A GENERALIZATION OF REDUCED RINGS

A. R. NASR-ISFAHANI$^A$ AND A. MOUSSAVI$^B$

$^A$Department of Mathematics, University of Isfahan, P.O. Box: 81746-73441, Isfahan, Iran. E-mail: Corresponding author: a_nasr_isfahani@yahoo.com

$^B$Department of Mathematics, Tarbiat Modares University, P.O. Box:14115-175, Tehran, Iran. E-mail: moussavai_a@modares.ac.ir

Abstract. For a ring derivation $\delta$, we introduce and investigate a generalization of reduced rings and Armendariz rings which we call a $\delta$-Armendariz ring. Various classes of $\delta$-Armendariz rings is provided and a number of properties of this generalization are established. Radicals and minimal prime ideals of the differential polynomial ring $R[x;\delta]$, in terms of those of a $\delta$-Armendariz $R$, is determined. We prove that several properties transfer between $R$ and the differential polynomial ring $R[x;\delta]$, in case $R$ is $\delta$-Armendariz.

Keywords : differential polynomial ring, Jacobson radical, prime radical, 2-primal ring, Baer ring, Armendariz ring.

AMS Subject Classification: 16S36; 16N20; 16N40

1. Introduction

Throughout this paper $R$ denotes an associative ring with unity, $\delta$ is a derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x;\delta]$ the differential polynomial ring whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = ax + \delta(a)$ for any $a \in R$.

In [2], S. A. Amitsur proved that the Jacobson radical $J(R[x])$ of the polynomial ring $R[x]$ is the polynomial ring over the nil ideal $J(R[x]) \cap R$. In [22], D. A. Jordan considered the Jacobson radical and the nilpotent radical of a differential polynomial ring in the Noetherian case. In [16], M. Ferrero proved that $\alpha(R[x;\delta]) = (\alpha(R[x;\delta]) \cap R)[x;\delta]$, where $\alpha$ is a radical in a class of radicals.

One drawback of these theorems is that it does not determine what $\alpha(R[x;\delta]) \cap R$ really is, other than it is a nil ideal, in some special cases. An interesting problem is to determine whether $J(R[x]) \cap R$ in Amitsur’s Theorem is indeed equal to the the upper nil radical $\text{Nil}^*(R)$? In fact it turns out to be equivalent to another famous problem called Kothe’s Conjecture.

Recall that a ring satisfies Kothe’s Conjecture if the upper nilradical contains every nil left ideal. A ring is reduced if it contains no nonzero nilpotent elements.

The purpose of this paper is to introduce and investigate a generalization of reduced rings which we call a $\delta$-Armendariz ring. Although reduced rings are $\delta$-Armendariz for every derivation $\delta$ of $R$, but we will provide a fairly rich class of non-reduced $\delta$-Armendariz rings.

In section 2 we show that, if $R$ is a $\delta$-Armendariz ring, then both $R$ and $R[x;\delta]$ satisfy the Kothe’s conjecture, and that $\alpha(R[x;\delta]) \cap R = \text{Nil}^*(R)$, where $\alpha$ is a
radical in a class of radicals which includes the Wedderburn, lower nil, Levitzky, upper nil and Jacobson radicals. We determine the radicals and minimal prime ideals of the differential polynomial ring \( R[x;\delta] \), in case \( R \) is a \( \delta \)-Armendariz ring.

In [33] G. Marks examined several conditions on a noncommutative ring which imply that it is 2-primal (i.e., the rings prime radical coincides with the set of nilpotent elements of the ring). G.F. Birkenmeier et.al, proved [8, Proposition 2.6] that the 2-primal condition is inherited by ordinary polynomial extensions. Ore extensions do not generally preserve the 2-primal condition. When \( R \) is 2-primal, a differential polynomial ring need not be 2-primal ([20, Example 2.1], also [15, Example 2.1]). In section 3, among the others, we prove that, if \( R \) is a weak \( \delta \)-Armendariz ring, then \( R \) is 2-primal if and only if \( R[x;\delta] \) is 2-primal.

In section 4, we prove that several properties, including the symmetric, reversible, \( ZC_n \), \( IFF \), zip, Baer, quasi Baer, p.p. and p.q.-Baer property, transfer between \( R \) and the differential polynomial ring \( R[x;\delta] \), in case \( R \) is \( \delta \)-Armendariz. We show by an example that, there exists a commutative ring \( R \) and a derivation \( \delta \) such that \( R[x;\delta] \) is neither symmetric nor reversible nor \( ZC_n \).

In section 5, we will provide several classes of non-reduced \( \delta \)-Armendariz rings. We also show that, an Ore ring \( R \) is (weak)\( \delta \)-Armendariz if and only if its classical quotient ring \( Q \) is (weak)\( \delta \)-Armendariz.

## 2. ON RADICALS OF DIFFERENTIAL POLYNOMIAL RINGS

A ring \( R \) is called Armendariz if whenever polynomials \( f(x) = a_0 + a_1x + \cdots + a_n x^n \), \( g(x) = b_0 + b_1x + \cdots + b_m x^m \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( a_ib_j = 0 \) for each \( i,j \).

The study of Armendariz rings was initiated by Armendariz [5] and Rege and Chhawchharia [37]. The more comprehensive study of Armendariz rings was carried out recently (see, e.g., [3], [5], [21], [25], [28], [29] and [37]). The interest of this notion lies in its natural and its useful role in understanding the relation between the annihilators of the ring \( R \) and the annihilators of the polynomial ring \( R[x] \), (see [19]).

We start with the following.

**Theorem 2.1.** Let \( R \) be a ring and \( \delta \) a derivation of \( R \). Then the following statements are equivalent.

1. For each \( f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta] \), if \( f(x)g(x) = 0 \), it implies \( a_i \delta^i(b_j) = 0 \) for each \( 0 \leq i \leq n, 0 \leq j \leq m \).
2. For each \( f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta] \), if \( f(x)g(x) = 0 \), it implies \( a_0 b_j = 0 \) for each \( 0 \leq j \leq m \).
3. For each \( f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta] \), if \( f(x)g(x) = 0 \), it implies \( a_i b_j = 0 \) for each \( 0 \leq i \leq n, 0 \leq j \leq m \).
4. For each \( f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta] \), if \( f(x)g(x) = 0 \), it implies \( a_i x^i b_j x^j = 0 \) for each \( 0 \leq i \leq n, 0 \leq j \leq m \).

**Proof.** (1) \( \Rightarrow \) (2), is clear.

(2) \( \Rightarrow \) (3). Let \( f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta] \), and \( f(x)g(x) = 0 \).

By induction on the degree of \( f(x) \) we show that \( a_i b_j = 0 \) for each \( 0 \leq i \leq n, 0 \leq j \leq m \).

If \( n = 0 \), then \( f(x) = a_0 \), and the result is true since \( f(x)g(x) = 0 \).
Now assume that the result is true for each \( k \leq n \) and assume \( k = n + 1 \). Let \( f(x) = \sum_{i=0}^{n+1} a_i x^i \) and \( f(x)g(x) = 0 \). So by (2), for each \( 0 \leq j \leq m \), \( a_0 b_j = 0 \). Thus \( f(x)g(x) = a_0 g(x) + (a_1 + a_2 x + \cdots + a_{n+1} x^n) x g(x) \). So \((a_1 + a_2 x + \cdots + a_{n+1} x^n) x g(x) = 0\), and hence \((a_1 + a_2 x + \cdots + a_{n+1} x^n) (\delta b_j) + b_0 x + \delta (b_1) x + b_1 x^2 + \delta (b_2) x^2 + \cdots + b_{m-1} x^m + \delta (b_m) x^m + b_m x^{m+1} = 0\). So by the induction hypothesis for each \( 1 \leq i \leq n + 1, 0 \leq j \leq m - 1, a_i (b_j + \delta (b_{j+1})) = 0 \) and \( a_i b_m = 0 \), for each \( 1 \leq i \leq n \). So \( \delta (a_i) b_m + a_i \delta (b_m) = 0 \). Let \( p(x) = \delta (a_i) + a_i x \) and \( q(x) = b_m + b_m x \). Then \( p(x)q(x) = 0 \). So \( \delta (a_i) b_m = 0 \) by 2 and hence \( a_i \delta (b_m) = 0 \), for \( 1 \leq i \leq n + 1 \). Since for each \( 1 \leq i \leq n + 1, a_i (b_{m-1} + \delta (b_m)) = 0 \), so \( a_i b_{m-1} = 0 \). By the above argument, \( a_i \delta (b_{m-1}) = 0 \). On the other hand \( a_i (b_{m-2} + \delta (b_{m-1})) = 0 \) so \( a_i b_{m-2} = 0 \). By continuing in this way, we get \( a_i b_j = 0 \) for each \( 1 \leq i \leq n + 1 \) and \( 0 \leq j \leq m \). So the result follows.

(3) \( \Rightarrow \) (4). Let \( f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \delta], \) and \( f(x)g(x) = 0 \). So by (3), for each \( 0 \leq i \leq n, 0 \leq j \leq m \), \( a_i b_j = 0 \). By the argument used in the implication \( (2) \Rightarrow (3) \), we can see that \( a_i \delta^k (b_j) = 0 \) for each \( k \geq 0 \). So \( a_i x^k b_j x^j = 0 \) for each \( 0 \leq i \leq n, 0 \leq j \leq m \).

(4) \( \Rightarrow \) (1) is clear.

**Definition 2.2.** Let \( R \) be a ring with a derivation \( \delta \). We say that \( R \) is a \( \delta \)-skew Armendariz (or simply, \( \delta \)-Armendariz) ring, if it satisfies the equivalent conditions of Theorem 2.1.

It is easy to see that every reduced ring \( R \) is \( \delta \)-Armendariz for each derivation \( \delta \) of \( R \).

**Theorem 2.3.** Let \( R \) be a ring and \( \delta \) a derivation of \( R \). Then the following statements are equivalent.

1. For each \( f(x) = a_0 + a_1 x, g(x) = b_0 + b_1 x \in R[x; \delta], \) if \( f(x)g(x) = 0 \), it implies \( a_0 \delta (b_j) = 0 \) for each \( 0 \leq i \leq 1, 0 \leq j \leq 1 \).
2. For each \( f(x) = a_0 + a_1 x, g(x) = b_0 + b_1 x \in R[x; \delta], \) if \( f(x)g(x) = 0 \), it implies \( a_0 b_j = 0 \) for each \( 0 \leq j \leq 1 \).
3. For each \( f(x) = a_0 + a_1 x, g(x) = b_0 + b_1 x \in R[x; \delta], \) if \( f(x)g(x) = 0 \), it implies \( a_i b_j = 0 \) for each \( 0 \leq i \leq 1, 0 \leq j \leq 1 \).
4. For each \( f(x) = a_0 + a_1 x, g(x) = b_0 + b_1 x \in R[x; \delta], \) if \( f(x)g(x) = 0 \), it implies \( a_i x^k b_j x^j = 0 \) for each \( 0 \leq i \leq 1, 0 \leq j \leq 1 \).

**Proof.** The proof is similar to that of Theorem 2.1.

**Definition 2.4.** Let \( R \) be a ring with a derivation \( \delta \). We say that \( R \) is a weak \( \delta \)-skew Armendariz (or simply weak \( \delta \)-Armendariz) ring, if it satisfies the equivalent conditions of Theorem 2.3.

Each Armendariz ring is \( \delta \)-Armendariz, where \( \delta \) is the zero mapping. It is clear that every \( \delta \)-Armendariz ring is weak \( \delta \)-Armendariz and that every subring of a (weak) \( \delta \)-Armendariz ring is still (weak) \( \delta \)-Armendariz. However, a weak \( \delta \)-Armendariz ring is not necessarily \( \delta \)-Armendariz in general [28, Example 3.2].
Lemma 2.5. Let $R$ be a weak $\delta$-Armendariz ring and $ab = 0$. Then $a\delta^n(b) = \delta^n(ab) = 0$, for each $a, b \in R$ and $n \geq 0$.

Proof. It is enough to prove that $\delta(a)b = a\delta(b) = 0$. Since $ab = 0$, $\delta(a)b + a\delta(b) = 0$. Put $f(x) = \delta(a) + ax$ and $g(x) = b + bx$ in $R[x; \delta]$, then we have $f(x)g(x) = 0$. Since $R$ is weak $\delta$-Armendariz, so $\delta(a)b = a\delta(b) = 0$.

Lemma 2.6. Let $R$ be a weak $\delta$-Armendariz ring. If $ab = c^n = 0$ for some positive integer $n$, then $ac^{-1}b = 0$, for each $a, b, c \in R$.

Proof. The case $n = 1$ is clear. Assume $n \geq 2$. Take $f(x) = a - ac^{n-1}x$ and $g(x) = b + c^n\delta(b) + c^{n-1}bx \in R[x; \delta]$. Now we have $(a - ac^{n-1}x)(b + c^n\delta(b) + c^{n-1}bx) = ab + ac^{n-1}\delta(b) + ac^{n-1}bx - ac^{n-1}\delta(b) - ac^{n-1}c^{n-1}\delta(b)x - ac^{n-1}\delta(c^{n-1}\delta(b)) - ac^{n-1}c^{n-1}bx^2 - ac^{n-1}\delta(c^{n-1}\delta(b)x = -ac^{n-2}\delta(b)x - ac^{n-1}\delta(c^{n-1}\delta(b)) - ac^{n-2}c^{n-2}bx^2 - ac^{n-1}\delta(c^{n-1}\delta(b)x$. Since $R$ is weak $\delta$-Armendariz, $c^n = 0$, $n \geq 2$ and $ac^{n-1}\delta(b) = ac^{n-1}c^{n-1}b = 0$, we have $ac^{n-1}\delta(c^{n-1}\delta(b)) = ac^{n-1}\delta(c^{n-1}b) = 0$, by Lemma 2.5. Thus we have $f(x)g(x) = 0$ and hence $ac^{-1}b = 0$.

Theorem 2.7. If $R$ is a weak $\delta$-Armendariz ring and $ab = c^n = 0$ for some positive integer $n$, then $acb = 0$, for each $a, b, c \in R$.

Proof. If $n \neq 2^k$, then there exists a positive integer $k$ such that $2^k > n$. So $0 = c^n = c^n c^{2^k-n} = c^k$ and that $c^k = 0$. Therefore it is enough to assume that $n = 2^k$ and $ab = c^n = 0$. By Lemma 2.6, since $ab = (c^{2^{k-1}})2^2 = 0$, we have $ac^{2^{k-1}}b = 0$. Now take $p(x) = a - ac^{2^{k-2}}x$ and $q(x) = b + c^{2^{k-2}}\delta(b) + c^{2^{k-2}}bx$. We then have $p(x)q(x) = ab + ac^{2^{k-2}}\delta(b) + ac^{2^{k-2}}bx - ac^{2^{k-2}}\delta(b)x - ac^{2^{k-2}}c^{2^{k-2}}\delta(b) - ac^{2^{k-2}}c^{2^{k-2}}bx^2 - ac^{2^{k-2}}\delta(c^{2^{k-2}}b)x$. Since $ac^{2^{k-1}}b = 0$, by Lemma 2.5, $p(x)q(x) = 0$. Since $R$ is weak $\delta$-Armendariz, $ac^{2^{k-2}}b = 0$. Now we have $(a - ac^{2^{k-2}}x)(b + c^{2^{k-3}}\delta(b) + c^{2^{k-3}}bx) = ab + ac^{2^{k-3}}\delta(b) + ac^{2^{k-3}}bx - ac^{2^{k-3}}bx - ac^{2^{k-3}}\delta(b)x - ac^{2^{k-3}}c^{2^{k-3}}\delta(b)x - ac^{2^{k-3}}c^{2^{k-3}}bx^2 - ac^{2^{k-3}}\delta(c^{2^{k-3}}b)x$. Since we showed $ac^{2^{k-2}}b = 0$, by Lemma 2.5, we have $(a - ac^{2^{k-2}}x)(b + c^{2^{k-3}}\delta(b) + c^{2^{k-3}}bx) = 0$. So $ac^{2^{k-2}}b = 0$. By continuing $k - 1$ times in this way, we have $ac^{2^{k-(k-1)}}b = 0$. Hence $ac^{2^k}b = 0$. Thus we have $(a - ac^{2^{k-2}}x)(b + cb) = ab + ac^2(b) + acx = ab + ac\delta(b) + ac^2\delta(b)x - ac^2\delta(cb)x = 0$. So we have $acb = 0$, and the result follows.

The wedderburn radical, the lower nil radical, the Levitzky radical, the upper nil radical, the Jacobson radical and the set of all nilpotent elements of $R$ is denoted by $N_0(R), Nil_*(R), L-radR, Nil^*(R), J(R)$ and $Nil(R)$, respectively.

Theorem 2.8. If $R$ is a weak $\delta$-Armendariz ring, then $N_0(R) = Nil_*(R) = L-radR = Nil^*(R)$.

Proof. We have $N_0(R) \subseteq Nil_*(R) \subseteq L-radR \subseteq Nil^*(R)$. Let $a \in Nil^*(R)$. Then $a^n = 0$, for some positive integer $n$. In order to see that $a \in N_0(R)$, we show that
Proof. Let \( h \) be a hypothesis, for each coefficient \( a_m < n \).
By the above argument \( a^{n-2}RaR = 0 \). Continuing in this process, we get \( aRaRaRaRa = 0 \). Therefore we get \((RaR)^{2n-1} = 0\), and the result follows.

Proposition 2.9. Let \( R \) be a \( \delta \)-Armendariz ring. If \( f_1, \ldots, f_n \in R[x; \delta] \) with \( f_1 \cdots f_n = 0 \), then \( a_{i+1} \cdots a_n = 0 \), where \( a_i \) is any coefficient of \( f_i \), for each \( i \).

Proof. The proof is by induction on \( n \). By definition the result is true for \( n = 2 \).
Assume that the result is true for all \( m < n \). Let \( f_1 \cdots f_n = 0 \), and \( a_i \) be any coefficient of \( f_i \). Since \( f_1(f_2 \cdots f_n) = 0 \), by definition \( a_1b = 0 \) for each coefficient \( b \) of \( f_2 \cdots f_n \). So \( a_1f_2 \cdots f_n = 0 \) and \( (a_1f_2)(f_3 \cdots f_n) = 0 \). But \( a_1f_2 \) is a polynomial in \( R[x; \delta] \) with coefficient \( a_1b_1 \), where \( b_1 \) is a coefficient of \( f_2 \). By the induction hypothesis, for each coefficient \( a_2 \) of \( f_2 \) and each coefficient \( a_i \) of \( f_i \), with \( 3 \leq i \leq n \), \( a_1a_2 \cdots a_n = 0 \). So the result follows.

Lemma 2.10. Let \( R \) be a weak \( \delta \)-Armendariz ring. If \( a_1a_2 \cdots a_n = 0 \), then \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots \delta^{r_n}(a_n) = 0 \), for each \( a_i \in R \), each non-negative integer \( r_i \), positive integer \( n \), and \( 1 \leq i \leq n \).

Proof. Since \( a_1a_2 \cdots a_n = 0 \), \( \delta^{r_1}(a_1)a_2 \cdots a_n = 0 \), by Lemma 2.5. So \( \delta^{r_1}(a_1)\delta(a_2) \cdots a_n = 0 \), and hence \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n + \delta^{r_1}(a_1)a_2\delta(a_3) \cdots a_n = 0 \).
Since \( \delta^{r_1}(a_1)a_2 \cdots a_n = 0 \), \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n = 0 \), by Lemma 2.5. So \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n = 0 \) and hence \( \delta^{r_1}(a_1)\delta(a_2) \cdots a_n = 0 \). Since \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n = 0 \), \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n = 0 \), by Lemma 2.5. So \( \delta^{r_1}(a_1)\delta(a_2)a_3 \cdots a_n = 0 \). By continuing in this fashion we get \( \delta^{r_1}(a_1)\delta^{r_2}(a_2)a_3 \cdots a_n = 0 \). So by Lemma 2.5, \( \delta^{r_1}(a_1)\delta^{r_2}(a_2)a_3 \cdots a_n = 0 \).
Hence \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots a_n = 0 \), where \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots a_n = 0 \).
Since \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots a_n = 0 \), by Lemma 2.5 we have \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots a_n = 0 \). By continuing in this fashion we get \( \delta^{r_1}(a_1)\delta^{r_2}(a_2) \cdots a_n = 0 \), for each non-negative integer \( r_3 \). By a similar method the result follows.

Proposition 2.11. Let \( R \) be a \( \delta \)-Armendariz ring. If for \( f, g, h \in R[x; \delta] \), \( f^k = 0 \) for some positive integer \( k \), and \( gh = 0 \), then \( gfh = 0 \).

Proof. Let \( f = \sum_{j=0}^{m}a_jx^j, g = \sum_{i=0}^{n}b_ix^i, h = \sum_{k=0}^{l}c_kx^k \). Since \( f^k = 0 \), by Proposition 2.9. \( a_j = 0 \) for each \( 0 \leq j \leq m \). Since \( gh = 0 \) and \( R \) is \( \delta \)-Armendariz, for each \( 0 \leq i \leq n \) and \( 0 \leq k \leq t \) we have \( b_i c_k = 0 \). Hence by Theorem 2.7, for each \( 0 \leq i \leq n \), \( 0 \leq j \leq m \), and \( 0 \leq k \leq t \) we have \( b_ia_jc_k = 0 \). So by Lemma 2.10, for all positive integers \( r_i, s_j, t_k \), \( \delta^{r_i}(b_i)\delta^{s_j}(a_j)\delta^{t_k}(c_k) = 0 \). So \( gfh = 0 \) and the result follows.

Theorem 2.12. If \( R \) is a \( \delta \)-Armendariz ring, then
\[
N_0(R[x; \delta]) = \text{Nil}_*(R[x; \delta]) = L\text{-rad}(R[x; \delta]) = \text{Nil}_*(R[x; \delta]).
\]
Proof. Using Proposition 2.11, the proof is similar to that of Theorem 2.8. Although $J(R[x]) \cap R$ is nil by S. A. Amitsur in [2], it remains an open question whether $J(R[x; \delta]) \cap R$ is nil. Concerning this problem, D.A. Jordan, proved [22, Corollary 2.7] that if $R$ is a right Noetherian ring with identity then $J(R[x; \delta]) \cap R$ is nil if $R$ is commutative.

Now we show that $J(R[x; \delta]) \cap R$ is nil if $R$ is a weak $\delta$-Armendariz ring.

**Theorem 2.13.** If $R$ is a weak $\delta$-Armendariz ring, then $J(R[x; \delta]) \cap R$ is a nil ideal.

**Proof.** Let $a \in J(R[x; \delta]) \cap R$. Then $1-ax$ is invertible. So $(1-ax)(a_0 + a_1x + \cdots + a_nx^n) = 1$, for some $a_0 + a_1x + \cdots + a_nx^n \in R[x; \delta]$. Then we have

\[ a_0 - a\delta(a_0) = 1, a_1 - aa_0 - a\delta(a_1) = 0, a_2 - aa_1 - a\delta(a_2) = 0, \cdots, \]

\[ a_{n-1} - a_{a_{n-2}} - a\delta(a_{n-1}) = 0, a_n - a_{a_{n-1}} - a\delta(a_n) = 0, \]

and $aa_n = 0$. Since $aa_n = 0$, by Lemma 2.5, we get $a\delta(a_n) = 0$. So $a_n = aa_{n-1}$. Hence $a^2a_{n-1} - aa_{n-1} = 0$ and $a^2\delta(a_{n-1}) = 0$. Thus $aa_{n-1} - a^2a_{n-2} - a^2\delta(a_{n-1}) = 0$. So $aa_{n-1} - a^2a_{n-2}$ and $a^2a_{n-2} = a^2a_{n-1} = 0$. Continuing in this way we get $a^na_1 = 0$. Since $a_1 - aa_0 - a\delta(a_1) = 0$, we have $a^{n-1}a_1 - a^na_0 - a^n\delta(a_1) = 0$. So $a^{n-1}a_1 = a^na_0$. But $a^{n+1}a_0 = a^na_1 = 0$ so $a^n a_0 - a^{n+1}\delta(a_0) = a^n$. Hence $a^n a_0 = a^n$ and $a^{n+1} a_0 = 0$. Thus $a$ is nilpotent and $J(R[x; \delta]) \cap R$ is a nil ideal.

Recall that an ideal $I \subseteq R$ satisfying $\delta(I) \subseteq I$ is called a $\delta$-ideal. A ring $R$ is semiprime (resp. $\delta$-semiprime) if $R$ has no non-zero nilpotent ideal (resp. nilpotent $\delta$-ideal).

**Theorem 2.14.** A weak $\delta$-Armendariz ring $R$, is $\delta$-semiprime if and only if $R$ is semiprime.

**Proof.** By [27, Proposition 4.8], $R$ is $\delta$-semiprime if and only if $R[x; \delta]$ is semiprime. So it is enough to show that $R$ is semiprime, when $R[x; \delta]$ is semiprime. Assume that $R[x; \delta]$ is semiprime and that for $a \in R$, $aRa = 0$. Then for each $r \in R$, $ara = 0$, and hence $a\delta^m(a) = 0$, for each positive integer $m$, by Lemma 2.5. So for each $f = \sum_{i=0}^n a_i x^i \in R[x; \delta]$, $afa = 0$, and hence $aR[x; \delta]a = 0$. So $a = 0$, and that $R$ is semiprime.

**Proposition 2.15.** Semiprime weak $\delta$-Armendariz rings have no non-zero nil one sided ideal.

**Proof.** Let $I$ be a non-zero nil right ideal of $R$. If there exists $0 \neq a \in I$ such that $a^n = 0$, $a^{n+1} \neq 0$ for some $n > 2$, then since $aR$ is a nil right ideal, each element in $aR$ is nilpotent. So $aRa^{n+1} = 0$, by Theorem 2.7. Hence $a^{n+1}Ra^n = 0$. Since $R$ is semiprime, $a^{n+1} = 0$, which is a contradiction. So for each $r \in I$, $r^2 = 0$, and there exists $0 \neq a \in I$ such that $a^2 = 0$. We claim that $(aR)^3 = 0$. For each $r_1, r_2, r_3 \in R$, $ar_1r_2r_3 = 0$, since $ar_2r_1 \in I$. So $(r_1r_2)^3 = 0$ and $a^2 = 0$. So by Theorem 2.7, $ar_1ar_2ar_3 = 0$. Hence $ar_1ar_2r_3 = 0$. Thus $(aR)^3 = 0$. Since $R$ is semiprime, $aR = 0$ consequently $a = 0$, which is a contradiction. Therefore $R$ has
no non-zero nil right ideal. The left case is similar.

**Corollary 2.16.** Weak $\delta$-Armendariz rings satisfy the Kothe’s conjecture.

**Proof.** If $\text{Nil}^*(R) = 0$, then $R$ is semiprime and consequently $R$ has no non-zero nil one sided ideal, by Theorem 2.15.

**Corollary 2.17.** If $R$ is a weak $\delta$-Armendariz ring, then
\[
J(R[x]) = N_0(R)[x] = \text{Nil}_s(R)[x] = L\text{-rad}(R)[x] = \text{Nil}^*(R)[x].
\]

**Proof.** By [26, Exercise 10.25] and Theorem 2.8.

**Corollary 2.18.** If $R$ is a weak $\delta$-Armendariz ring, then
\[
\text{Nil}^*(M_n(R)) = M_n(\text{Nil}^*(R)) = M_n(\text{Nil}_s(R)) = \text{Nil}_s(M_n(R)).
\]

**Proof.** By [26, Exercise 10.25 and Theorem 10.21] and Theorem 2.8.

**Proposition 2.19.** If $R$ is a semiprime $\delta$-Armendariz ring, then $R[x; \delta]$ has no non-zero nil one sided ideal.

**Proof.** Let $I$ be a non-zero nil left ideal of $R[x; \delta]$. Let $J$ be the set of all leading coefficients of elements of $I$ together with 0. Then $J$ is a left ideal of $R$. To see that $J$ is nil, let $a \in J$. So for some $f \in I$, $a$ is leading coefficient of $f$. Let $f = a_0 + \cdots + a_{n-1}x^{n-1} + ax^n$. Since $I$ is nil, $f^t = 0$ for some $t$. By Proposition 2.9, $a^t = 0$, whence $J$ is a nil left ideal. Hence $J = 0$, by Proposition 2.15, consequently $I = 0$.

**Corollary 2.20.** If $R$ is a $\delta$-Armendariz ring, then $R[x; \delta]$ satisfies the Kothe’s conjecture.

**Proof.** Let $\text{Nil}^*(R[x; \delta]) = 0$. So $R[x; \delta]$ is semiprime. Thus $R$ is semiprime, by [27, Proposition 4.8] and Theorem 2.14. Now $R$ is $\delta$-Armendariz and semiprime, so $R[x; \delta]$ has no non-zero nil one sided ideal, by Proposition 2.19. Therefore $R[x; \delta]$ satisfies the Kothe’s conjecture.

An ideal $I \subseteq R$ is called $\delta$-prime if it is a proper $\delta$-ideal and the only time a product of two $\delta$-ideals is contained in $I$ is when at least one of the two is contained in $I$. Recall from [22, Lemma 1.3] that if $Q \subseteq R[x; \delta]$ is a prime ideal then $Q \cap R \subseteq R$ is a $\delta$-prime ideal, and if $P \subseteq R$ is a $\delta$-prime ideal then $P[x; \delta] \subseteq R[x; \delta]$ is a prime ideal.

**Lemma 2.21.** (1) If $Q$ is a minimal prime ideal of $R[x; \delta]$, then $Q \cap R$ is a minimal $\delta$-prime ideal of $R$.

(2) If $P$ is a minimal $\delta$-prime ideal of $R$, then $P[x; \delta]$ is a minimal prime ideal of $R[x; \delta]$.

**Proof.** (1) Let $Q$ be a minimal prime ideal of $S = R[x; \delta]$. By [22, Lemma 1.3], $Q \cap R$ is a $\delta$-prime ideal of $R$. If $Q \cap R$ is not minimal, $P \subseteq Q \cap R$ and $P \neq Q \cap R$ for some $\delta$-prime ideal $P$ of $R$, then $PS \subseteq Q$ and $PS \neq Q$. But $PS$ is a prime ideal of $S$ and this contradicts the minimality of $Q$. 

A GENERALIZATION OF REDUCED RINGS
(2). Let $P$ be a minimal $\delta$-prime ideal of $R$. Then by [22, Lemma 1.3], $PS$ is a prime ideal of $S$. If $PS$ is not minimal, there exists a prime ideal $Q$ of $S$ with $Q \subseteq PS$ and $Q \neq PS$. So $Q \cap R \subseteq P$. Since $Q \neq PS$, $Q \cap R \neq P$. But $Q \cap R$ is $\delta$-prime, and this is a contradiction, so $PS$ is a minimal prime ideal of $S$.

**Proposition 2.22.** If $R$ is a semiprime weak $\delta$-Armendariz ring with acc on right annihilators, then every minimal prime ideal of $R$ is a minimal $\delta$-prime ideal of $R$ and vise versa.

**Proof.** Let $R$ be a semiprime weak $\delta$-Armendariz ring with acc on right annihilators. By [34, 2.2.14], a prime ideal of $R$ is minimal if and only if it is annihilator ideal. Let $P$ be a minimal prime ideal of $R$ and $P = \text{ran}_R(U)$ for some subset $U$ of $R$. So $UP = 0$, and hence for each $r \in P, u \in U, ur = 0$. Since $R$ is weak $\delta$-Armendariz, $u\delta(r) = 0$ by Lemma 2.5, so $U\delta(r) = 0$. Hence $\delta(P) \subseteq \text{ran}_R(U) = P$. Thus $P$ is $\delta$-prime. Next we show that, $P$ is minimal $\delta$-prime. Let $P' \subseteq P$ be a $\delta$-prime ideal. If $\text{ran}_R(P) \subseteq P'$, then $\text{ran}_R(P) \subseteq P$, and hence $\text{ran}_R(P) \text{ran}_R(P) = 0$. Since $R$ is semiprime, $\text{ran}_R(P) = 0$, which is a contradiction. Hence $\text{ran}_R(P) \not\subseteq P'$. But $\text{ran}_R(P), P = 0$, and $P$ is a $\delta$-ideal of $R$. Since $R$ is weak $\delta$-Armendariz, $\text{ran}_R(P)$ is a $\delta$-ideal. Hence $P \subseteq P'$, since $P'$ is $\delta$-prime. Therefore $P$ is a minimal $\delta$-prime ideal of $R$. Conversely assume that $P$ is a minimal $\delta$-prime ideal of $R$. We have $\text{rad}(P) = \bigcap Q_i$, where $P \subseteq Q_i$ is a minimal prime ideal of $R$. So $P \subseteq \text{rad}(P) \subseteq Q_i$, for each $i$. Since $Q_i$ is minimal prime, it is also minimal $\delta$-prime, by the first part of the proof. So $P = Q_i$, and hence $\text{rad}(P) = P$, consequently $P$ is a minimal prime ideal of $R$.

**Theorem 2.23.** Let $R$ be a ring with a derivation $\delta$. Then the following statements are equivalent:

1. $R$ is a $\delta$-Armendariz ring;
2. $\varphi : \{r_R(U) \mid U \subseteq R\} \to \{r_S(U) \mid U \subseteq S\}; A \mapsto AS$ is bijective and $ab = 0$ implies that $a\delta(b) = 0$, for each $a, b \in R$;
3. $\psi : \{\ell_R(U) \mid U \subseteq R\} \to \{\ell_S(U) \mid U \subseteq S\}; B \mapsto SB$ is bijective and $ab = 0$ implies that $a\delta(b) = 0$, for each $a, b \in R$;

where $S = R[x; \delta]$.

**Proof.** (1) $\Rightarrow$ (2). Consider the maps $\varphi : \{r_R(U) \mid U \subseteq R\} \to \{r_S(U) \mid U \subseteq S\}$ defined by $A \mapsto AS$ for every $A \in \{r_R(U) \mid U \subseteq R\}$, and $\varphi' : \{r_S(U) \mid U \subseteq S\} \to \{r_R(U) \mid U \subseteq R\}$ defined by $B \mapsto B \cap R$. Observe that $r_R(U)R[x; \delta] = r_S(U)$ for each $U \subseteq R$, so $\varphi'$ is well defined. Now we see that $r_S(V) \cap R = r_R(V_0)$ for each $V \subseteq S$, where $V_0$ is the set of coefficients of all elements of $V$. Let $r \in r_R(V_0)$ and $f = a_0 + a_1x + \cdots + a_nx^n \in V$. Then $a_ir = 0$ for each $0 \leq i \leq n$, by the assumption. So $r \in r_S(V) \cap R$, by Lemma 2.5. If $r \in r_S(V) \cap R$, then for each $f = a_0 + a_1x + \cdots + a_nx^n \in V$, we have $fr = 0$. Since $R$ is $\delta$-Armendariz, $a_ir = 0$ for each $0 \leq i \leq n$, so $r \in r_R(V_0)$. Thus $r_S(V) \cap R = r_R(V_0)$ and hence $\varphi'$ is well defined. It is easy to see that $\varphi \varphi' = \varphi' \varphi$. (2) $\Rightarrow$ (1). Let $f = a_0 + a_1x + \cdots + a_nx^n \in R[x; \delta]$. So $r_S(f) = AR[x; \delta]$ for some right ideal $A$ of $R$, by (2). If $g = b_0 + b_1x + \cdots + a_mx^m \in R[x; \delta]$ and $fg = 0$, then $g \in A[x; \delta]$. So $b_0, b_1, \ldots, b_m \in A \subseteq r_S(f)$. So $(a_0 + a_1x + \cdots + a_nx^n)b_i = 0$ for each $0 \leq i \leq n$, so $a_nb_i = 0$ and hence $a_n\delta(b_i) = 0$, for each $j$. So $a_nx^nb_i = 0$ for
A GENERALIZATION OF REDUCED RINGS

9

each 0 ≤ i ≤ n. Continuing in this way, we get a_i b_i = 0 for each 0 ≤ i ≤ n and 0 ≤ j ≤ m. Hence R is δ-Armendariz ring.

Similarly we can prove (1) ⇔ (3).

Kerr [24] constructed an example of a commutative Goldie ring R whose polynomial ring R[x] has an infinite ascending chain of annihilator ideals.

Corollary 2.24. Let R be a δ-Armendariz ring. Then R satisfies the ascending chain condition on right (left) annihilators if and only if so does R[x; δ].

Theorem 2.25. Let R be a semiprime δ-Armendariz ring with acc for right annihilators. Then R[x; δ] has finitely many minimal prime ideals, say Q_1, Q_2, ..., Q_n with Q_1Q_2...Q_n = 0, and such that Q_i = p_i[x; δ] = q_i[x; δ] for i = 1, ..., n, where \{p_1, ..., p_n\} is the set of all minimal prime ideals and \{q_1, ..., q_n\} is the set of all minimal δ-prime ideals of R.

Proof. Since R is semiprime with acc on right annihilators, R has finitely many minimal prime ideals p_1, ..., p_n, by [34, 2.2.15]. By [27, Proposition 4.3] R[x; δ] is semiprime. Since R is δ-Armendariz, R[x; δ] has acc for right annihilators, by Corollary 2.24. So by [34, 2.2.15], it has finitely many minimal prime ideals, say Q_1, Q_2, ..., Q_m with Q_1Q_2...Q_m = 0. If P is a minimal prime ideal of R, then P is a minimal δ-prime ideal of R, by Proposition 2.22, and P[x; δ] is a minimal prime ideal of R[x; δ], by Lemma 2.21. If Q is a minimal prime ideal of R[x; δ], then Q ∩ R is a minimal prime ideal of R, by Lemma 2.21 and Proposition 2.22. So m = n and the result follows.

Corollary 2.26. Let R be a reduced ring with acc for right annihilators, then R[x; δ] has finitely many minimal prime ideals, say Q_1, Q_2, ..., Q_n, such that Q_i = P_i[x; δ] for i = 1, ..., n, where \{P_1, ..., P_n\} is the set of all minimal prime ideals of R, and Q_1Q_2...Q_n = 0.

3. DIFFERENTIAL POLYNOMIAL RINGS OF 2-PRIMAL RINGS

The classes of rings under consideration are defined as follows. A ring R is symmetric if for all a, b, c ∈ R we have abc = 0 implies that acb = 0. A ring R is called reversible if for all a, b ∈ R we have ab = 0 implies that ba = 0. Recall by H.E. Bell [7] that a ring R has the insertion of factors property (or simply, IFP) if ab = 0 implies aRb = 0 for each a, b ∈ R.

Note that every reduced ring is symmetric and every symmetric ring is reversible and every symmetric ring is 2-primal. But the converse is not true in general.

G.F. Birkenmeier, H.E. Heatherly, and E.K. Lee proved [8, Proposition 2.6] that the 2-primal condition is inherited by ordinary polynomial extensions. Attempts to extend this result to skew polynomial rings and differential operator rings were explored in [20], [31] and [32].

When R is 2-primal, a differential polynomial ring need not be 2-primal ([20, Example 2.1], also [15, Example 2.1]).
In [31] and [32] G. Marks investigated conditions on ideals of a 2-primal ring \( R \) that will ensure that a skew polynomial ring \( R[x; \alpha] \) or a differential polynomial ring \( R[x; \delta] \) be 2-primal.

In this section, we prove that, if \( R \) is a weak \( \delta \)-Armendariz ring, then \( R \) is 2-primal if and only if \( R[x; \delta] \) is 2-primal.

We show by an example that, there exists a commutative ring \( R \) and a derivation \( \delta \) such that \( R[x; \delta] \) is neither symmetric nor reversible nor \( \mathbb{Z}C_n \) and does not have IFFP. However we show that IFFP and symmetric, reversible, \( \mathbb{Z}C_n \) properties transfer between \( R \) and the differential polynomial ring \( R[x; \delta] \), in case \( R \) is \( \delta \)-Armendariz.

In [14], M. Ferrero, K. Kishimoto and K. Motose, defined \( \mathcal{D}(R) \), and proved [15, Theorem 1.1] that, \( \mathcal{D}(R) \) is equal to the intersection of all \( \delta \)-prime ideals of \( R \). They proved [14, Corollary 2.2] that, \( \text{Nil}_*\alpha(R[x; \delta]) = \mathcal{D}(R)[x; \delta] = (\text{Nil}_*\alpha(R[x; \delta]) \cap R)[x; \delta] \).

For each ideal \( I \) of \( R \), they also defined \( M(I) \) and showed that, 
\( \mathcal{D}(R) \subseteq M(\text{Nil}_*\alpha(R)) \subseteq \text{Nil}_*\alpha(R) \), and then asked whether \( \mathcal{D}(R) = M(\text{Nil}_*\alpha(R)) \) ? They showed that \( \mathcal{D}(R) = M(\text{Nil}_*\alpha(R)) \) under a certain finiteness condition of \( \mathcal{D}(R) \) on \( M(\text{Nil}_*\alpha(R)) \).

Now we show that, if \( R \) is a weak \( \delta \)-Armendariz ring, then \( \mathcal{D}(R) = M(\text{Nil}_*\alpha(R)) = \text{Nil}_*\alpha(R) \).

**Theorem 3.1.** If \( R \) is a weak \( \delta \)-Armendariz ring, then \( \mathcal{D}(R) = M(\text{Nil}_*\alpha(R)) = \text{Nil}_*\alpha(R) \).

**Proof.** By [14, Section 4], \( \mathcal{D}(R) \subseteq M(\text{Nil}_*\alpha(R)) \subseteq \text{Nil}_*\alpha(R) \). Let \( \alpha \in \text{Nil}_*\alpha(R) \). Since \( R \) is weak \( \delta \)-Armendariz, \( N_0(R) = \text{Nil}_*\alpha(R) \), by Theorem 2.8. So the ideal \( RaR \) is nilpotent, and \((RaR)^n = 0 \) for some positive integer \( n \). Let \( I \) be the \( \delta \)-ideal of \( R \) generated by \( \alpha \). We show that \( I^n = 0 \). Since \((RaR)^n = 0 \), \( r_1ar_2a \cdots r_nar_{n+1} = 0 \), for each \( r_1, \cdots, r_{n+1} \in R \). Hence by Lemma 2.10, \( r_1\delta^{s_1}(a)r_2\delta^{s_2}(a) \cdots r_n\delta^{s_n}(a)r_{n+1} = 0 \), for each positive integer \( s_i \). Hence \( I^n = 0 \). Now suppose that \( P \) is a \( \delta \)-prime ideal of \( R \). We have \( I^n \subseteq P \) and \( I \) is a \( \delta \)-ideal, so \( I \subseteq P \). So by [15, Theorem 1.1], \( I \subseteq \mathcal{D}(R) \) and that \( a \in \mathcal{D}(R) \). So \( \mathcal{D}(R) = \text{Nil}_*\alpha(R) \) and the result follows.

**Corollary 3.2.** If \( R \) is a \( \delta \)-Armendariz ring, then \( N_0(R) = N_0(R[x; \delta]) \cap R, N_\alpha(R) = N\alpha(R[x; \delta]) \cap R \subseteq \mathcal{D}(R)[x; \delta] = (\text{Nil}_*\alpha(R[x; \delta]) \cap R)[x; \delta] \), by [14, Corollary 2.2]. Using Theorem 3.1, Theorem 2.8 and Theorem 2.12, the result follows.

**Corollary 3.3.** If \( R \) is a \( \delta \)-Armendariz ring, then
\[ N_0(R) = N\alpha(R) = L-rad(R) = N\alpha(R) = J(R[x; \delta]) \cap R. \]

**Proof.** It is enough to prove that \( N\alpha(R) = J(R[x; \delta]) \cap R \). Let \( a \in J(R[x; \delta]) \cap R \). Then \( \langle a \rangle \subseteq J(R[x; \delta]) \cap R \). So by Theorem 2.13, \( \langle a \rangle \) is a nil ideal of \( R \). So \( a \in N\alpha(R) \). But \( N\alpha(R[x; \delta]) \subseteq J(R[x; \delta]) \), so \( N\alpha(R[x; \delta]) \cap R \subseteq J(R[x; \delta]) \cap R \). By Corollary 3.2, \( N\alpha(R[x; \delta]) \cap R = N\alpha(R) \) and hence \( N\alpha(R) \subseteq J(R[x; \delta]) \cap R \).
It is natural to ask, for a $\delta$-Armendariz ring is that true that $J(R[x; \delta]) \cap R = J(R)$? The answer is negative. Consider a reduced local ring $R$ which is not a division ring. Then for each derivation $\delta$ of $R$, $R$ is $\delta$-Armendariz so $J(R[x; \delta]) \cap R$ is nil and hence is zero, but $J(R) \neq 0$.

**Corollary 3.4.** If $R$ is a $\delta$-Armendariz ring, then $J(R[x; \delta]) = N_0(R[x; \delta]) = N_0(R)[x; \delta] = Nil^*_\delta(R[x; \delta]) = Nil^*_\delta(R)[x; \delta] = L-rad(R[x; \delta]) = (L-radR)[x; \delta] = Nil^*_\delta(R)[x; \delta]$.

**Proof.** The result follows by [14, Theorem 3.2] and Corollaries 3.2 and 3.3.

**Corollary 3.5.** If $R$ is a weak $\delta$-Armendariz ring, then $R$ is 2-primal if and only if $R[x; \delta]$ is 2-primal.

**Proof.** Assume that $f(x) = \sum_{i=0}^{n} a_i x^i \in Nil^*(R[x; \delta])$. So for some $m$, $f^m = 0$, and by Proposition 2.9, $a_i^m = 0$ for each $0 \leq i \leq n$. Hence $a_i \in Nil^*(R)$ for each $0 \leq i \leq n$. Thus $f \in Nil^*_\delta(R)[x; \delta] = Nil^*_\delta(R[x; \delta])$, by Corollary 3.4. Therefore $R[x; \delta]$ is 2-primal.

In [1] S.A. Amitsur asked if $R$ is a nil ring, whether the polynomial ring $R[x]$ is nil? A. Smoktunowicz answers in negative by an example in [39]. But for $\delta$-Armendariz rings we have the following result.

**Corollary 3.6.** If $R$ is a nil $\delta$-Armendariz ring, then $N_0(R[x; \delta]) = R[x; \delta]$, and hence $R[x; \delta]$ is a Jacobson radical ring.

**Corollary 3.7.** If $R$ is a nil $\delta$-Armendariz ring, then $R[x; \delta]$ is a nil ring.

So by the above result we see that Amitsur’s question is true for Armendariz rings.

A ring is said to be abelian if all its idempotent elements are central.

**Lemma 3.8.** Let $R$ be a weak $\delta$-Armendariz ring. Then for each idempotent element $e \in R$, we have $\delta(e) = 0$.

**Proof.** Since $e = e^2$, we have $\delta(e) = \delta(e)e + e\delta(e)$. Let $f(x) = \delta(e) + ex$ and $g(x) = (e - 1) + (e - 1)x \in R[x; \delta]$. Then $f(x)g(x) = 0$. Since $R$ is weak $\delta$-Armendariz, $\delta(e)e = \delta(e)$ and hence $\delta(e) = 0$. Now suppose that $h(x) = \delta(e) - (1 - e)x$ and $k(x) = e + ex \in R[x; \delta]$. Then $h(x)k(x) = 0$. Hence $\delta(e)e = 0$ and so $\delta(e) = \delta(e)e = 0$.

**Proposition 3.9.** Every weak $\delta$-Armendariz ring is abelian.

**Proof.** Let $R$ be a weak $\delta$-Armendariz ring and let $e^2 = e, a \in R$. Consider the polynomials $f(x) = e - ea(1-e)x$ and $g(x) = 1 - e + ea(1-e)x \in R[x; \delta]$. Then we have $f(x)g(x) = 0$. Since $R$ is weak $\delta$-Armendariz, $eae(1-e) = 0$. So $ea = eae$. Next let $h(x) = 1 - e - (1-e)ax$ and $k(x) = e + (1-e)ax \in R[x; \delta]$. We have
b(x)k(x) = 0 \text{ so } (1 - e)(1 - e)ae = 0, \text{ since } R \text{ is weak } \delta\text{-Armendariz. Hence } ae = eae \text{ and so } ae = ea. \text{ Thus } R \text{ is abelian.}

A ring } R \text{ is said to be Dedekind finite, if } ab = 1 \text{ implies } ba = 1, \text{ for each } a, b \in R.

**Proposition 3.10.** Every weak \(\delta\)-Armendariz ring is Dedekind finite.

**Proof.** If for \(a, b \in R\), \(ab = 1\), then we have \((ba)^2 = baba = ba\). Since \(R\) is weak \(\delta\)-Armendariz, \(ba \in Z(R)\), by proposition 3.9. We have \(a(ba)b = (ab)(ab) = 1\). Since \(ba \in Z(R)\), \(1 = a(ba)b = (ab)(ba)\). Thus \(ba = 1\).

It is a natural question that, whether \(\text{Nil}^*(R) = \text{Nil}(R)\), for an \(\delta\)-Armendariz ring? The answer is negative, by the following example.

**Example 3.11.** ([21, Example 14]) Let \(F\) be a field and \(A = F[a, b, c]\) be the free algebra of polynomials with zero constant terms in noncommuting indeterminates \(a, b, c\) over \(F\). Let \(I\) be the ideal of \(R = A + F\), generated by \(ac, cc\) and \(crc\), for all \(r \in A\). Then \(R/I\) is a non-reduced semiprime Armendariz ring.

Next we show that under a certain condition, a \(\delta\)-Armendariz ring will be 2-primal.

A ring is called **locally finite** if every finite subset in it generates a finite semigroup multiplicatively. Finite rings are clearly locally finite, and an algebraic closure of a finite field is locally finite but not finite.

**Theorem 3.12.** If \(R\) is a locally finite weak \(\delta\)-Armendariz ring, then

\[ N_0(R) = \text{Nil}_*(R) = L-radR = \text{Nil}^*(R) = \text{Nil}(R). \]

**Proof.** Let \(R\) be a locally finite weak \(\delta\)-Armendariz ring. By Theorem 2.8, \(N_0(R) = \text{Nil}_*(R) = L-radR = \text{Nil}^*(R)\). So it is enough to show that \(\text{Nil}^*(R) = \text{Nil}(R)\). We first prove that, for each \(r \in R\) there exists a positive integer \(m_r\) such that \(r^{m_r} \in Z(R)\). Let \(r \in R\). Since \(R\) is locally finite there exist positive integers \(m, k \geq 1\), such that \(r^m = r^{m+k}\). So we have \(r^m = r^{m_r} = r^{m_r+2} = \ldots = r^{m_r+mk} = r^{m(k+1)}\). Thus we have \(r^{km} = r^{(k-1)m_r} = r^{(k-1)m_r+mk} = r^{2mk} = (r^{km})^2\). Hence \(r^{km}\) is an idempotent element of \(R\). Since \(R\) is weak \(\delta\)-Armendariz, by Proposition 3.9, \(R\) is abelian, and so \(r^{km} \in Z(R)\). Now suppose \(a \in \text{Nil}(R)\), so \(a^n = 0\) for some positive integer \(n\). To show that \(a \in \text{Nil}^*(R)\), it is enough to show that \(\{a\}\) is a nil ideal. Let \(r, s \in R\). We show that \(sar\) is nilpotent. Now there exists a positive integer \(m\) such that \((rs)^m \in Z(R)\). Since \(a^n = 0\), so we have \((rs)^ma^{n-1} = 0\). Hence \((a^{m})r^a = r^{m-1}a^{n-1} = 0\). Since \(a^n = 0\) and \(R\) is weak \(\delta\)-Armendariz, \((rs)^ma^{n-2}a^{n-1} = 0\), by Theorem 2.7. So \((a^{m})r^a = r^{m-2}a^{n-1} = 0\). Again by Theorem 2.7, \((a^{m})r^a = r^{m-2}a^{n-1} = 0\). Continuing in this process, we get \((a^{m})r^a = a^{n-1} = 0\), and hence \(a^{n}a^{n-1} = 0\). Since \((rs)^m \in Z(R)\), we have \((a^{m})r^a = a^{n}a^{n-2} = 0\). After \(n\)-times doing in this way, we have \((a^{m})r^a = a^{n} = 0\). Thus we get \((a^{m})r^{m+1} = 0\) and hence \(s(a^{m})r^{m+1} = 0\). Therefore we have \((sar)^{m(n-1)+2} = 0\), and the result follows.
Corollary 3.13. Every locally finite weak $\delta$-Armendariz ring is 2-primal.

Corollary 3.14. For a locally finite ring $R$, the following statements are equivalent:

1. $R$ is semiprime $\delta$-Armendariz;
2. $R$ is semiprime weak $\delta$-Armendariz;
3. $R$ is reduced.

Corollary 3.15. If $R$ is a locally finite $\delta$-Armendariz ring, then $J(R[x; \delta]) = N_0(R[x; \delta]) = Nil_*(R[x; \delta]) = L-rad(R[x; \delta]) = Nil^*(R[x; \delta]) = Nil(R[x; \delta])$.

Proof. By Corollary 3.4, it is enough to show that $Nil(R[x; \delta]) \subseteq Nil^*(R[x; \delta])$. Let $f = \sum_{i=0}^{m} a_i x^i \in Nil(R[x; \delta])$ and let for some positive integer $n$, $f^n = 0$. By Proposition 2.9, since $R$ is $\delta$-Armendariz, $a_i^n = 0$ for $0 \leq i \leq m$. So $a_i \in Nil(R)$, and by Theorem 3.12, $a_i \in Nil^*(R)$. Hence $f \in Nil^*(R[x; \delta]) = Nil^*(R[x; \delta])$ and the result follows.

In [21, Example 2], C. Huh, Y. Lee and A. Smoktunowicz showed that there is a ring $R$ with $IFP$ such that the polynomial ring $R[x]$ does not satisfy the insertion factor of property.

Theorem 3.16. If $R$ is a $\delta$-Armendariz ring, then $R$ has $IFP$ if and only if $R[x; \delta]$ has $IFP$.

Proof. Assume that $R$ has $IFP$ and $f(x)g(x) = 0$ with $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \delta]$. So $a_i b_j = 0$ for each $0 \leq i \leq n, 0 \leq j \leq m$. Since $R$ has $IFP$, for each $r \in R$, $a_i r b_j = 0$. On the other hand by Lemma 2.10, $a_i \delta^s(r) \delta^t(b_j) = 0$ for each $0 \leq i \leq n, 0 \leq j \leq m$ and positive integers $s, t$. Therefore for each $h(x) \in R[x; \delta]$, $f(x)h(x)g(x) = 0$ and hence $R[x; \delta]$ has $IFP$.

Theorem 3.17. If $R$ is a $\delta$-Armendariz ring, then $R$ is reversible if and only if $R[x; \delta]$ is a reversible ring.

Proof. Let $R$ be reversible and $fg = 0$, with $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j$. This implies that $a_i b_j = 0$ for each $i, j$. So $b_j a_i = 0$, and hence $gf = 0$, by Lemma 2.5.

Theorem 3.18. If $R$ is a $\delta$-Armendariz ring, then $R$ is symmetric if and only if $R[x; \delta]$ is a symmetric ring.

Proof. Let $R$ be a symmetric ring and $fhg = 0$, for $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j, h = \sum_{k=0}^{l} c_k x^k \in R[x; \delta]$. By Proposition 2.9, $a_i b_j c_k = 0$, for each $i, j, k$. Since $R$ is symmetric, $a_i c_k b_j = 0$. So by Lemma 2.10, $a_i \delta^t(c_k) \delta^s(b_j) = 0$ for each $i, j, k, t, s$. This implies $fhg = 0$ and hence the result follows.

By Anderson and Camillo [4], a ring $R$ satisfies $ZC_n$ if for $a_1, a_2, \ldots, a_n \in R$ with $a_1 a_2 \cdots a_n = 0$ it implies that $a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} = 0$ for each $\sigma \in S_n$ and $n \geq 2$, where $S_n$ denotes the permutation group on $n$ letters.
Theorem 3.19. If $R$ is a $\delta$-Armendariz ring, then $R$ satisfies $ZC_n$ if and only if $R[x;\delta]$ satisfies $ZC_n$.

Proof. Assume that $R$ satisfies $ZC_n$ and that $f_1f_2\cdots f_n = 0$ with $f_1, f_2, \cdots, f_n \in R[x;\delta]$. Since $R$ is $\delta$-Armendariz, $a_1a_2\cdots a_n = 0$, for each coefficient $a_i$ of $f_i$, by Proposition 2.9. Since $R$ satisfies $ZC_n$, for each $\sigma \in \Sigma_n$ we have $a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} = 0$. By Lemma 2.10, $f_{\sigma(1)}f_{\sigma(2)}\cdots f_{\sigma(n)} = 0$, and the result follows.

The trivial extension of $R$, which is denoted by $T(R, R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$, is a ring with matrix addition and multiplication.

Example 3.20. Let $R = T(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\delta$ be the derivation on $R$, given by $\delta\left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ for each $a, b \in \mathbb{Z}_2$. Then $R$ is a commutative ring, but we show that $R[x;\delta]$ is neither symmetric nor reversible nor $ZC_n$ and does not have IFP. We have

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

This shows that $R[x;\delta]$ is not reversible and hence is neither symmetric nor $ZC_n$. Next we have

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

But,

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

which shows that $R[x;\delta]$ does not have IFP.

4. DIFFERENTIAL POLYNOMIAL RINGS OF QUASI-BAER RINGS

A ring $R$ is called $\delta$-Baer (resp. Baer), if the right annihilator of every $\delta$-subset (resp. subset) of $R$ is generated by an idempotent, as a right ideal. $R$ is called $\delta$-quasi Baer (resp. quasi-Baer), if the right annihilator of every $\delta$-ideal (resp. ideal) of $R$ is generated by an idempotent, as a right ideal.

Kaplansky [23], introduced the Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [12] introduced the quasi-Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra.
over an algebraically closed field. A ring $R$ is called a right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of $R$ is generated (as a right (resp. left) ideal) by an idempotent of $R$). $R$ is called a p.p ring (also called a Rickart ring) if it is both right and left p.p. 

In [10] G.F. Birkenmeier, J.Y. Kim and J.k. Park, defines a ring to be called right (resp. left) principally quasi-Baer (or simply right (resp. left) p.q.-Baer) if the right annihilator of a principal right (resp. left) ideal of $R$ is generated by an idempotent. Further works on quasi-Baer rings appears in [9-10], [12], [18-19], [23] and [35-36].

In this section we study on the relationship between the Baer, quasi Baer, p.p. and p.q.-Baer property of a ring $R$ and those of the differential polynomial ring $R[x; \delta]$ in case $R$ is $\delta$-Armendariz. By (Armendariz et al. [6, Example 11]), there is a ring $R$ and a derivation $\delta$ of $R$ such that $R[x; \delta]$ is a Baer ring, but $R$ is neither p.p. nor p.q.-Baer (and hence nor quasi Baer).

We first prove some properties of differential polynomial rings over $\delta$-Armendariz rings.

**Lemma 4.1.** Let $R$ be a $\delta$-Armendariz ring. If $e^2 = e \in R[x; \delta]$, where $e = e_0 + e_1 x + \cdots + e_n x^n$, then $e = e_0$.

**Proof.** Since $e(1 - e) = 0 = (1 - e)e$, we have $(e_0 + e_1 x + \cdots + e_n x^n)(1 - e_0 - e_1 x - \cdots - e_n x^n) = 0$ and $((1 - e_0) - e_1 x - \cdots - e_n x^n)(e_0 + e_1 x + \cdots + e_n x^n) = 0$. Hence, $e_0(1 - e_0) = 0$, $e_0 e_i = 0$ and $(1 - e_0) e_i = 0$ for $1 \leq i \leq n$, since $R$ is $\delta$-Armendariz. Thus $e_i = 0$ for $1 \leq i \leq n$, and so $e = e_0 = e_0^2$.

**Theorem 4.2.** If $R$ is a $\delta$-Armendariz ring, then $R[x; \delta]$ is an abelian ring.

**Proof.** By Lemma 3.8, for each idempotent $e \in R$, $\delta(e) = 0$. By Lemma 4.1, the set of idempotent elements of $R[x; \delta]$ is a subset of the set of idempotent elements of $R$. By Proposition 3.9, $R$ is abelian and hence the result follows.

For a ring $R$ with a derivation $\delta$, there exists a derivation on $S = R[x; \delta]$ which extends $\delta$. For example, consider an inner derivation $\overline{\delta}$ on $S$ by $x$ defined by $\overline{\delta}(f(x)) = xf(x) - f(x)x$ for all $f(x) \in S$. Then $\overline{\delta}(f(x)) = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_n)x^n$ for all $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in S$ and $\overline{\delta}(r) = \delta(r)$ for all $r \in R$, which means that $\overline{\delta}$ is an extension of $\delta$. Such a derivation $\overline{\delta}$ on $S$ is called an extended derivation of $\delta$.

In [18], Han, Hirano and Kim proves that, when $R$ is a $\delta$-semiprime ring, then $R$ is a $\delta$-quasi Baer ring if and only if $R[x; \delta]$ is a quasi Baer ring if and only if $R[x; \delta]$ is $\overline{\delta}$-quasi Baer, for every extension $\overline{\delta}$ of $\delta$. Note that if $R$ is a $\delta$-semiprime ring, then $R[x; \delta]$ is a semiprime ring.
In [35], the authors show the left-right symmetry of $\delta$-quasi Baer condition by proving that $R$ is $\delta$-quasi Baer if and only if the left annihilator of every $\delta$-left ideal is generated, as a left ideal, by an idempotent. We see that $\delta$-Baer condition is also symmetric. We then prove our main result showing that the semiprime condition for $R[x;\delta]$, in Han et al.’s result [18], is redundant. Indeed, it is shown that $R$ is $\delta$-quasi Baer if and only if $R[x;\delta]$ is quasi Baer if and only if $R[x;\delta]$ is $\delta$-quasi Baer for any extended derivation $\delta$ of $\delta$. A rich source of examples of $\delta$-(quasi) Baer rings which are not quasi-Baer is provided. It is natural to ask for conditions under which a $\delta$-(quasi) Baer ring to be quasi-Baer.

Now we show that for $\delta$-Armendariz rings, the condition $\delta$-quasi Baer is equivalent to that of quasi-Baer:

**Theorem 4.3.** Let $R$ be a $\delta$-Armendariz ring. Then $R$ is a $\delta$-quasi Baer ring if and only if $R$ is a quasi-Baer ring.

**Proof.** Let $R$ be $\delta$-quasi Baer and $I$ an ideal of $R$. By [35, Theorem 2.6], $R[x;\delta]$ is quasi Baer. So for some idempotent $e \in R[x;\delta]$ we have $r_{R[x;\delta]}IR[x;\delta] = eR[x;\delta]$. Since $R$ is $\delta$-Armendariz, $e \in R$, by Lemma 4.1. Now we show that $r_R(I) = eR$. It is clear that $r_R(I) \supseteq eR$. Assume that $b \in r_R(I)$. So for each $r \in R$ and $a \in I$, $arb = 0$. So by Lemma 2.5, $arb(b) = 0$, for each $i \geq 0$. Hence $arx^ib = 0$ for each $i \geq 0$. Thus $b \in r_{R[x;\delta]}IR[x;\delta]$. Hence $b = eb$. Therefore $r_R(I) = eR$, and hence $R$ is quasi-Baer.

**Corollary 4.4.** Let $R$ be a $\delta$-Armendariz ring. Then the following are equivalent:

1. $R$ is quasi Baer.
2. $R$ is $\delta$-quasi Baer.
3. $R[x;\delta]$ is quasi Baer.
4. $R[x;\delta]$ is $\delta$-quasi Baer, for every extension $\delta$ of $\delta$.

**Proof.** The result follows by Theorem 4.3 and [35, Theorem 2.6].

**Theorem 4.5.** Let $R$ be a $\delta$-Armendariz ring. Then the following are equivalent:

1. $R$ is Baer.
2. $R$ is $\delta$-Baer.
3. $R[x;\delta]$ is Baer.
4. $R[x;\delta]$ is $\delta$-Baer, for every extension $\delta$ of $\delta$.

**Proof.** $(1) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (3)$. Since $R$ is $\delta$-Baer, so $R$ is $\delta$-quasi Baer, and by Corollary 4.4, $R[x;\delta]$ is quasi Baer. Now we show that $R[x;\delta]$ is reduced. Let $a \in R$ with $a^2 = 0$. Set $U = \{a, \delta(a), \delta^2(a), \ldots\}$. Then $U$ is a $\delta$-subset of $R$. So $r_R(U) = eR$, for some $e^2 = e \in R$. Since $R$ is $\delta$-Armendariz, by Lemma 2.5, $\delta^n(a)a = 0$ for each positive integer $n$. Hence $a \in r_R(U)$. By Proposition 3.9, $R$ is abelian, so $a = ea = ae = 0$. Thus $R[x;\delta]$ is reduced quasi Baer and hence it is a Baer ring.

$(3) \Rightarrow (4)$. It is clear.

$(4) \Rightarrow (3)$. Since $R[x;\delta]$ is $\delta$-Baer, so it is $\delta$-quasi Baer, whence by Corollary 4.4, $R[x;\delta]$ is quasi Baer. By a same method as in the proof of $(2) \Rightarrow (3)$, $R[x;\delta]$ is
reduced and hence it is Baer.

(3) ⇒ (1). Let \( U \subseteq R \). Then, \( r_{R[x, \delta]}(U) = eR[x, \delta] \), for some \( e^2 = e \in R \), by Lemma 4.1. Thus \( r_R(U) = r_{R[x, \delta]}(U) \cap R = eR \).

**Corollary 4.6.** ([Han et al.\[18\]]) Let \( R \) be an reduced ring. Then the following are equivalent:

1. \( R \) is Baer.
2. \( R \) is \( \delta \)-Baer.
3. \( R[x, \delta] \) is Baer.
4. \( R[x, \delta] \) is \( \delta \)-Baer, for every extension \( \delta \) of \( \delta \).

**Proof.** The result follows by Theorem 4.5, since every reduced ring is \( \delta \)-Armendariz.

**Corollary 4.7.** ([Kim and Lee \[25, Theorem 10\]]) Let \( R \) be an Armendariz ring. Then \( R \) is a Baer ring if and only if \( R[x] \) is a Baer ring.

**Theorem 4.8.** Let \( R \) be a \( \delta \)-Armendariz ring. Then \( R \) is a right p.q.-Baer ring if and only if \( R[x, \delta] \) is a right p.q.-Baer ring.

**Proof.** Let \( R \) be a right p.q.-Baer ring and \( f = a_0 + a_1x + \cdots + a_nx^n \in S = R[x, \delta] \). We show that \( r_S(fS) \) is generated by an idempotent as a right ideal of \( S \). Let \( J = a_0R + a_1R + \cdots + a_nR \). Since \( R \) is right p.q.-Baer, \( r_R(J) = eR \) with \( e = e^2 \in R \), by [10, Proposition 1.7.]. We show that \( r_S(fS) = eS \). Since \( R \) is \( \delta \)-Armendariz, \( S \) is abelian, by Theorem 4.2. So we can see that \( eS \subseteq r_S(fS) \).

Now let \( g \in r_S(fS) \) with \( g = b_0 + b_1x + \cdots + b_mx^m \). Let \( I = a_nR \). Since \( g \in r_S(fS) \), for each \( r \in R \) we have \( frg = 0 \), and hence \( a_nrb_n = 0 \). Thus \( b_m \in r_R(I) \). Since \( R \) is right p.q.-Baer, \( r_R(I) = tR \) for some idempotent \( t \in R \), and hence \( b_m = tb_m \). But \( ft \in fS \), and \( R[x, \delta] \) is abelian, by Theorem 4.2, so \( ft = a_0t + a_1tx + \cdots + a_nx^n \). So \( ft = a_0t + a_1tx + \cdots + a_n(t-1)x^{n-1} \). We have for each \( r \in R \), \( frtg = 0 \). By the same method, we have \( a_{n-1}rb_n = 0 \) and \( a_{n-1}rb_n = 0 \). So we can see that \( a_iRb_n = 0 \) for \( 0 \leq i \leq n \). Thus \( b_m \in r_R(J) = eR \) and hence \( b_m = eb_m \). Since \( fg = f(b_0 + b_1x + \cdots + b_{m-1}x^{m-1}) + efb_mx^m = 0 \), so \( f(b_0 + b_1x + \cdots + b_{m-1}x^{m-1}) = 0 \). By the same method we can show that \( b_{m-1} \in r_R(J) = eR \), so \( b_{m-1} = eb_{m-1} \). Inductively, we can see that, for each \( 0 \leq i \leq m, b_i \in r_R(J) = eR, \) so \( b_i = eb_i \). Hence \( g = ebg_0 + ebg_1x + \cdots + ebg_mx^m = eg \). Therefore \( g \in eS \) and so \( r_S(fS) \subseteq eS \). Conversely, let \( R[x, \delta] \) be a right p.q.-Baer ring and \( I = aR \). So \( r_S(IS) = eS \) for some \( e^2 = e \in R \), by Lemma 4.1. Using Lemma 2.5, it implies that, \( r_R(I) = r_S(IS) \cap R = eS \cap R \), hence \( r_R(I) = eR \).

By [10, Proposition 1.7.], a ring \( R \) is right p.q.-Baer, if and only if the right annihilator of every finitely generated right ideal of \( R \) is generated by an idempotent.

**Proposition 4.9.** Let \( R \) be a \( \delta \)-Armendariz ring and \( \delta^n = 0 \) for some \( n \geq 0 \). Then the following statements are equivalent:

1. For every finitely generated \( \delta \)-right ideal \( I \) of \( R \), \( r_R(I) = eR \) for some idempotent \( e \in R \).
2. \( R \) is right p.q.-Baer.
3. \( S = R[x, \delta] \) is right p.q.-Baer.
Corollary 4.11. (Kim and Lee [25, Theorem 11]) Let \( R \) be a \( \delta \)-Armendariz ring. Then \( R \) is a p.p.-ring if and only if \( R[x; \delta] \) is a p.p.-ring.

The following example (see Birkenmeier et al. [9, Example 1.6]) shows that in 4.3, 4.4, 4.5, 4.8, 4.9 and 4.10, the condition \( \delta \)-Armendariz is not superfluous.

Example 4.12. (Armendariz et al. [6, Example 11]) There is a ring \( R \) and a derivation \( \delta \) of \( R \) such that \( R[x; \delta] \) is a Baer (hence quasi-Baer) ring, but \( R \) is neither quasi-Baer nor p.p. nor p.q.-Baer. In fact let \( R = \mathbb{Z}_2[t]/(t^2) \) with the derivation \( \delta(t) = 1 \) where \( \overline{t} = t + (t^2) \in R \) and \( \mathbb{Z}_2[t] \) is the polynomial ring over the field \( \mathbb{Z}_2 \) of two elements. Consider the Ore extension \( R[x; \delta] \). If we set \( e_{11} = \overline{t}x \), \( e_{12} = \overline{t} \), \( e_{21} = \overline{t}x^2 + x \), and \( e_{22} = 1 + \overline{t}x \) in \( R[x; \delta] \), then they form a system of matrix units in \( R[x; \delta] \). Now the centralizer of these matrix units in \( R[x; \delta] \) is \( \mathbb{Z}_2[x^2] \). Therefore \( R[x; \delta] \approx M_2(\mathbb{Z}_2[x^2]) \approx M_2(\mathbb{Z}_2)[y] \) where \( M_2(\mathbb{Z}_2)[y] \) is the polynomial ring over \( M_2(\mathbb{Z}_2) \). So the ring \( R[x; \delta] \) is a Baer ring, but \( R \) is not quasi-Baer. Now take \( f(x) = \overline{t} + \overline{t}x \), \( g(x) = \overline{t}x \in R[x; \delta] \). We see that \( f(x)g(x) = 0 \), but \( TT \neq 0 \) whence \( R \) is not weak \( \delta \)-Armendariz.

The concept of zip rings initiated by Zelmanowicz [40]. Zelmanowicz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring. Faith [13] called a ring \( R \) right zip provided that if the right annihilator \( r_R(X) \) of a subset \( X \) of \( R \) is zero, \( r_R(Y) = 0 \) for a finite subset \( Y \subseteq X \); equivalently, for a left ideal \( L \) of \( R \) with \( r_R(L) = 0 \), there exists a finitely generated left ideal \( L_1 \subseteq L \) such that \( r_R(L_1) = 0 \). \( R \) is zip if it is right and left zip. F. Cédó in [11]...
gives an example of a zip ring \( R \) for which \( R[x] \) is not zip.

**Proposition 4.13.** If \( R[x; \delta] \) is a right zip ring, then \( R \) is a right zip ring.

**Proof.** Suppose that \( S = R[x; \delta] \) is a right zip ring. Let \( A \subseteq R \) with \( r_R(A) = 0 \). If \( f(x) = b_0 + b_1x + b_nx^n \in r_S(A) \), then \( af(x) = 0 \) for any \( a \in A \). Thus \( ab_n = 0 \), and so \( b_i \in r_R(A) = 0 \), for \( i = 1, 2, \ldots, n \). Therefore, \( f(x) = 0 \), so \( r_S(A) = 0 \). Since \( R[x; \delta] \) is right zip, there exists a finite set \( B \subseteq A \) such that \( r_S(B) = 0 \). Therefore \( r_R(B) = r_S(B) \cap R = 0 \).

**Theorem 4.14.** Let \( R \) be a \( \delta \)-Armendariz ring. If \( R \) is a right zip ring, then \( R[x; \delta] \) is a right zip ring.

**Proof.** Suppose that \( R \) is a right zip ring and put \( S = R[x; \delta] \). Let \( A \subseteq R[x; \delta] \) with \( r_S(A) = 0 \). Now let \( B \) be the set of all coefficients of elements in \( A \). If \( a \in r_R(B) \), then \( ba = 0 \) for any \( b \in B \). By Lemma 2.5, \( f(x)a = 0 \) for any \( f(x) \in A \), and so \( a \in r_S(A) = 0 \). That is, \( r_R(B) = 0 \). Since \( R \) is right zip, there exists a finite set \( B_0 \subseteq B \) such that \( r_R(B_0) = 0 \). For each \( b \in B_0 \), there exists \( g_b(x) \in A \) such that some of coefficients of \( g_b(x) \) is \( b \). Let \( A_0 \) be a minimal subset of \( A \) such that \( g_b(x) \in A_0 \) for each \( b \in B_0 \). Then \( A_0 \) is a nonempty finite subset of \( A \). Let \( B_1 \) be the set of all coefficients of elements in \( A_0 \). Then \( B_0 \subseteq B_1 \) and so \( r_R(B_1) \subseteq r_R(B_0) = 0 \). If \( f(x) = a_0 + a_1x + + a_kx^k \in r_S(A_0) \), then \( g(x)f(x) = 0 \) for any \( g(x) = b_0 + b_1x + + b_\ell \in A_0 \). Since \( R \) is \( \delta \)-Armendariz, \( a_i \in r_R(B_1) = 0 \) for \( 0 \leq i \leq k \), and so \( f(x) = 0 \). Hence \( r_S(A_0) = 0 \).

**Corollary 4.15.** If \( R \) is a reduced zip ring with a derivation \( \delta \), then \( R[x; \delta] \) is a reduced zip ring.

Notice that, reduced rings need not be right (left) zip in general.

5. SOME CLASSES OF NON-REDUCED \( \delta \)-ARMENDARIZ RINGS

For a reduced ring \( R \), T.K. Lee and Y. Zhou, in [29], introduced some Armendariz subrings of \( T_n(R) \) which contain all known Armendariz subrings of \( T_n(R) \). For this purpose they introduced some notations as follows. We notice that the method which is employed by T.K. Lee and Y. Zhou does not work in our case, and we have to provide some different methods in order to prove our results.

Let \( R \) be a ring. In this section, the \((i, j)\)-th entry of a matrix \( A \in M_n(R) \) is denoted by \( A_{i,j} \). Define \( V_n = \sum_{i=1}^{n-1} E_{i,i+1} \), for \( n \geq 2 \), where \( E_{i,j} \) is the matrix units for all \( i, j \). For even integers \( n = 2k \geq 2 \), define \( A_n(R) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} RE_{i,j} \), \( A_n(R) = R^{2k+1} + RV_n^2 + \cdots + RV_n^{k-1} + A_n^0(R) \). For odd integers \( n = 2k + 1 \geq 3 \), define \( A_n^0(R) = \sum_{i=1}^{k+1} \sum_{j=k+1}^{n} RE_{i,j} \), \( A_n(R) = R^{2k+1} + RV_n^2 + \cdots + RV_n^{k-1} + A_n^0(R) \). Let \( \delta \) be a derivation of \( R \), then for each \( n \), \( \delta : A_n(R) \rightarrow A_n(R) \), given by \( \delta((a_{ij})) = (\delta(a_{ij})) \), is a derivation.
Theorem 5.1. Let $R$ be a reduced ring. For each integer $n \geq 2$, let $A = (a_{ij})$, $B = (b_{ij}) \in A_n(R)$. If $AB = 0$, then $a_{ij}b_{ij} = 0$, for each $1 \leq \ell \leq n$ and $1 \leq i, j \leq n$.

Proof. The proof is by induction on $n$. If $n = 2$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in A_2(R)$, with $AB = 0$. So $ac = 0, ad + bc = 0$, and hence $ada + bca = 0$. So $ac = 0$, consequently $bc = 0$, and the result follows.

Now inductively assume that the result is true for integers less than $n$. Now we consider the following two cases:

Case (I) : $n$ is an even integer $n = 2k$. Then we have:

$A = a_1I_n + a_2V_n + a_3V_n^2 + \cdots + a_kV_n^{k-1} + \sum_{i=1}^k \sum_{j=k+i} a_{ij}E_{ij},$

$B = b_1I_n + b_2V_n + b_3V_n^2 + \cdots + b_kV_n^{k-1} + \sum_{i=1}^k \sum_{j=k+i} b_{ij}E_{ij},$ with $a_i, a_{ij}, b_i, b_{ij} \in R$

for each $i, j$. Since $AB = 0$, we have $(AB)_{12} = 0$, and that $a_1b_1 = 0$. Also $(AB)_{12} = 0,$ so $a_1b_2 + a_2b_1 = 0$. By multiplying from right by $a_1$, we get $a_1b_2 = a_2b_1 = 0$. Similarly, by a similar method, after $k$ steps, $(AB)_{1k} = 0,$ gives $a_1b_k + a_2b_{k-1} + \cdots + a_kb_1 = 0$. Multiplying from right by $a_1$, we get $a_1b_k = 0$. So $a_2b_{k-1} + \cdots + a_kb_1 = 0$. Also $a_2b_{k-1} + \cdots + a_kb_1 = 0$. By this way we conclude that each term is zero, and hence $a_ib_j = 0$, for each $2 \leq \ell \leq k + 1$ and each positive integers $i, j$ with $i + j = \ell$. On the other hand since $AB = 0$, entries on the $k + 1$-th diagonal of the matrix $AB$ is zero. So $(AB)_{1,k+1} = a_1b_{k+1} + a_2b_k + a_3b_{k-1} + \cdots + a_kb_2 + a_{k+1}b_1 = 0$. Multiplying from right by $a_1$, we get $a_1b_{k+1} + a_1b_1 = 0$, since for each $2 \leq \ell \leq k + 1$ and each positive integers $i, j$ with $i + j = \ell$, $a_ib_j = 0$. So $a_1b_{k+1} = 0$. Multiply from left by $b_1$, by a similar argument, we get $a_kb_1b_1 = 0$. We have $(AB)_{2,k+2} = a_1b_{k+2} + a_2b_k + a_3b_{k-1} + \cdots + a_kb_2 + a_{k+2}b_1 = 0$. Multiplying from right by $a_1$, we get $a_1b_{k+2} + a_1b_1 = 0$. Multiply from left by $b_1$, we get $a_2b_{k+2} = 0$. Continuing in this way we get $(AB)_{k,k+k} = (AB)_{k,n} = a_kb_{k,n} + a_2b_k + a_3b_{k-1} + \cdots + a_kb_2 + a_{k,n}b_1 = 0$. Multiplying from right by $a_1$, we get $a_1b_{k+2} + a_1b_1 = 0$. Multiply from left by $b_1$, we get $a_2b_{k+2} = 0$. Now we see that entries of the $k + 2$-th diagonal is zero. Hence we have $(AB)_{1,k+2} = a_1b_{k+2} + a_2b_{k+2} + a_3b_k + \cdots + a_kb_3 + a_{k+1}b_2 + a_{1,k+2}b_1 = 0$. Multiply from right by $a_1$, we get $a_1b_{k+2} + a_2b_{k+2} + a_3b_k = 0$. But $a_2b_{k+2} = 0$, so $a_1b_{k+2} = 0.$ Now multiply form left by $b_1$ we get $b_1a_{k+1}b_1 + b_1a_{1,k+2}b_1 = 0$. Since $a_1b_{k+2} = 0$, we have $(AB)_{2,k+3} = 0$, so $a_1b_{1,k+3} = 0$ and $b_1a_{1,k+3} = 0.$ After $n$ times we get $a_1b_{1,j} = 0$ and $a_1b_1 = 0$. Next set $A' = a_2I_{n-1} + a_3V_{n-1} + \cdots + a_kV_{n-1}^{k-2} + \sum_{i=1}^k \sum_{j=k+i-1} a_{i,j}E_{i,j},$ and
\[ B' = b_2 I_{n-1} + b_3 V_{n-1} + \cdots + b_k V_{n-1}^{k-2} + \sum_{i=1}^{k} \sum_{j=k+1}^{n} b_{ij} E_{ij}, \]

where \( A', B' \) is obtained by deleting the main diagonals of \( A \) and \( B \), respectively. Since \( AB = 0 \), and for each \( 1 \leq i, j \leq n \) \( a_{ij}b_{ij} = 0 \) and \( a_{ij}b_1 = 0 \), so \( A'B' = 0 \), where \( A', B' \in A_{n-1}(R) \).

So \( a_{ij}b_{ij} = 0 \) for each \( 2 \leq \ell \leq n \) and \( 2 \leq i, j \leq n \), and the result follows.

Case \((II)\) : \( n \) is an odd integer, \( n = 2k + 1 \). Then we have

\[ A = a_1 I_n + a_2 V_n + a_3 V_n^2 + \cdots + a_k V_n^{k-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij} E_{ij}, \]

\[ B = b_1 I_n + b_2 V_n + b_3 V_n^2 + \cdots + b_k V_n^{k-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{n} b_{ij} E_{ij}, \]

with \( a_i, a_{ij}, b_i, b_{ij} \in R \) for each \( i, j \). By a similar method employed in the above case, we obtain \( a_{ij}b_{ij} = 0 \), for each \( 2 \leq \ell \leq k + 1 \) and positive integers \( i, j \) with \( i + j = \ell \). On the other hand since \( AB = 0 \), \( n \) entries of the \( n \)-th column of the matrix \( AB \) is zero, so we have

\[ (AB)_{k+1,n} = a_1 b_{k+1,n} + a_2 b_{k+1,n} + a_3 b_{k+1,n} + \cdots + a_k b_{k+1,n} = 0. \]

Multiply from right by \( a_1 \) we get \( a_1 b_{k+1,n} = 0 \). Multiply from left by \( b_1 \) we get \( a_{k+1,1} b_1 = 0 \).

Continuing in this way we get \( a_{ij}b_{ij} = 0 \), and \( a_{ij}b_{1} = 0 \) for each \( 1 \leq i \leq n \). Now by the above argument we see that all the terms in the sum of \( (AB)_{k+1,n} \) is zero.

So \( (AB)_{k,n} = a_1 b_{k,n} + a_2 b_{k+1,n} + a_3 b_{k+1,n} + \cdots + a_k b_{k+1,n} = 0 \). Thus \( a_{ij}b_{ij} = 0 \), \( a_{ij}b_{1,j} = 0 \), \( a_{ij}b_{1,j} = 0 \), \( a_{ij}b_{1,j} = 0 \), and \( a_{ij}b_{1,j} = 0 \).

Multiplying from right by \( a_2 \) we get \( a_2 b_{k+1,n} = 0 \). So \( a_2 b_{k+1,n} = 0 \), \( a_2 b_{k+1,n} = 0 \), \( a_2 b_{k+1,n} = 0 \), \( a_2 b_{k+1,n} = 0 \), and \( a_2 b_{k+1,n} = 0 \).

Continuing in this way, we see that all the terms in this sum is also zero.

Also, we can see by the same way that, all the terms in the sum of \( (AB)_{1,n} \) is zero.

Now by deleting the \( n \)-th row and the \( n \)-th column of \( A \) and \( B \), respectively we get \( A' \) and \( B' \). So \( A' = a_1 I_{n-1} + a_2 V_{n-1} + \cdots + a_k V_{n-1}^{k-1} + \sum_{i=1}^{k-1} \sum_{j=k+1}^{n} a_{ij} E_{ij} \), and

\[ B' = b_1 I_{n-1} + b_2 V_{n-1} + \cdots + b_{k-1} V_{n-1}^{k-2} + \sum_{i=1}^{k-1} \sum_{j=k+1}^{n} b_{ij} E_{ij}. \]

Since \( AB = 0 \) and all the terms in \( (AB)_{i,j} \) is zero, and \( a_{ij}b_{ij} = 0 \) for each i, so \( A'B' = 0 \), where \( A', B' \in A_{n-1}(R) \). So by the induction hypothesis \( a_i b_{i,j} = 0 \) for each \( 1 \leq \ell, i, j \leq n-1 \), and the result follows.

**Proposition 5.2.** Let \( R \) be a reduced ring, and \( n = 2k \geq 4 \). Let \( A, B \in A_n(R) \) with \( AB = 0 \). Then \( a_i b_{i,j} = 0 \), for each \( 1 \leq \ell \leq n \) and \( 1 \leq i, j \leq n \).

**Proof.** There exist \( A', B' \in A_n(R) \) and \( r, s \in R \) such that \( A = A' + rE_{1,k} \) and \( B = B' + sE_{1,k} \). Then we have \( AB = AB = (A' + rE_{1,k})(B' + sE_{1,k}) = A'B' + A'sE_{1,k} + rE_{1,k}B' = A'B' + a_{1,1}E_{1,k} + r_{b,k}E_{1,k} + \sum_{i=1}^{k} b_{i,1+k} E_{i+1,k+1} + \cdots + \sum_{i=1}^{k} b_{i,n} E_{n+1,k+1} = 0 \),

where \( a_{ij} \) is the \( (i, j) \)-th entry of \( A' \) and \( b_{ij} \) is the \( (i, j) \)-th entry of \( B' \). By a similar argument as in Theorem 5.1, we see that \( a_{1,1} s = r_{b,k} = r_{b,k+1} = \cdots = r_{b,n} = 0 \).

So \( A'B' = 0 \), and by Theorem 5.1, and the above identities the result follows.

Now, we provide a rich class of non-reduced \( \delta \)-Armendariz rings.

**Theorem 5.3.** Let \( R \) be a ring, and \( \delta \) be a derivation on \( R \). Then the following are equivalent.

1. \( R \) is reduced.
2. For each positive integer \( n \), \( A_n(R) \) is \( \delta \)-Armendariz.
3. For some \( n \geq 2 \), \( A_n(R) \) is \( \delta \)-Armendariz.
Proof. (1) ⇒ (2). It is easy to see that, the mapping
\[ \Phi : A_n(R)[x; \delta] \to A_n(R[x; \delta]), \]
given by \( \Phi(A_0 + A_1x + \cdots + A_kx^k) = (f_{ij}), \)
where \( f_{ij}(x) = a_{ij}^{(0)} + a_{ij}^{(1)} x + \cdots + a_{ij}^{(k)} x^k, \)
and \( a_{ij}^{(\ell)} \) is the \((i, j)\)-th entry of \( A_\ell, \)
is an isomorphism. Now assume that \( f, g \in A_n(R)[x; \delta] \)
and \( f(x) = a_0 + a_1x + \cdots + a_n x^n; \)
\( g(x) = b_0 + b_1x + \cdots + b_m x^m \in A_n(R)[x; \delta], \)
and \( fg = 0, \) with \( B_\ell = (b_{ij}), A_\ell = (a_{ij}). \)
But, since \( f_{ij}, g_{ij} \in A_n(R[x; \delta]) \) by the above isomorphism,
where \( (f_{ij})(g_{ij}) = 0. \) Since \( f_{ij}, g_{ij} \in R[x; \delta], \)
and \( R[x; \delta] \) is reduced, by Theorem 5.1, for each \( 1 \leq \ell \leq n \)
and \( 1 \leq i, j \leq n, \)
\( f_{ij}g_{ij} = 0. \) Thus for each \( 0 \leq k \leq m, \)
\( A_0B_k = 0. \) So \( A_n(R) \) is \( \delta \)-Armendariz.
(2) ⇒ (3) is clear.

(3) ⇒ (1) If \( n = 2, r \in R \) with \( r^2 = 0, \) then set \( f(x) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) - \left( \begin{array}{cc} r & 0 \\ 0 & r \end{array} \right) x, \)
\( g(x) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} r & 0 \\ 0 & r \end{array} \right) x, \)
where \( f, g \in A_2(R)[x; \delta], \) and \( fg = 0. \) So
\( \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} r & 0 \\ 0 & r \end{array} \right) = 0. \) So \( r = 0. \) If \( n \geq 3 \)
and \( r \in R, r^2 = 0. \) Set \( h(x) = V_n^2 - rv_n x \) and \( k(x) = V_n^{n-2} + rV_n^{n-3} x, \)
where \( h(x), k(x) \in A_n(R)[x; \delta], \) with \( h(x)k(x) = 0. \) So \( V_n^2 rV_n^{n-3} = 0, \)
and hence \( rV_n^{n-1} = 0. \) Thus \( r = 0, \) and the result follows.

Theorem 5.4. Let \( R \) be a ring, and \( \delta \) be a derivation on \( R. \)
Then the following are equivalent.

1. \( R \) is reduced.
2. For each positive integer \( n = 2k \geq 4, \)
   \( A_n(R) + RE_{1k} \) is \( \delta \)-Armendariz.
3. For some \( n = 2k \geq 4, \)
   \( A_n(R) + RE_{1k} \) is \( \delta \)-Armendariz.

Proof. The proof is similar to that of Theorem 5.3.

Let \( R \) be a ring and let
\[
T(R, n) := \left\{ \begin{pmatrix}
    a_1 & a_2 & a_3 & \cdots & a_n \\
    0 & a_1 & a_2 & \cdots & a_{n-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & a_1
\end{pmatrix} \mid a_i \in R \right\},
\]
with \( n \geq 2. \) Then \( T(R, n) \) is a subring of the triangular matrix ring \( T_n(R). \)
We can denote elements of \( T(R, n) \) by \( (a_1, a_2, \cdots, a_n). \)
In the case \( n = 2 \) it is the trivial extension of \( R. \)

For a derivation \( \delta \) of \( R, \) the natural extension \( \hat{\delta} : T(R, n) \to T(R, n) \)
defined by \( \hat{\delta}(a_i) = (\delta(a_i)), \) is a derivation of \( T(R, n). \)

In the following result, since \( T(R, n) \) is a subring of \( A_n(R), \)
we deduce from Theorem 5.3, that \( T(R, n) \) is a non-reduced \( \delta \)-Armendariz ring,
for every reduced ring \( R \) and any derivation \( \delta \) on \( R. \)
Corollary 5.5. Let $R$ be a ring with a derivation $\delta$. Then the following are equivalent.

1. $R$ is reduced.
2. $T(R,n)$ is $\delta$-Armendariz, for each positive integer $n$.
3. $T(R,n)$ is weak $\delta$-Armendariz, for each positive integer $n$.
4. There exist an integer $n \geq 2$ such that $T(R,n)$ is weak $\delta$-Armendariz.

Corollary 5.6. Let $R$ be a ring with a derivation $\delta$. Then the following are equivalent.

1. $R$ is reduced.
2. The trivial extension $T(R,R)$ of $R$, is weak $\hat{\delta}$-Armendariz.
3. $T(R,R)$ is $\hat{\delta}$-Armendariz.
4. $T(R,R)$ is weak Armendariz.
5. $T(R,R)$ is Armendariz.

Corollary 5.7. Let $R$ be a ring with a derivation $\delta$. Then the following are equivalent.

1. $R$ is reduced.
2. $R[x]/\langle x^n \rangle$ is $\hat{\delta}$-Armendariz, for each positive integer $n$.
3. There exist an integer $n \geq 2$ such that $R[x]/\langle x^n \rangle$ is weak $\hat{\delta}$-Armendariz.
4. $R[x]/\langle x^n \rangle$ is weak $\hat{\delta}$-Armendariz, for each positive integer $n$.

where $R[x]$ is the polynomial ring over $R$ and $\langle x^n \rangle$ is an ideal of $R[x]$ generated by $x^n$.

Proof. Observe that $T(R,n) \cong R[x]/\langle x^n \rangle$.

Corollary 5.8. (Anderson and Camillo, [3, Theorem 5]). Let $R$ be a ring. Then the following are equivalent.

1. $R$ is a reduced ring.
2. $R[x]/\langle x^n \rangle$ is an Armendariz ring, for each positive integer $n$.
3. $R[x]/\langle x^n \rangle$ is a weak Armendariz ring, for each positive integer $n$.
4. There exist an integer $n \geq 2$ such that $R[x]/\langle x^n \rangle$ is a weak Armendariz ring.

where $R[x]$ is the polynomial ring over $R$ and $\langle x^n \rangle$ is an ideal of $R[x]$ generated by $x^n$.

Example 5.9. ([33, Example 3.4]) Let $p$ be a prime number and $F$ be a field of characteristic $p$. Let $G$ be a cyclic $p$-group. If $G$ is of order $p^n$, and $R = FG$, then $R \cong \frac{F[t]}{tp^n} \cong \frac{F[q]}{q^{p^n}} = T(F,p^n)$. Since $F$ is reduced, $R$ is $\delta$-Armendariz by Corollary 5.5.

For an $(R,R)$-bimodule $M$, the trivial extension of $R$ and $M$, denoted by $T(R,M)$, is the subring $\left\{ \left( \begin{array}{cc} a & m \\ 0 & a \end{array} \right) \mid a \in R, m \in M \right\}$ of the formal upper triangular ring $\left( \begin{array}{cc} R & M \\ 0 & R \end{array} \right)$.

Let $R$ be a ring, and $\delta$ be a derivation on $R$. Let $M$ be an $(R,R)$-bimodule. An additive mapping $\tau : M \rightarrow M$ is called a generalized derivation related to $\delta$, if for
each $r \in R$ and $m \in M$, $\tau(rm) = \delta(r)\tau(m)$ and $\tau(mr) = \tau(m)r + m\delta(r)$. In this case, if we take $T(R, M)$ and $d : T(R, M) \rightarrow T(R, M)$, given by,

\[ d \left( \begin{array}{cc} r & m \\ 0 & r \end{array} \right) = \left( \begin{array}{cc} \delta(r) & \tau(m) \\ 0 & \delta(r) \end{array} \right), \]

is a derivation of $T(R, M)$. It is easy to see that $T(R, M)[x; d] \cong T(R[x; \delta], M[x; \tau])$, where $M[x; \tau]$ is an $(R[x; \delta], R[x; \delta])$-bimodule. Elements of $M[x; \tau]$ are of the form $m_0 + m_1x + \cdots + m_ex^e$, and for each $m \in M$, $x^em = mx + \tau(m)$.

Now let $d$ be the derivation defined in the above argument. Then we have the following result.

**Theorem 5.10.** Let $R$ be a domain and $\delta$ be a derivation of $R$. Let $M$ be a torsion free $(R, R)$-bimodule and $\tau : M \rightarrow M$ be a generalized derivation related to $\delta$. Then $T(R, M)$ is a $d$-Armendariz ring.

**Proof.** Let $f, g \in T(R, M)[x; d]$ and that $fg = 0$, with $f = A_0 + A_1x + \cdots + A_nX^n$ and $g = B_0 + B_1x + \cdots + B_tx^t$. By the isomorphism $T(R, M)[x; d] \cong T(R[x; \delta], M[x; \tau])$, we have

\[ \left( \begin{array}{cc} f_1 & g_1 \\ 0 & f_1 \end{array} \right) \left( \begin{array}{cc} f_2 & g_2 \\ 0 & f_2 \end{array} \right) = 0, \]

with $f_1 = a_0 + a_1x + \cdots + a_nx^n$, $g_1 = m_0 + m_1x + \cdots + m_nx^n$, $f_2 = b_0 + b_1x + \cdots + b_tx^t$ and $g_2 = s_0 + s_2x + \cdots + s_tx^t$, where $A_i = \left( \begin{array}{cc} a_i & m_i \\ 0 & a_i \end{array} \right)$, $B_i = \left( \begin{array}{cc} b_i & s_i \\ 0 & b_i \end{array} \right)$. So we have $f_1f_2 = 0, f_1g_2 + g_1f_2 = 0$. Since $f_1, g_2, g_1f_2 = 0$, we get $f_2f_1 = 0$ and hence $f_1g_2 = 0$. But $M$ is torsion free, so $f_1 = 0$ or $g_2 = 0$. If $f_1 = 0$, then $g_1f_2 = 0$. Since $M$ is torsion free, $f_2 = 0$ or $g_1 = 0$. If $f_2 = 0$, since $f_1 = 0$ we have $A_0B_j = 0$ for each $j$ and the result follows. If $g_1 = 0$, since $f_1 = 0$ so $A_0 = 0$ and the result follows. If $f_1 \neq 0$, then $g_2 = 0$ and hence $g_1f_2 = 0$. If $f_2 = 0$, so $B_j = 0$ for each $j$ and the result follows. If $f_2 \neq 0$, then $g_1 = 0$. So for each $j$, $A_0B_j = 0$ and the result follows.

**Corollary 5.11.** Let $D$ be a division ring and $\delta$ be a derivation of $D$. Let $V$ be a vector space over $D$ and $\tau : V \rightarrow V$ be a generalized derivation related to $\delta$. Then $T(D, V)$ is a $d$-Armendariz ring.

Let $A_i$ be a ring and $\delta_i$ a derivation of $A_i$ for each $i \in I$. Then the product $\prod_{i \in I} A_i$ with the derivation $\delta : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} A_i$ given by

\[ \delta(\{a_i\}_{i \in I}) = \{\delta_i(a_i)\}_{i \in I}, \]

is a (weak) $\delta_i$-Armendariz ring if and only if each $R_i$ is a (weak) $\delta_i$-Armendariz ring for each $i \in I$.

Notice that for any idempotent $e \in R$, if $R$ is a $\delta$-Armendariz ring then so is $eRe$, but the converse is not true in general (See example 15 of [25]).

**Proposition 5.12.** Let $R$ be a ring $R$ and $\delta$ a derivation of $R$. Then the following are equivalent.

1. $R$ is $\delta$-Armendariz.
2. For each idempotent $e \in R$, $eR$ and $(1-e)R$ are $\delta$-Armendariz, $R$ is abelian and $\delta(e) = 0$.
3. For some idempotent $e \in R$, $eR$ and $(1-e)R$ are $\delta$-Armendariz, $R$ is abelian.
and $\delta(e) = 0$.

**Proof.** Since $eR$ and $(1 - e)R$ are subrings of $R$, so the implications $(1 \Rightarrow 2)$ and $(2 \Rightarrow 3)$ are clear. Now assume that (3) holds. Let $e \in R$ be an idempotent of $R$ for which $eR$ and $(1 - e)R$ are $\delta$-Armendariz. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \delta]$ with $f(x)g(x) = 0$. Let $f_1(x) = ef(x)$, $f_2(x) = (1 - e)f(x)$, $g_1(x) = eg(x)$ and $g_2(x) = (1 - e)g(x)$. Since $R$ is abelian and $\delta(e) = \delta(1 - e) = 0$, we have $f(x)g(x) = f_1(x)g_1(x) + f_2(x)g_2(x)$.

Let $f(x)g(x) = f_1(x)g_1(x) + f_2(x)g_2(x) = 0$. Hence $ef(x)g(x) = f_1(x)g_1(x) = 0$ and $(1 - e)f(x)g(x) = f_2(x)g_2(x) = 0$. Since $eR$ and $(1 - e)R$ are $\delta$-Armendariz, we have $e a_0 b_1 = 0$ and that $(1 - e)a_0 (1 - e)b_j = 0$. Since $R$ is abelian, $a_0 b_j = 0$ for each $0 \leq j \leq m$. Therefore $R$ is $\delta$-Armendariz.

Now we concern the classical right quotient rings of $\delta$-Armendariz rings. A ring $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $a b_1 = b a_1$. It is a well-known fact that $R$ is a right Ore ring if and only if there exists the classical right quotient ring of $R$.

Note that every Ore domain has a classical quotient ring and is a $\delta$-Armendariz ring for every derivation $\delta$ of $R$. Now we show that $\delta$-Armendariz condition extends to its quotient ring.

Let $R$ be a ring with a classical left quotient ring $Q$. Then each derivation $\delta$ of $R$, extends to $Q$, by setting $\tilde{\delta}(c^{-1}r) = c^{-1}(\delta(r) - \delta(c)c^{-1}r)$, for each $r \in R$ and each regular element $c \in R$. We denote the set of all regular elements of $R$ by $C$.

We also note that, since a domain $R$ cannot have a classical right (left) quotient ring in general; hence the hypothesis of the following result is necessary.

**Lemma 5.13.** Let $R$ be a right Ore ring, $\delta$ a derivation of $R$ and $\tilde{\delta}$ the extended derivation of the classical right quotient ring $Q$ of $R$. Then for each element $g \in Q = Q[x; \delta]$ there exists a regular element $c \in C$ such that $gc \in S = R[x; \delta]$.

**Proof.** If $g \in Q$, then for some $c \in C$ and $a \in R$ we have $g = ac^{-1}$ and $gc \in S$. Now assume inductively that for all elements $g \in T$ of degree less than $n$ the assertion is hold and let $g = q_0 + \cdots + q_n x^n \in T$. Let $q_n = ac^{-1}$ with $a \in R$ and $c \in C$. Then $qc = (q_0 + \cdots + q_{n-1} x^{n-1})c + ax^n + h$ for some $h \in T$ with $\deg(h) \leq n - 1$. Now we have $qc = h' + ax^n$ with $h' \in T$ and $\deg(h') < n$. By induction hypothesis there exist some $d \in C$ with $h'd \in S$. Thus we have $gcd = h'd + ax^n d \in S$ where $cd \in C$ and the result follows.

**Theorem 5.14.** Let $R$ be an Ore ring and $\delta$ a derivation of $R$. Then $R$ is a (weak) $\delta$-Armendariz ring if and only if the classical quotient ring $Q$ of $R$ is a (weak) $\tilde{\delta}$-Armendariz ring.

**Proof.** Assume that $R$ is a $\delta$-Armendariz ring and $g(x) = p_0 + p_1 x + \cdots + p_m x^m$, $f(x) = q_0 + q_1 x + \cdots + q_n x^n \in Q[x; \tilde{\delta}]$ such that $f(x)g(x) = 0$. Let $q_i = a_i c_i^{-1}$ and $p_j = a_j' c_j'^{-1}$ with $a_i, a_j' \in R$, $c_i, c_j' \in C$, for $0 \leq i \leq n$ and $0 \leq j \leq m$. Then there exist $b_i, b_j' \in R, s \in C$ such that $q_i = d_i^{-1} b_i$ and $p_j = s^{-1} b_j'$, for $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we have $(b_0 + b_1 x + \cdots + b_n x^n)s^{-1}(b_0' + b_1' x + \cdots + b_m' x^m) = 0$.

By Lemma 5.13, there are $e \in C$ and $d_0 + d_1 x + \cdots + d_t x^t \in S$, such that
Now we have \( b_0 + b_1 x + \cdots + b_n x^n = (d_0 + d_1 x + \cdots + d_t x^t)e^{-1} \). Hence \((b_0 + b_1 x + \cdots + b_n x^n)s^{-1}(b_0 + b_1 x + \cdots + b_n x^n) = (b_0 + b_1 x + \cdots + b_n x^n)(d_0 + d_1 x + \cdots + d_t x^t)e^{-1} = 0 \).

This implies that \((b_0 + b_1 x + \cdots + b_n x^n)(d_0 + d_1 x + \cdots + d_t x^t) = 0 \). Hence \( b_0 d_i = 0 \) for \( 0 \leq i \leq t \), since \( R \) is \( \delta \)-Armendariz. But \( s^{-1}b_0' = d_0 e^{-1} + d_1 \delta(e^{-1}) + \cdots + d_t \delta^t(e^{-1}) \).

and \( b_0 d_i = 0 \) for \( 0 \leq i \leq t \), so \( b_0 s^{-1}b_0' = 0 \) and hence \( d^{-1}b_0 p_0 = 0 \). So \( q_0 p_0 = 0 \). Now we have \( d^{-1}b_0' = d_1 e^{-1} + 2d_2 \delta(e^{-1}) + \cdots + t d_t \delta^t(e^{-1}) \).

But for \( 0 \leq i \leq t \), \( b_0 d_i = 0 \), so \( b_0 s^{-1}b_0' = 0 \). Thus \( d^{-1}b_0 p_1 = 0 \), and hence \( q_0 p_1 = 0 \). By a similar method we can get \( q_0 p_i = 0 \) for each \( 0 \leq i \leq m \). Therefore \( Q \) is \( \delta \)-Armendariz. A similar method can be employed for the weak \( \delta \)-Armendariz case.

Notice that Huh, Lee, and Smoktunowicz in [21, Theorem 12], prove that, if there exists the classical right quotient ring \( Q \) of a ring \( R \). Then \( R \) is Armendariz if and only if \( Q \) is Armendariz. 

**Corollary 5.15.** Suppose that there exists the classical right quotient ring \( Q \) of a ring \( R \). Then \( R \) is (weak) Armendariz if and only if \( Q \) is (weak) Armendariz.

As a well-known fact, \( R \) is a semiprime Goldie ring if and only if there exists the classical quotient ring of \( R \) which is semisimple Artinian.

**Theorem 5.16.** Let \( R \) be a semiprime right Goldie ring. Then the following are equivalent:

1. \( R \) is \( \delta \)-Armendariz.
2. \( Q \) is \( \delta \)-Armendariz.
3. \( R \) is weak \( \delta \)-Armendariz.
4. \( Q \) is weak \( \delta \)-Armendariz.
5. \( R \) is reduced.
6. \( Q \) is reduced;
7. \( R \) is Armendariz.
8. \( Q \) is Armendariz.
9. \( R \) is weak Armendariz.
10. \( Q \) is weak Armendariz.
11. \( Q \) is 2-primal.
12. \( R \) is 2-primal.
13. \( Q \) is a finite direct product of division rings, where \( Q \) is the classical right quotient ring of \( R \).

**Proof.** The equivalence (1) \( \Leftrightarrow \) (2), (3) \( \Leftrightarrow \) (4), (7) \( \Leftrightarrow \) (8) and (9) \( \Leftrightarrow \) (10) follows by Theorem 5.14. The implication (2) \( \Rightarrow \) (4) follows by definition. For the implication (4) \( \Rightarrow \) (6), \( Q \) is abelian by Proposition 3.9. So \( Q \) is an abelian semisimple ring and hence is reduced. The implication (6) \( \Rightarrow \) (8) and (8) \( \Rightarrow \) (10) are clear. The implication (10) \( \Rightarrow \) (11) is the same as (4) \( \Rightarrow \) (6), since in this case \( Q \) is reduced and hence is 2-primal. The implication (11) \( \Rightarrow \) (12) follows the fact that every subring of a 2-primal ring is 2-primal. The implication (12) \( \Rightarrow \) (13) follows the fact that a semiprime 2-primal ring is reduced. Hence \( Q \) is a reduced semisimple ring and the result follows. For the implication (13) \( \Rightarrow \) (1), since \( Q \) is a finite sum of division rings, so it is reduced and hence is \( \delta \)-Armendariz which implies that \( R \) is
δ-Armendariz.

By this result we may obtain [25, Proposition 18 and Corollary 19] as corollaries.

Using the fact that abelian von Neuman regular rings are reduced, we obtain the following.

**Theorem 5.17.** Let $R$ be a Von Neuman regular ring and $\delta$ a derivation of $R$. Suppose that there exists the classical quotient ring $Q$ of the ring $R$. Then the following statements are equivalent;

1. $R$ is $\delta$-Armendariz.
2. $Q$ is $\delta$-Armendariz.
3. $R$ is weak $\delta$-Armendariz.
4. $Q$ is weak $\delta$-Armendariz.
5. $R$ is reduced.
6. $Q$ is reduced.

Notice that, in the special case when $\delta = 0$, we obtain a generalization of [21, Corollary 17] and [3, Theorem 6].

**ACKNOWLEDGEMENT.** The first author would like to thank the Banach Algebra Center of Excellence for Mathematics, University of Isfahan.

**REFERENCES**