemma 3.4. Suppose that $X$ is a topological space such that every two disjoint semi-closed and preclosed subsets of $X$ can be separated by open subsets of $X$. The following conditions are equivalent:

(i) For every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, there exists a continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h^{-1}(1) = F_2$ (resp. $h^{-1}(1) = F_1$).

(ii) Every preclosed (resp. semi-closed) subset of $X$ is a countable intersection of open subsets of $X$.

(iii) Every preopen (resp. semi-open) subset of $X$ is a countable union of closed subsets of $X$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $F$ is a preclosed (resp. semi-closed) subset of $X$. Since $\emptyset$ is a semi-closed (resp. preclosed) subset of $X$, by (i) there exists a continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = F$. Set $G_n = \{ x \in X : h(x) < \frac{1}{n} \}$. Then for every $n \in \mathbb{N}$, $G_n$ is an open subset of $X$ and $\bigcap_{n=1}^{\infty} G_n = \{ x \in X : h(x) = 0 \} = F$.

(ii) $\Rightarrow$ (i) Suppose that $F_1$ and $F_2$ are two disjoint subsets of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed. By Lemma 3.3, there exists a continuous function $f : X \to [0, 1]$ such that, $f^{-1}(0) = F_1$ and $f(F_2) = \{ 1 \}$. Set $G = \{ x \in X : f(x) < \frac{1}{2} \}$, $F = \{ x \in X : f(x) = \frac{1}{2} \}$, and $H = \{ x \in X : f(x) > \frac{1}{2} \}$. Then $G \cup F$ and $H \cup F$ are two closed subsets of $X$ and $(G \cup F) \cap F_2 = \emptyset$. By Lemma 3.3, there exists a continuous function $g : X \to [\frac{1}{2}, 1]$ such that, $g^{-1}(1) = F_2$ and $g(G \cup F) = \{ \frac{1}{2} \}$.

Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then $h$ is well-defined and a continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0, 1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every open subset of $X$ is a closed subset of $X$ and complement of every closed subset of $X$ is an open subset of $X$, the equivalence is hold.

Corollary 3.5. If for every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed (resp. semi-closed) and $F_2$ is semi-closed (resp. preclosed), there exists a continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ then $X$ has the strong $c$-insertion property for $(pc, sc)$ (resp. $(sc, pc)$).

Proof. Since for every two disjoint subsets $F_1$ and $F_2$ of $X$, where $F_1$ is preclosed (resp. semi-closed) and $F_2$ is semi-closed (resp. preclosed), there exists a continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{ x \in X : h(x) < \frac{1}{2} \}$ and $G_2 = \{ x \in X : h(x) > \frac{1}{2} \}$. Then $G_1$ and $G_2$ are two disjoint open subsets of $X$ that contain $F_1$ and $F_2$, respectively. Hence by Corollary 3.4, $X$ has the weak $c$-insertion property for $(pc, sc)$ and $(sc, pc)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ is $pc$ (resp. $sc$) and $f$ is $c$. Since $f - g$ is $pc$ (resp. $sc$), therefore the lower cut set $A(f - g, 2^{-n}) = \{ x \in X : (f - g)(x) \leq 2^{-n} \}$ is a preclosed (resp. semi-closed) subset of $X$. By Lemma 3.4, we can choose a sequence
\( \{F_n\} \) of closed subsets of \( X \) such that, \( \{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n \) and for every \( n \in \mathbb{N} \), \( F_n \) and \( A(f - g, 2^{-n}) \) are disjoint subsets of \( X \). By Lemma 3.2, \( F_n \) and \( A(f - g, 2^{-n}) \) can be completely separated by continuous functions. Hence by Theorem 2.2, \( X \) has the strong \( c-\)insertion property for \((pc, c)\) (resp. \((sc, c)\)).

By an analogous argument, we can prove that \( X \) has the strong \( c-\)insertion property for \((c, sc)\) (resp. \((c, pc)\)). Hence, by Theorem 2.3, \( X \) has the strong \( c-\)insertion property for \((pc, sc)\) (resp. \((sc, pc)\)).

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References