KAPLANSKY CLASSES AND COTORSION THEORIES OF COMPLEXES

JAVAD ASADOLLAHI AND RASOOL HAFEZI

Abstract. In this paper we provide arguments for constructing Kaplansky classes in the category of complexes out of a Kaplansky class of modules. This leads to several complete cotorsion theories in such categories. Our method gives a unified proof for most of the known cotorsion theories in the category of complexes and can be applied to the category of quasi-coherent sheaves over a scheme as well as the category of the representations of a quiver.

1. Introduction

Let $P$ be a projective module and $x \in P$. Kaplansky [K] proved that there exists a countably generated summand of $P$ that contains $x$. Enochs and López-Ramos generalized this idea and introduced the notion of a Kaplansky class [EL, Definition 2.1]. A class $\mathcal{K}$ of $R$-modules is called a Kaplansky class if there exists a cardinal number $\kappa$ such that for every $M \in \mathcal{K}$ and for each $x \in M$, there exists a submodule $F$ of $M$ that contains $x$ with the property that $\text{Card}(F) \leq \kappa$ and both $F$ and $M/F$ are in $\mathcal{K}$. One of the important results related to this class of modules states that if a Kaplansky class $\mathcal{K}$ is closed under direct limits, then the pair $(\mathcal{K}, \mathcal{K}^\perp)$ is cogenerated by a set, i.e. there exists a set $S$ such that $S^\perp = \mathcal{K}^\perp$.

When we know a cotorsion theory of modules is cogenerated by a set, we immediately get that $\mathcal{K}$ is precovering and $\mathcal{K}^\perp$ is preenveloping, or equivalently, the cotorsion theory is complete, see [ET]. In fact, the flat cover conjecture is proved by showing that the famous cotorsion theory $(\text{Flat } R, \text{Cot } R)$ is cogenerated by a set, where $\text{Cot } R$ denotes the class of cotorsion modules, see [BBE].

On the other hand, there is a lot of interest in the complete cotorsion theories in the category of complexes. It not only proves the existence of certain covers and envelopes in the category of complexes [AERO], but also is related to the model category and also to the existence of certain adjoints. In fact, a famous result of Hovey [H02] relates the notion of cotorsion theory to the notion of model category, ‘a model category structure on an abelian category $\mathcal{A}$ that respects the abelian structure in a simple way is equivalent to two compatible complete cotorsion pairs on $\mathcal{A}$’, quoted from [H02, Introduction]. One of the upshots of this result was that the study of cotorsion pairs in the category of complexes attracted more attentions. Beside this, a recent result of a group of authors [BEIJR, Theorem 3.5] show that there is a tight connection between the complete cotorsion theories in the category of complexes of modules $\mathcal{C}(R)$ and the existence of adjoint functors on the corresponding homotopy categories. Hence, there has been several attempts to get cotorsion theories in

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The category of (unbounded) chain complexes over $R$, out of a cotorsion theory in $\text{Mod}_R$, see e.g. [AERO], [G04], [G07], [G08], [WL], [St] and [LCh].

Our aim in this paper is to construct several Kaplansky classes in the category of complexes starting from a Kaplansky class in the module category. Our argument will give a unified simple proof for most of the existing cotorsion theories in $\mathbb{C}(R)$. Furthermore, this method can be applied to prove the existence of some cotorsion theories in the category of quasi-coherent sheaves over a scheme as well as the category of representations of any quivers, see Remark 3.10 below.

2. Preliminaries

Let $R$ be an associative ring with identity. Let $\text{Mod}_R$ denote the category of (right) $R$-modules. We denote the category of (unbounded) chain complexes over $R$, $\mathbb{C}(\text{Mod}_R)$, by $\mathbb{C}(R)$. Given a complex $(X,d)$ (or $X$ for short)
\[ \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots \]
in $\mathbb{C}(R)$, we shall denote its $(n+1)$th syzygy $\ker d_n$ by $Z_nX$ and its $n$th boundary $\text{Im} d_{n+1}$ by $B_nX$. The subcategory of $\mathbb{C}(R)$ consisting of all exact complexes will be denoted by $\mathbb{E}(R)$.

Let $\mathcal{C}$ be a full subcategory of $\text{Mod}_R$. Consider the following subcategories of $\mathbb{C}(R)$:
- $\mathcal{C}_T$: consisting of all complexes $X$ with all terms in $\mathcal{C}$, that is, $X_n \in \mathcal{C}$ for all $n$.
- $\mathcal{C}_B$: consisting of all complexes $X$ with all boundaries in $\mathcal{C}$, that is, $B_nX \in \mathcal{C}$ for all $n$.
- $\mathcal{C}_Z$: consisting of all complexes $X$ with all cycles in $\mathcal{C}$, that is, $Z_nX \in \mathcal{C}$ for all $n$.

We may consider any intersections of these subcategories together as well as with $\mathbb{E}(R)$. We shall combine subscripts to denote the intersections. So, for example, $\mathcal{C}_T \cap \mathbb{E}(R)$ that consists of all exact complexes with terms in $\mathcal{C}$ will be denoted by $\mathcal{C}_{T \mathcal{E}}$. The category of all complexes with the property that all terms, all syzygies and all boundaries belong to $\mathcal{C}$ is the intersection of all the above three subcategories and will be denoted by $\mathcal{C}_{T \mathcal{E} \mathcal{Z}}$. It is clear that $\mathcal{C}_{B \mathcal{E}} = \mathcal{C}_{Z \mathcal{E}}$ and if $\mathcal{C}$ is closed under extension, $\mathcal{C}_{T \mathcal{E} \mathcal{Z}} = \mathcal{C}_{Z \mathcal{E}}$.

Clearly if we let $\mathcal{C}$ to be a certain class of modules, we will get some familiar classes of complexes. For example, if $\mathcal{C} = \text{Flat}_R$, the class of flat $R$-modules, (resp. $\mathcal{C} = \text{Prj}_R$, the class of projective $R$-modules or $\mathcal{C} = \text{Inj}_R$, the class of injective $R$-modules) then $\mathcal{C}_{Z \mathcal{E}}$ is the class of all flat (resp. projective or injective) complexes in $\mathbb{C}(R)$. As another example, if $\mathcal{C} = \text{GPrj}_R$ (resp. $\mathcal{C} = \text{GInj}_R$) the class of Gorenstein projective (resp. Gorenstein injective) $R$-modules, then $\mathcal{T}_R$ is the class of Gorenstein projective (resp. Gorenstein injective) complexes in $\mathbb{C}(R)$, see [EEI] and [YL].

Some of the above classes has already been studied in literature with different notations. For example, Gillespie in [G08] used the notations $\tilde{\mathcal{C}}$, $d\tilde{\mathcal{C}}$ and $e\tilde{\mathcal{C}}$ to denote $\mathcal{C}_{Z \mathcal{E}}$, $\mathcal{C}_T$ and $\mathcal{C}_{T \mathcal{E}}$, respectively. See also [LCh], for different notations.

2.1. Let $M$ be an object of $\mathcal{A}$ and $\lambda$ be an ordinal number. A set $\{M_\alpha\}_{\alpha<\lambda}$ of subobjects of $M$ is called a continuous chain of subobjects if $M_\alpha$ is a subobject of $M_\beta$, for all $\alpha \leq \beta < \lambda$, and $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$ whenever $\gamma < \lambda$ is a limit ordinal.

It is proved by Eklof [E, Theorem 1.2] that if a module $M$ is a direct union of a continuous chain of submodules $\{M_\alpha\}_{\alpha<\lambda}$ such that $\text{Ext}^1(M_0,C) = 0$ and $\text{Ext}^1(M_{\alpha+1}/M_\alpha,C) = 0$ for all $\alpha < \lambda$, then $\text{Ext}^1(M,C) = 0$, see also [ET, Lemma 1]. It is known that this result also holds for any Grothendieck category with a projective generator. Recall that a Grothendieck
category is an abelian category with a generator and with the property that direct limits are exact.

2.2. A class $\mathcal{K}$ of $R$-modules is called a $\kappa$-Kaplansky class if there exists a cardinal number $\kappa$ such that for every $M \in \mathcal{K}$ and for any subset $S \subseteq M$ with $\text{Card}(S) \leq \kappa$, there exists a submodule $F$ of $M$ that contains $S$ with the property that $\text{Card}(F) \leq \kappa$ and both $F$ and $M/F$ are in $\mathcal{K}$. We say that $\mathcal{K}$ is a Kaplansky class if it is a $\kappa$-Kaplansky class for some regular cardinal $\kappa$, see [EL, Definition 2.1].

Besides the class of projective modules, the well-known Kaplansky classes are the class of injective and also the class of flat $R$-modules. Furthermore, it is proved [EL, Propositions 2.6 and 2.10] that the class of Gorenstein flat modules over any ring is Kaplansky while the class of Gorenstein injective (left) $R$-modules is Kaplansky, if the ground ring is left noetherian.

**Notation.** Let $X$ be a complex in $C(R)$. By the cardinality of $X$, $\text{Card}(X)$, we mean $\text{Card}(\prod_{n \in \mathbb{Z}} X_n)$. By a subset $S$ of $X$ we mean a family $(S_n)_{n \in \mathbb{Z}}$ such that $S_n$ is a subset of $X_n$, for $n \in \mathbb{Z}$. We define the cardinality of $S$ to be $\text{Card}(\bigcup_{n \in \mathbb{Z}} S_n)$, the disjoint union of the sets $S_n$, $n \in \mathbb{Z}$.

Throughout, we assume that all cardinals are regular, that is, are infinite cardinals which are not the sum of a smaller number of smaller cardinals. We let $\omega$ denote the first limit ordinal.

### 3. Kaplansky classes of complexes

The following result and its corollary seems to be known in the literature. It follows by the same argument as in [EL, Theorem 2.8]. They also appeared, without proof, in [LCh, Lemma 3.9, Theorem 3.10]. For the convenience of the reader we present a simple proof.

**Theorem 3.1.** Let $\mathcal{C}_x$ be a Kaplansky class in $C(R)$ which is closed under well ordered direct limits. Then the pair $(\mathcal{C}_x, \mathcal{C}_x^\perp)$ is cogenerated by a set.

**Proof.** Let $\kappa$ be the cardinal number such that $\mathcal{C}$ is $\kappa$-Kaplansky. We claim that any object $X$ of $\mathcal{C}_x$ can be written as a direct union of a continues chain of subobjects $\{X_\alpha\}_{\alpha < \lambda}$ of $X$ such that $\text{Card}(X_0)$ and $\text{Card}(X_{\alpha+1}/X_\alpha)$ are both $\leq \kappa$, whenever $\alpha + 1 < \lambda$. If we prove the claim, then we may consider the desired set to be the set of all representative of objects $Y$ in $\mathcal{C}_x$ with $\text{Card}(Y) \leq \kappa$. To prove the claim, let $X \in \mathcal{C}_x$ be an arbitrary object and consider $x \in X$. Since $\mathcal{C}_x$ is Kaplansky, there exists an object $X_0 \subseteq X$ in $\mathcal{C}_x$ such that $x \in X_0$, $\text{Card}(X_0) \leq \kappa$ and $X/X_0 \in \mathcal{C}_x$. Now consider $X/X_0$ and pick an element $x \in X \setminus X_0$. Again one may use the Kaplansky property to obtain $X_1$ such that $x \in X_1$, $X_0 \subseteq X_1 \subseteq X$, $\text{Card}(X_1/X_0) \leq \kappa$ and $X_1/X_0$ and $X/X_1$ are in $\mathcal{C}_x$. So following this procedure we have $X_\omega = \bigcup_{\alpha < \omega} X_\alpha$. But $X_\omega \in \mathcal{C}_x$, as $\mathcal{C}_x$ is closed under well ordered direct limits. Hence we may continue the process. But this process should be stopped otherwise we get an upper bound on the ordinal numbers, which is impossible. Therefore the claim.

Recall that a cotorsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects of $\mathcal{A}$ such that $\mathcal{X}^\perp = \mathcal{Y}$ and $\mathcal{X} = \mathcal{Y}^\perp$, where the orthogonals are taken with respect to $\text{Ext}^1$. $(\mathcal{X}, \mathcal{Y})$ is called complete if it has both enough projective and enough injective objects. This in particular implies that every module has an $\mathcal{X}$-precover and a $\mathcal{Y}$-preenvelope. $(\mathcal{X}, \mathcal{Y})$ is
called perfect if every module has an \( \mathcal{X} \)-cover and \( \mathcal{Y} \)-envelope. We say that \((\mathcal{X}, \mathcal{Y})\) in \(\mathcal{A}\) is cogenerated by a set if there exists a set \(S \subseteq \mathcal{X}\) such that \(S^\bot = \mathcal{Y}\). A well-known result of Eklof and Trlifaj [ET, Theorem 10] says that every cotorsion theory in \(\text{Mod}R\) which is cogenerated by a set of modules is complete. If furthermore, \(\mathcal{X}\) is closed under direct limits, then it is a perfect cotorsion theory.

**Corollary 3.2.** Let \(\mathcal{C}_x\) be a Kaplansky class in \(\mathcal{C}(R)\) that contains all projective objects and is closed under direct sums, direct summands, extensions and continuous well-ordered unions. Then \((\mathcal{C}_x, \mathcal{C}^\bot_x)\) is a perfect cotorsion theory.

*Proof.* In view of the above theorem, the pair \((\mathcal{C}_x, \mathcal{C}^\bot_x)\) is cogenerated by a set. Now the result follows from Corollaries 2.13, 2.12 and 2.11 of [AERO]. \(\Box\)

So it is important to have Kaplansky classes in \(\mathcal{C}(R)\). In our next theorem, we will present an argument to get such classes out of a given Kaplansky class of modules. To this end, we need the following lemma. Recall that a subcategory \(\mathcal{X}\) of \(\mathcal{A}\) is said to be resolving if it contains the projective objects and is closed under extensions and kernels of epimorphisms.

**Lemma 3.3.** Let \(\mathcal{K}\) be a resolving class of modules which is \(\kappa\)-Kaplansky. Consider the short exact sequence

\[
0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} L \rightarrow 0
\]

in \(\mathcal{K}\). Assume that there exists a subset \(T \subseteq N\), such that \(\text{Card}(T) \leq \kappa\). Then there exists a diagram

\[
\begin{array}{ccccccc}
0 & 0 & 0 \\
\downarrow & & & \downarrow \\
0 & \rightarrow & M' & \rightarrow & N' & \rightarrow & L' & \rightarrow & 0 \\
\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & & \\
0 & \rightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & L & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
M/M' & \rightarrow & N/N' & \rightarrow & L/L' & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & & & & \\
0 & 0 & 0 & & & & & \\
\end{array}
\]

such that \(T \subseteq N', \max\{\text{Card}(M'), \text{Card}(N'), \text{Card}(L')\} \leq \kappa\) and all terms, except probably \(M/M', \) belongs to \(\mathcal{K}\).

*Proof.* Since \(g(T)\) is a subset of \(L\) with the property that \(\text{Card}(g(T)) \leq \kappa\), there exists a submodule \(L'\) of \(L\) such that \(g(T) \subseteq L', \text{Card}(L') \leq \kappa\) and both \(L'\) and \(L/L'\) belong to \(\mathcal{K}\).
Consider the pull back diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow M \rightarrow N_1 \rightarrow L' \rightarrow 0 \\
\downarrow f \\
0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \\
\downarrow g \\
0 \rightarrow L/L' \rightarrow L/L' \\
\end{array}
\]

Since \(g(T) \subseteq L'\), it follows from the above diagram that \(T \subseteq N_1\). Moreover, since \(K\) is resolving and \(M\) and \(L'\) are in \(K\), so is \(N_1\). On the other hand, since \(\text{Card}(L') \leq \kappa\), there exists a set \(T_1 \subseteq N_1\) such that \(g(T_1) = L'\) and \(\text{Card}(T_1) \leq \kappa\). Therefore, there exists a submodule \(N'\) of \(N_1\) such that \(T \cup T_1 \subseteq N'\), \(\text{Card}(N') \leq \kappa\) and \(N'\) and \(N_1/N'\) are in \(K\). So we have the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow \ker g| \rightarrow N'/ N_1 \rightarrow 0 \\
\downarrow f \\
0 \rightarrow M \rightarrow N_1 \rightarrow L' \\
\downarrow g \\
0 \rightarrow L'/ L_1' \\
\end{array}
\]

Since \(K\) is resolving, \(\ker g| \in K\) and clearly \(\text{Card}(\ker g|) \leq \kappa\). To complete the proof, consider the short exact sequence

\[0 \rightarrow N_1/N' \rightarrow N/N' \rightarrow N_1 \rightarrow 0.\]

By our construction, the end terms are in \(K\), hence so is the middle term, as \(K\) is resolving. □

In case \(C\) is the class of flat modules, parts (1) and (2) of the following result has been proved in [AERO, Proposition 3.1]. Here we prove it for an arbitrary Kaplansky class by a shorter and simpler proof. In fact, the novelty of our proof is that we do not need to construct the subcomplexes, as they did in their proof. Instead we will work with subsets of complexes. As we will see, this efficiently simplifies the argument.

**Theorem 3.4.** Let \(C \subseteq \text{Mod}R\) be a Kaplansky class of modules which is closed under the direct unions of continuous chains of submodules. Then we have the following.

1. \(C_T\) is a Kaplansky class of complexes.
2. \(C_{TE}\) is a Kaplansky class of complexes.
3. \(C_{TB} = C_{TZE}\) is a Kaplansky class of complexes, provided \(C\) is resolving.

Note that all of these classes are closed under the direct union of continuous chains of submodules.

**Proof.** Let \(\kappa\) be the cardinal number such that \(C\) is \(\kappa\)-Kaplansky and let \(\kappa'\) be a cardinal larger than \(\max\{\kappa, \omega, \text{Card}(R)\}\). In all parts, we plan to show that \(C_x\) is \(\kappa'\)-Kaplansky, where
$x$ is any of the elements of the set \{T, TE, TBE\}. So assume that $X$ is an object of $\mathcal{C}_x$. Let $S$ be a subset of $X$ with $\text{Card}(S) \leq \kappa'$. We show that there exists a complex $X'$ in $\mathcal{C}_x$ such that $S \subseteq X'$, $\text{Card}(X') \leq \kappa'$ and $X'$ and $X/X'$ belongs to $\mathcal{C}_x$. Note that we may well order the set $\{i\}_{i \leq n < \omega} \in \omega$ as $\{i_n\}_{n \leq n < \omega}$.

1. To begin, let $X^1$ be the subcomplex of $X$ generated by $S$. Let us denote it by $\langle S \rangle$. It is clear that $\text{Card}(X^1) \leq \kappa'$. For any integer $n \geq 1$, we will inductively construct a subset $X^n$ of $X$, satisfying the following four properties:

   (i) for any integer $i \in \mathbb{Z}$, the $i$th term of $X^n$, denoted $(X^n)_i$, is a submodules of the $i$th term of $X$;

   (ii) $X^i$ is a subset (not necessarily subcomplex) of $X^n$, for any $i \leq n$;

   (iii) $\text{Card}(X^n) \leq \kappa'$;

   (iv) $(X^n)_i$ and $X_i/(X^n)_i$ both belong to $\mathcal{C}$.

Assume inductively that $n > 1$ and we have already constructed $X^{n-1}$. Let $(X^{n-1})_i$ be the $i$th term of $X^{n-1}$. Since $\text{Card}((X^{n-1})_i) \leq \kappa'$, there exists an $R$-module $C \subseteq \mathcal{C}$ such that $\text{Card}(C) \leq \kappa$, $(X^{n-1})_i \subseteq C$ and $X_i/C \in \mathcal{C}$. Define $X^n$ as follows: for $i \neq i_n$, we set $(X^n)_i = (X^{n-1})_i$ and for $i = i_n$, we set $(X^n)_i = C$. Note that we do not need $X^n$ to be a subcomplex of $X$. Now set $X^\omega = \bigcup_{i < \omega} X^i$. To continue we define $X^{\omega+1}$ to be the subset of $X$ generated by $X^\omega$, and continue this zig-zag procedure to get $X^{\omega+n}$. Finally we set $X' = \bigcup_{k \in \mathbb{N}} X^{k \omega+n}$.

2. Follow the argument in (1) and construct $X^\omega$. Now we plan to continue the process one more step to force the desired complex $X'$ to be exact. This will imply that $X/X'$ is also exact. We construct $X^{\omega+1}$ as follows. Since $X$ is an exact complex and $\text{Card}(X) \leq \kappa'$, we can take $T \subseteq X_{i+1}$ such that $\text{Ker}\partial_{i+1}(X^\omega)_i \subseteq \partial_{i+1}(T)$ and $\text{Card}(T) \leq \kappa'$. Now define $X^{\omega+1}$ with the same terms as in $X^\omega$, for all $i$ not equal to $i_1 + 1$ and $(X^{\omega+1})_{i_1+1} = (X^{\omega+1})_{i_1+1} + T$. This implies that $\text{Ker}\partial_{i_1}(X^{\omega+1})_i \subseteq \partial_{i_1+1}((X^{\omega+1})_{i_1+1})$. We may continue this process to construct $X^{\omega+n}$, for any $n$. Set $X^{2\omega} = \bigcup_{i < 2\omega} X^i$. Now again go back to part (1), set $X^{2\omega+1} = (X^{2\omega})$ and follow the same argument. Finally set $X' = \bigcup_{k \in \mathbb{N}} X^{k \omega+n}$. It is routine to check that $X'$ is the desired complex.

3. Follow the argument in part (2) and construct $X^\omega$. To construct $X^{\omega+1}$, note that $X^\omega_{i+1}$ is a submodule of $X_{i_1}$. So by Lemma 3.3, there exists an $R$-submodule $X^{\omega+1}_{i_1}$ of $X_{i_1}$ in $\mathcal{C}$ such that $\text{Card}(X^{\omega+1}_{i_1}) \leq \kappa'$. Define $X^{\omega+1}$, as follows. For $i = i_1$, set $(X^{\omega+1})_{i_1} = X^{\omega}_{i_1}$ and for $i$ not equal to $i_1$, set $(X^{\omega+1})_i = (X^\omega)_i$. We may continue this process and construct $X^{\omega+n}$, for any $n$. Set $X^{2\omega} = \bigcup_{i < 2\omega} X^i$. Go back to part (2), let $X^{2\omega+1}$ be the subcomplex of $X$ generated by $X^{2\omega}$ and follow the same argument. Finally set $X' = \bigcup_{k \in \mathbb{N}} X^{k \omega+n}$. By construction, it follows that $X' \in \mathcal{C}_{TBZ}$ and $\text{Card}(X') \leq \kappa'$. It also follows from the short exact sequence

$$0 \to X' \to X \to X/X' \to 0$$

of complexes that $X/X'$ is exact. So to complete the proof we only show that the kernels and images of $X/X'$ are in $\mathcal{C}$. Since $X/X'$ is exact, we just need to consider the images. The above short exact sequence of complexes, implies easily that $B(X/X') \cong BX/BX'$. But the latter module belongs to $\mathcal{C}$, because of the construction of Lemma 3.3.

\[\square\]

**Remark 3.5.** One may follow the argument in part (2) to show that the pair $(E(R), E(R)^{1})$ is cogenerated by a set and hence is a complete cotorsion theory. This fact has already been proved in [AERO, Corollary 4.5].
Remark 3.6. There are some results in [LCh], related to the above theorem. With our notations, they proved that if $\mathcal{C}$ is a class of $R$-modules closed under pure submodules and pure quotients, then $\mathcal{C}_{ZE}$ is a Kaplansky class of complexes [LCh, Corollary 3.4]. If furthermore, we know that $\mathcal{C}$ is closed under direct limits, then $\mathcal{C}_{TE}$ is a Kaplansky class of complexes [LCh, Corollary 3.7].

It is known that if a class of modules is closed under pure submodules and pure quotients it is Kaplansky [EO]. This means that part (2) of the above theorem covers Corollary 3.7 of [LCh].

On the other hand, there are examples of Kaplansky classes that are closed under direct unions of continuous chains of modules but not closed under pure submodules and pure quotients. One formal example is the class $\mathcal{C}$ of infinitely generated modules over an arbitrary ring $R$. Clearly $\mathcal{C}$ is Kaplansky and closed under unions of continuous chains of submodules, but not under direct limits. Now if we assume that $R$ is an artin algebra of finite representation type, then also each infinitely generated module has a finitely generated summand and hence $\mathcal{C}$ is not closed under pure submodules, see eg. page 2 in [GT]. Another class which has been better understood only recently, is the class of flat Mittag-Leffler modules over a ring. This is always a Kaplansky class closed under pure submodules and transfinite extensions, but not under direct limits unless the ring is perfect. This has been studied for instance in [HT] and recently generalized in [ST]. Thus the above theorem extends results given in [LCh].

3.7. DECONSTRUCTIBLE CLASSES. There is a tight connection between the notion of Kaplansky classes and deconstructible classes, introduced and studied by Eklof in 2006. Let us recall briefly the definition of a deconstructible class. A filtration of an $R$-module $M$ is a continuous chain of submodules $\{M_\alpha : \alpha < \sigma\}$ of $M$ whose union is $M$ such that $M_0 = 0$ and $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ for all limit ordinals $\beta < \sigma$. Let $\mathcal{S}$ be a class of $R$-modules. A module $M$ is said to be $\mathcal{S}$-filtered if it has a filtration with the property that $M_{\alpha+1}/M_\alpha \in \mathcal{S}$ for all $\alpha + 1 < \sigma$. The class of all $\mathcal{S}$-filtered modules will be denoted by $\text{Filt} - \mathcal{S}$.

A class $\mathcal{F}$ of modules is called deconstructible if there exists a set $\mathcal{S}$ such that $\mathcal{F} = \text{Filt} - \mathcal{S}$. For example, the class of all $R$-modules is deconstructible. To see this just take $\mathcal{S}$ to be the set of all modules $M$ with $\text{Card}(M) \leq \text{Card}(R)$. Also, the class $\mathcal{F}$ of all flat modules is deconstructible. Moreover it is known that the class $\text{GPrj} R$ (resp. $\text{GFlat} R$) consisting of all Gorenstein projective (resp. Gorenstein flat) modules are deconstructible, provided $R$ is an Iwanaga-Gorenstein ring.

It is known that each deconstructible class of modules is Kaplansky and the converse holds for classes closed under direct limits, but not in general, see Lemmas 6.7 and 6.9 and Example 6.8 of [HT]. In fact, we have the following sequence of strict inclusions

$$\left\{\begin{array}{c}
\text{Kaplansky classes} \\
\text{closed under direct limit}
\end{array}\right\} \subsetneq \left\{\begin{array}{c}
\text{Deconstructible} \\
\text{classes}
\end{array}\right\} \subsetneq \left\{\begin{array}{c}
\text{Kaplansky} \\
\text{classes}
\end{array}\right\}.$$ 

Štovíček [St, Theorem 4.2] proves that if $\mathcal{C}$ is a deconstructible class of objects in a Grothendieck category $\mathcal{G}$, then $\mathcal{C}_T$ and $\mathcal{C}_{ZE}$ are deconstructible classes in $\mathcal{C}(\mathcal{G})$. Instead of starting with a deconstructible class we start with a Kaplansky class closed under direct limit and get a Kaplansky class of complexes which is closed under direct limit. Since $\mathcal{C}_T$ and $\mathcal{C}_{ZE}$ are closed under direct limits, it is clear that they are deconstructible. So our result
reverses Theorem 4.2 of [St].

Given a module $M \in \text{Mod } R$ and integer $i \in \mathbb{Z}$, we let $S^i(M)$ denote the complex with $M$ in the $i$-th place and 0 in the other places.

**Theorem 3.8.** Let $\mathcal{C}$ be a class of $R$-modules such that the pair $(\mathcal{C}, \mathcal{C}^\perp)$ is cogenerated by a set. Then the pair $(\mathcal{C}_{\text{TBZ}}, \mathcal{C}^\perp_{\text{TBZ}})$ is cogenerated by a set.

**Proof.** Let $S_0$ be a set such that $S_0^\perp = C^\perp$. Set $S = \{S^i(M) : M \in S_0\}$. We claim that $S^\perp = C^\perp_{\text{TBZ}}$. It is clear that $S \subseteq C_{\text{TBZ}}$ and so $C^\perp_{\text{TBZ}} \subseteq S^\perp$. To see the reverse inclusion, pick $Y \in S^\perp$. Let $X \in C_{\text{TBZ}}$. There exists a short exact sequence

$$0 \to \bigoplus_{i \in \mathbb{Z}} S^i(\ker f_i) \to X \to \bigoplus_{i \in \mathbb{Z}} S^i(\text{im} f_i) \to 0.$$ 

Now apply the functor $\text{Hom}_{\mathcal{C}(R)}(-, Y)$ on this sequence to get the exact sequence

$$\text{Ext}_{\mathcal{C}(R)}^1\left(\bigoplus_{i \in \mathbb{Z}} S^i(\text{im} f_i), Y\right) \to \text{Ext}_{\mathcal{C}(R)}^1(X, Y) \to \text{Ext}_{\mathcal{C}(R)}^1\left(\bigoplus_{i \in \mathbb{Z}} S^i(\ker f_i), Y\right).$$

So to complete the proof it is enough to show that

$$\text{Ext}_{\mathcal{C}(R)}^1\left(S^i(\text{im} f_i), Y\right) = 0 = \text{Ext}_{\mathcal{C}(R)}^1\left(S^i(\ker f_i), Y\right),$$

for all $i \in \mathbb{Z}$.

By Lemma 3.1 of [G04], for an $R$-module $M$, we have a natural isomorphism

$$\text{Ext}_{\mathcal{C}(R)}^1(S^i(M), Y) \cong \text{Ext}_{R}^1(M, Z_i Y).$$

Now since $Y \in S^\perp$, we have $\text{Ext}_{\mathcal{C}(R)}^1(S^i(M), Y) = 0$, for all integers $i$ and all $M \in S_0$. Therefore, in view of the above isomorphism, we have $\text{Ext}_{R}^1(M, Z_i Y) = 0$, for all $i \in \mathbb{Z}$ and all $M \in S_0$. Since $S_0^\perp = C^\perp$, we deduce that $\text{Ext}_{R}^1(M, Z_i Y) = 0$, for all $i \in \mathbb{Z}$ and all $M \in \mathcal{C}$. Hence another use of the above natural isomorphism, implies that $\text{Ext}_{\mathcal{C}(R)}^1(S^i(M), Y) = 0$, for all $i$ and all $M \in \mathcal{C}$. Now the result follows because $X \in C_{\text{TBZ}}$ and so $\ker f_i$ and $\text{im} f_i$ are in $\mathcal{C}$, for all integers $i$. 

$\square$
Corollary 3.9. Let $\mathcal{C} \subseteq \text{Mod} R$ be a Kaplansky class of modules which is closed under the direct unions of continuous chains of submodules. Then the pair $(\mathcal{C}_x, \mathcal{C}_x^\perp)$ is cogenerated by a set, whenever $x$ is any element of the set $\{T, TE, \text{TBZ}\}$. If moreover, $\mathcal{C}$ is resolving, the pair $(\mathcal{C}_{\text{TBZ}}, \mathcal{C}_{\text{TBZ}}^\perp)$ is cogenerated by a set.

Proof. This follows from Theorem 3.4 and Theorem 3.8. Just note that when $\mathcal{C}$ is Kaplansky, the pair $(\mathcal{C}, \mathcal{C}^\perp)$ is cogenerated by a set. □

Remark 3.10. (1). Let $\mathcal{C}$ be a class of $R$-modules, where $R$ is a ring. Let $Q$ be any quiver with the set of vertices denoted by $V$. Finally, let $\mathcal{C}_T$ denotes the class of all representations $M$ of $Q$ such that $M(v) \in \mathcal{C}$, for all $v \in V$. This is consistent with our previous notation, in case $Q = A_\infty$. When $\mathcal{C} = \mathcal{F}$ is the class of flat modules, $\mathcal{F}_T$ has been introduced and studied in [EO] as the class of the componentwise flat representations. They proved that the pair $(\mathcal{F}_T, \mathcal{F}_T^\perp)$ is cogenerated by a set and hence any representation of $Q$ admits an $\mathcal{F}_T$-cover and $\mathcal{F}_T^\perp$-envelope, see [EO, Theorem 4.5 and Corollary 4.6]. Actually our method can be applied to generalize this result to an arbitrary Kaplansky class of modules which is closed under the direct unions of continuous chains of submodules. In fact, we well-order the set of vertices of $Q$ and instead of working with subrepresentations, we work with subsets. As we saw, this simplifies the argument.

(2). Let $X$ be a scheme with the structure sheaf $\mathcal{O}$ and $V$ be a family of open sets that covers $X$ as well as all intersections $u \cap v$, with $u, v \in V$. For each $v \in V$, choose a deconstructible class $\mathcal{F}_v \subseteq \text{Mod-}\mathcal{O}(v)$. It is proved [EGPT, Theorem 3.7] that the class of all quasi-coherent sheaves defined as
\[ \mathcal{F} = \{ X' \in \text{Qco}(X) \mid X'(v) \in \mathcal{F}_v \text{ for each } v \in V \} \]
is deconstructible in $\text{Qco}(X)$. Our method here show that we may start with any Kaplansky class which is closed under direct limits and reach to a Kaplansky class in $\text{Qco}(X)$ which is closed under direct limits, that is, one can get Kaplansky classes of sheaves locally. In particular, if for any $v \in V$ we let $\mathcal{F}_v$ to be the class of flat $\mathcal{O}(v)$-modules, we recover the main result of [EE].

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