ROOT SYSTEMS ARISING FROM AUTOMORPHISMS

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Abstract. We study a combinatorial approach of producing new root systems from the old ones in the context of affine root systems and their new generalizations. The appearance of this approach in the literature goes back to the outstanding work of V. Kac in the realization of affine Kac-Moody Lie algebras. In recent years, this approach has been appeared in many other works, including the study of affinization of extended affine Lie algebras and invariant affine reflection algebras.

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0. Introduction

This paper is about root systems which naturally arise in the study of affinization of extended affine Lie algebras, and their new generalizations such as invariant affine reflection algebras. In recent years there had been several attempts (see for example [W], [P], [Az]) to construct new classes of Lie algebras through a method, known as twisting by automorphisms or affinization, due to V. Kac [K]. In 2002, B. Allison, S. Berman and A. Pianzola [ABP] considered an extended affine Lie algebra \((\mathcal{L}, (\cdot, \cdot), \mathfrak{h})\) and a finite period automorphism \(\sigma\) of \(\mathcal{L}\) which satisfies certain conditions. They generalized the outstanding work of V. Kac [K, Ch. 8] (see also [He, Section X.5]) on realization of affine Lie algebras to the class of extended affine Lie algebras. This work motivated several other works, including [ABY], [AK], [You1] and [AHY], therein the authors have considered fixed points of automorphisms of certain classes of infinite dimensional Lie algebras.

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Let \((\mathcal{L}, (\cdot, \cdot), \mathfrak{h})\) be an extended affine Lie algebra, or even an invariant affine reflection algebra with root systems \(R\), and \(\sigma\) be an automorphism of \(\mathcal{L}\) of period \(m\) satisfying certain natural conditions. One knows that \(\sigma\) induces an automorphism of \(R\), denoted again by \(\sigma\). It also induces certain gradings on \(\mathcal{L}\), \(\mathfrak{h}\), the algebra of fixed points of \(\mathcal{L}\) and \(\mathfrak{h}\) under \(\sigma\), and some other related spaces. The main object of this study is the set \(\pi(R) := \{\pi(\alpha) := (\frac{1}{m})\sum_{i=0}^{m-1} \sigma^i(\alpha) \mid \alpha \in R\}\). It is surprising and interesting that the ingredients mentioned above, including the ground Lie algebra \(\mathcal{L}\) and its root system \(R\), the considered automorphism \(\sigma\) of period \(m\), different involved gradings, the corresponding fixed point Lie subalgebra and its root system, are all related by the set \(\pi(R)\). In spite of its essential role and its long lasting appearance (since early work of V. Kac \([K]\) in 1968) this set is not studied independently.

The objective of this work is to study the set \(\pi(R)\), which we always assume contains some non-isotropic roots. In Section 1, we recall from \([Yos]\) the notion of a locally extended affine root system (LEARS for short), a generalization of the notion of an extended affine root system defined by K. Saito \([S]\). These root systems, according to their defining axioms, contain no isotropic roots. In order to include such roots in the root system, we define a new set of axioms for a LEARS and show that there is a one to one correspondence between the class of LEARS defined by Y. Yoshii and the one defined here (see Proposition 1.3). Similar to extended affine root systems, one can describe a LEARS in terms of some involved ingredients (see (1.7), or \([AYY]\) for a more detailed exposition). In Section 2, we show that starting from a finite order automorphism of a LEARS, the corresponding set \(\pi(R)\) satisfies all axioms of a LEARS except possibly the root string property (see Proposition 2.6). For types \(A_1\) and \(BC_1\), it turns out that \(\pi(R)\) is a LEARS of the same type of \(R\) (see Proposition 2.7).

In Section 3, we consider an automorphism of an EALA with the corresponding root system \(R\) and study the set \(\pi(R)\) with respect to the induced automorphism. Then we show that \(\pi(R)\) is an extended affine root system, not necessarily reduced. An interesting result is that \(\pi(R)\) appears naturally as the set of roots of a toral type extended affine Lie algebra. As a corollary, it is shown that starting from a finite root system \(R\) and a finite order automorphism of \(R\), the set \(\pi(R)\) is a finite root system, provided that \(\pi(R) = \pi(R^\times)\).

The remaining sections are devoted to the study of automorphisms, their orbits and the corresponding sets \(\pi(R)\) for a LEARS \(R\) of type \(B\). In Section 4, we study orbits of the action of a given finite order automorphism \(\sigma\). It follows from this study that if \(\sigma\) does not satisfy an orbit-length condition
then \( \pi(R) \) will not be a LEARS (see Lemma 4.6). In Section 5, we study the isotropic orbits (orbits mapped by \( \pi \) to isotropic elements). We believe that this is the first detailed study of such objects and we hope that this furnishes the ground for a systematic study of automorphisms of LEARS’s (or even EARS’s).

In Section 6, we prove that if \( R \) is a LEARS of type \( B \) and \( \sigma \) is a finite order automorphism of \( R \) satisfying an orbit-length condition, then the set \( \pi(R) \) is a LEARS, where we fully describe its internal structure in the form (1.7). To illustrate the variety of the possible finite order automorphisms and to elaborate on the results obtained throughout this work, the paper is concluded with a section containing a variety of examples. In each example we construct a finite order automorphism of a LEARS of type \( B \) and compute explicitly the set \( \pi(R) \), giving in details all the involved ingredients appearing in Theorem 6.2.

1. Locally extended affine root systems

For a subset \( A \) of a vector space \( V \) we denote the \( \mathbb{Z} \)-span of \( A \) in \( V \) by \( \langle A \rangle \). If \( V \) is equipped with a symmetric bilinear form, we set

\[
A^\times := \{ \alpha \in A \mid (\alpha, \alpha) \neq 0 \} \quad \text{and} \quad A^0 := \{ \alpha \in A \mid (\alpha, \alpha) = 0 \}.
\]

We call an element \( \delta \) of \( A^0 \), non-isolated if \( \delta \in A^\times - A^\times \). Following [Ho], we also set

\[
IRC(A) = ((A^\times - A^\times) \cap V^0) \cup A^\times.
\]

In 2008, Yoshii [Yos] defined the notion of a locally extended affine root system (LEARS for short). This notion generalizes the notion of a locally finite root system and also the notion of (non-isotropic parts of) an extended affine root system (EARS for short). Here we recall Yoshii’s definition, which is a generalization of Saito’s axioms for an extended affine root system (see [S]).

**Definition 1.1.** Let \( V \) be a non-trivial vector space over \( \mathbb{Q} \) with a positive semidefinite bilinear form \( (\cdot, \cdot) \), and let \( R \subset V \). The triple \((R, (\cdot, \cdot), V)\) (or simply \( R \)) is called a locally extended affine root system or a LEARS for short if

- \((R1)\) \((\alpha, \alpha) \neq 0\) for all \( \alpha \in R \), and \( R \) spans \( V \),
- \((R2)\) \((\alpha, \beta^\vee) := 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}\) for all \( \alpha, \beta \in R \),
- \((R3)\) \(w_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in R\) for all \( \alpha, \beta \in R \),
- \((R4)\) \( R \) can not be written as a union of two its non-empty subsets of \( R \) which are orthogonal with respect to the form \((R \text{ is indecomposable})\).

A LEARS \( R \) is called reduced if
(R5) $2\alpha \notin R$ for all $\alpha \in R$.

Since we are interested in a set of axioms which also contains isotropic roots we propose the following definition for a LEARS, and we explain the relation between our axioms of those of Yoshii.

**Definition 1.2.** Let $\mathcal{V}$ be a non-trivial vector space over $\mathbb{Q}$ with a positive semidefinite bilinear form $(\cdot, \cdot)$, and let $R \subset \mathcal{V}$. The triple $(R, (\cdot, \cdot), \mathcal{V})$ (or simply $R$) is called a *locally extended affine root system* or a LEARS for short if it satisfies the following four axioms:

1. **(LR1)** $R = -R$, 
2. **(LR2)** $R$ spans $\mathcal{V}$, 
3. **(LR3)** for $\alpha \in R^\times$ and $\beta \in R$, there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that $(\beta + Z\alpha) \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\}$ and $d - u = (\beta, \alpha^\vee)$, 
4. **(LR4)**(a) $R^\times$ cannot be written as a union of two its non-empty subsets which are orthogonal with respect to the form ($R$ is indecomposable), 
   (b) elements of $R^0$ are non-isolated ($R^0 \subseteq R^\times - R^\times$).

Moreover, $R$ is called *reduced* if it satisfies **(LR5)** $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.

**Proposition 1.3.** There is a one to one correspondence between the class of (reduced) irreducible LEARS (in the sense of Definition 1.2) onto the class of Yoshii’s (reduced) LEARS. Namely given any irreducible (reduced) LEARS $R$, the set $R^\times$ is a Yoshii’s (reduced) LEARS, and conversely given any Yoshii’s (reduced) LEARS, the set $\text{IRC}(R)$ is an irreducible (reduced) LEARS (in the sense of Definition 1.2).

**Proof.** The equivalence of axiom (LR4)(a) with (R4) and axiom (LR5) with (R5), in the two sets of axioms is clear. If a set $R$ satisfies axioms (LR1)-(LR3) and (LR4)(b), then clearly the set $R^\times$, satisfies (R1)-(R3). Conversely, suppose $R$ satisfies (R1)-(R3). Then an argument analogous to the proof of [Ho, Proposition 5.9], shows that $\text{IRC}(R)$ satisfies axioms (LR1)-(LR3) and (LR4)(b).

From now on by a LEARS, we mean the one from Definition 1.2. Let $(R, (\cdot, \cdot), \mathcal{V})$ be a LEARS. Here we record some facts from [Yos] about $R$. Let $\mathcal{V}^0$ be the radical of the form and $\bar{R}$ be the image of $R$ in $\bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$, under the canonical map. It follows that $\bar{R}$ is a *locally finite root system*, a LEARS with $\dim(\mathcal{V}) = 0$, whose type is called the type of $R$. A locally finite root system with $\dim(\mathcal{V}) < \infty$ is just a usual finite root system. For details on locally finite root systems see [LN].

**Remark 1.4.** An *extended affine root system* in the sense of [S] and [Yos] is a Yoshii’s LEARS for which $\dim(\mathcal{V}/\mathcal{V}^0)$ is finite and $\langle R \rangle$ is free. However,
in the notation of [AABGP], a (reduced) extended affine root system is a (reduced) LEARS for which $\dim(\mathcal{V}) < \infty$ and $\langle R \rangle$ is free of finite rank. It should be mentioned that in [AABGP], $\mathcal{V}$ is considered to be the real span of roots. However, one can change the base filed from $\mathbb{R}$ to $\mathbb{Q}$ without any difficulties.

Let $\hat{R}$ be an irreducible locally finite root system of type $X$ in a vector space $\hat{\mathcal{V}}$ over $\mathbb{Q}$, with the sets of short, long and extra long roots $\hat{R}_{sh}$, $\hat{R}_{lg}$ and $\hat{R}_{ex}$ respectively ($\hat{R}_{lg}$ and $\hat{R}_{ex}$ might be empty sets). Let $\mathcal{V}^0$ be a $\mathbb{Q}$-vector space and extend the form on $\hat{\mathcal{V}}$ to $\mathcal{V} := \hat{\mathcal{V}} \oplus \mathcal{V}^0$ so that $\mathcal{V}^0$ becomes the radical of the form. Let $S, L, E$ be subsets of $\mathcal{V}^0$ satisfying

\begin{equation}
\emptyset \neq F \in \{S, L, E\} \implies F + 2F \subseteq F, \\
S \text{ spans } \mathcal{V}^0, \ 0 \in S, \ 0 \in L \text{ (if } L \neq \emptyset), \\
S + L \subseteq S, \ kS + L \subseteq L \text{ if } \hat{R}_{lg} \neq \emptyset, \hat{R}_{ex} = \emptyset \\
S + E \subseteq S, \ 4S + E \subseteq E \text{ if } \hat{R}_{lg} = \emptyset, \hat{R}_{ex} \neq \emptyset \\
S + L \subseteq S, \ 2S + L \subseteq L, \ L + E \subseteq L, \ 2L + E \subseteq E \text{ if } \hat{R}_{lg} \neq \emptyset, \hat{R}_{ex} \neq \emptyset.
\end{equation}

(Here $k = 3$ if $X = G_2$ and $k = 2$ otherwise.) Furthermore assume that

\begin{equation}
\begin{align*}
S & \text{ is a group if } R \text{ is of type } A \neq A_1, C \neq C_2, D, E, F_4 \text{ or } G_2; \\
L & \text{ is a group if } R \text{ is of type } B \neq B_2, F_4, G_2 \text{ or } BC \neq BC_1, BC_2.
\end{align*}
\end{equation}

Finally set

\begin{equation}
R := \begin{cases} 
(S + S) \cup (\hat{R} + S) & \text{if } X = A, D, E \\
(S + S) \cup (\hat{R}_{sh} + S) \cup (\hat{R}_{lg} + L) & \text{if } X = B, C, F_4, G_2 \\
(S + S) \cup (\hat{R}_{sh} + S) \cup (\hat{R}_{ex} + E) & \text{if } X = BC_1 \\
(S + S) \cup (\hat{R}_{sh} + S) \cup (\hat{R}_{lg} + L) \cup (\hat{R}_{ex} + E) & \text{if } X = BC \neq BC_1.
\end{cases}
\end{equation}

Then by Proposition 1.3 and [Yos, Theorem 7], $R$ is a LEARS if and only if $R$ is of this form (Yoshii’s Theorem is stated one sided (“if” part), however we understood through a private communication with Yoshii that the converse is also true.) Note that, as locally finite root systems, $\hat{R} \cong R$.

\section{Root systems arising from automorphisms}

We now fix an irreducible LEARS $(R, (\cdot, \cdot), \mathcal{V})$ of the form (1.7). Let $m$ be a positive integer and $\sigma$ be a period $m$ automorphism of $R$. That is $\sigma$ is a linear isomorphism of $\mathcal{V}$ satisfying:

\begin{itemize}
  \item (AR1) $\sigma^m = 1$. \\
  \item (AR2) $\sigma(\alpha) \in R$ for all $\alpha \in R$.
\end{itemize}

\begin{lemma}
$\sigma(\mathcal{V}^0) = \mathcal{V}^0$. Moreover $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in \mathcal{V}$.
\end{lemma}
Proof. Let $\delta \in R^0$. Then it follows from (1.7) and (1.5) that $Z\delta \subseteq R^0$. Since $\sigma$ satisfies (AR2), we have $Z\sigma(\delta) \subseteq R^0$. But this is possible only if $\sigma(\delta) \in R^0$. Therefore $\sigma(R^0) \subseteq R^0$, forcing $\sigma(\mathcal{V}^0) = \mathcal{V}^0$. Thus $\sigma$ induces an automorphism $\overline{\sigma}$ of $\overline{\mathcal{V}}$ with $\overline{\sigma}(\overline{R}) = \overline{R}$. Next we note that $\overline{R}$ is a locally finite root system in the vector space $\overline{\mathcal{V}}$ equipped with the induced form $(\cdot, \cdot)$ and so using [LN, Corollary 4.7], one gets that $\overline{\sigma}$ preserves the length. Now this together with [LN, Lemma 3.7(ii)] completes the proof. \hfill $\square$

Consider the linear map $\pi : \mathcal{V} \to \mathcal{V}$ defined by

$$\pi(\alpha) := \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha); \quad \alpha \in \mathcal{V}. \tag{2.2}$$

It maps $\mathcal{V}$ onto the fixed point subspace

$$\mathcal{V}^\sigma := \{\alpha \in \mathcal{V} \mid \sigma(\alpha) = \alpha\}$$

and splits $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}^\sigma \oplus \mathcal{V}^c$, where $\mathcal{V}^c = \ker(\pi)$. Moreover

$$\pi \circ \sigma = \sigma \circ \pi = \pi. \tag{2.3}$$

Form this and Lemma 2.1 we have, for any $\alpha, \beta \in \mathcal{V},$

$$\pi(\alpha), \pi(\beta) = \frac{1}{m} \sum_{i=0}^{m-1} (\sigma^i(\alpha), \pi(\beta)) = \frac{1}{m} \sum_{i=0}^{m-1} (\alpha, \pi(\beta)) = (\alpha, \pi(\beta)), \tag{2.4}$$

and by symmetry $(\pi(\alpha), \pi(\beta)) = (\pi(\alpha), \beta)$. Therefore

$$(\mathcal{V}^\sigma, \mathcal{V}^c) = (\pi(\mathcal{V}^\sigma), \pi(\mathcal{V}^c)) = (\mathcal{V}^\sigma, \pi(\mathcal{V}^c)) = (\mathcal{V}^\sigma, \{0\}) = \{0\}.$$}

Since by Lemma 2.1, $\sigma(\mathcal{V}^0) = \mathcal{V}^0$ and consequently $\pi(\mathcal{V}^0) \subseteq \mathcal{V}^0$, we have

$$\mathcal{V}^0 = (\mathcal{V}^0)^\sigma \oplus (\mathcal{V}^0)^c$$

where $(\mathcal{V}^0)^\sigma := \mathcal{V}^\sigma \cap \mathcal{V}^0$ and $(\mathcal{V}^0)^c := \mathcal{V}^c \cap \mathcal{V}^0$.

The automorphism $\sigma$ induces an automorphism $\overline{\sigma}$ of $\overline{R}$. Also the map $\pi$ induces a map $\overline{\pi} : \overline{\mathcal{V}} \to \overline{\mathcal{V}}$ with $\overline{\pi}(\overline{\alpha}) = \overline{\pi(\alpha)}$. In fact $\overline{\pi}$ maps $\overline{\mathcal{V}}$ onto $\overline{\mathcal{V}}(\sigma) := \{\overline{\alpha} \in \overline{\mathcal{V}} \mid \overline{\sigma}(\overline{\alpha}) = \overline{\alpha}\}$.

Throughout this work we assume that

$$\pi(R)^\times \neq \emptyset.$$}

Here we need to state a modified version of a lemma from [ABP] which its proof remains exactly the same as [ABP, Lemma 3.34] (replacing finite with locally finite and changing the base filed from $\mathbb{R}$ to $\mathbb{Q}$).

Lemma 2.5. Let $\Omega$ be a locally finite root system in a nonzero $\mathbb{Q}$-space $X$. Let $Y$ be a nonzero subspace of $X$, and let $p : X \to Y$ be the orthogonal projection onto $Y$. Then

(i) If $\alpha \in \Omega^\times$, then there exists $\beta \in \Omega^\times$ so that $p(\beta) \neq 0$ and $(\beta, \alpha) \neq 0$. 

(ii) Let $\alpha \in \Omega^\times$, and $\beta \in \Omega^\times$. Then $\alpha \mathcal{V}^{\sigma\beta} \cap \mathcal{V}^c \neq \emptyset$. 

(iii) Let $\alpha \in \Omega^\times$ and $\beta \in \Omega^\times$. Then $\mathcal{V}^{\sigma\beta} \cap \mathcal{V}^c \neq \emptyset$. 

(iv) Let $\alpha \in \Omega^\times$ and $\beta \in \Omega^\times$. Then $\mathcal{V}^{\sigma\beta} \cap \mathcal{V}^c \neq \emptyset$.
(ii) $p(\Omega^x) \setminus \{0\}$ cannot be written as a union of two its nonempty orthogonal subsets.

**Proposition 2.6.** Let $\pi(R)^x \neq \emptyset$. Then,

(i) If $\langle R \rangle \subseteq \mathcal{V}$ is free (of finite rank), then so is $\langle \pi(R) \rangle \subseteq \mathcal{V}^\sigma$,

(ii) $\pi(R)^x$ is indecomposable with respect to the form,

(iii) Elements of $\pi(R)^0$ are non-isolated,

(iv) $\pi(R)^x$ spans $\mathcal{V}^\sigma$,

(v) $\pi(\mathcal{V})^0 = \mathcal{V}^\sigma \cap \mathcal{V}^0 = \pi(\mathcal{V}^0)$,

(vi) $\pi(R^0)$ (and so $\pi(R)^0$) spans $\pi(\mathcal{V})^0$.

In particular, the triple $(\mathcal{V}^\sigma, \langle \cdot, \cdot \rangle, \pi(R))$ satisfies all axioms of a LEARS, except possibly (LR3).

**Proof.** For $\alpha \in R$ we have $\pi(\alpha) \in \frac{1}{m} \langle R \rangle$, so (i) follows. We now show that (ii) holds. Note that the map $\pi$ induces a map $\bar{\pi} : \mathcal{V}/\mathcal{V}^0 \rightarrow \mathcal{V}/\mathcal{V}^0$ with $\bar{\pi}(v) := \pi(v)$ for $v \in \mathcal{V}/\mathcal{V}^0$. Now if $\pi(R)^x$ is written as a disjoint union of two nonempty orthogonal subsets, then $\overline{\pi(R)} \setminus \{0\}$ can be written as a disjoint union of two nonempty orthogonal subsets. However, as $\pi(R)$ is a locally finite root system, this contradicts Lemma 2.5(ii).

For (iii), we must show that the elements of $\pi(R)^0$ are non-isolated. For this, let $\delta := \pi(\alpha)$ for some $\alpha \in R$ and $(\pi(\alpha), \pi(\alpha)) = 0$. We consider two cases $\alpha \in R^0$ and $\alpha \in R^x$, separately. Consider a decomposition for $R$ as in (1.7) and take $\mathcal{V}^0$ to be the radical of $(\cdot, \cdot)_{\mathcal{V} \times \mathcal{V}}$.

(a) $\alpha \in R^0 = S + S$. Then $\alpha = \delta_1 + \delta_2$ for some $\delta_1, \delta_2 \in S \subseteq R^0$. Since we have already assumed $\pi(R)^x \neq \emptyset$, there exists $\beta \in R$ such that $(\pi(\beta), \pi(\beta)) \neq 0$. Since $\tilde{R}_{sh} + S$ and $R$ have the same $\mathbb{Q}$-span and $\pi(R)^0 \subseteq \mathcal{V}^0$, we may assume that $\beta \in \tilde{R}_{sh} + S$. Also $\pi(\mathcal{V}^0) \subseteq \mathcal{V}^0$, so we may assume that $\beta = \dot{\alpha} \in \tilde{R}_{sh}$. Set

$$\gamma := \dot{\alpha} + \delta_1 \in \tilde{R}_{sh} + S \subseteq R^x.$$ 

Then $(\pi(\gamma), \pi(\gamma)) = (\pi(\dot{\alpha}), \pi(\dot{\alpha})) \neq 0$. So $\bar{\gamma} := \pi(\gamma) \in \pi(R)^x$. Also

$$\gamma + \alpha = \dot{\alpha} + \delta_1 + \delta_1 + \delta_2 \in \dot{\alpha} + 2S + S \subseteq \dot{\alpha} + S \subseteq R^x.$$ 

Finally, $\bar{\gamma} + \tilde{\delta} = \pi(\gamma + \alpha) \in \pi(R)^x$.

(b) $\alpha \in R^x$. By Lemma 2.5(i), there exists $\beta \in R^x$ such that $(\alpha, \beta) \neq 0$ and $(\pi(\beta), \pi(\beta)) \neq 0$. Now it follows from (R3) (the root string property) that either $\beta + \alpha \in R$ or $-\beta + \alpha \in R$. Take $\bar{\beta} := \pi(\beta)$ in the first case and $\tilde{\beta} = \pi(-\beta)$ in the latter case. Then $\bar{\beta} \in \pi(R)^x$ and $\bar{\beta} + \tilde{\delta} = \pi(\pm \beta + \alpha) \in \pi(R)^x$. This finishes the proof of part (iii).

Since $R$ spans $\mathcal{V}$, $\pi(R)$ spans $\mathcal{V}^\sigma = \pi(\mathcal{V})$ so (iv) follows from (iii). To see (v), just follow the definition of each term. Finally (vi) follows from (v).
and the fact that $R^0$ spans $V^0$. Note that $\pi(R^0) \subseteq \pi(R)^0$ and so $\pi(R)^0$ also spans $\pi(V)^0$. The last statement is now clear. □

The question is now, in which cases $\pi(R)$ satisfies axioms (LA3), that is $\pi(R)$ is a LEARS. As we will see in Section 4, the answer is not always affirmative. However, in some cases we know the exact answer. Let us start with simplest ones.

**Proposition 2.7.** Let $R$ be a LEARS of type $X = A_1, BC_1$ in $V$ and $\sigma$ be an automorphism of $V$ satisfying (A1)-(A2). If $\pi(R)^x \neq \emptyset$, then $\pi(R) \subseteq \pi(V)$ is also a LEARS of type $X$.

**Proof.** By Proposition 2.6, it is only enough to verify axiom (LR3). However, we found it more convenient to express $\pi(R)$ in the form (1.7). Then using [Yos, Theorem 7], we obtain all axioms at once.

By (1.7), in the case $A_1$, we have $R = (S + S) \cup (\pm \epsilon + S)$ where $S$ satisfies

\[ 0 \in S, \quad S \text{ spans } V^0, \quad S \pm 2S \subseteq S, \]

and $(\epsilon, \epsilon) \neq 0$. Since $\pi(R)^x \neq \emptyset$, we have $(\pi(\epsilon), \pi(\epsilon)) \neq 0$. Moreover

\[ \pi(R) = (\pi(S) + \pi(S)) \cup (\pm \pi(\epsilon) + \pi(S)), \]

where clearly $\pi(S)$ satisfies the same properties of $S$ in (2.8). Thus by [Yos, Theorem 7], $\pi(R)$ is a LEARS of type $A_1$ in $\pi(V)$.

For the case $X = BC_1$, we obtain

\[ \pi(R) = (\pi(S) + \pi(S)) \cup (\pm \pi(\epsilon) + \pi(S)) \cup (2\pi(\epsilon) + \pi(E)). \]

So a similar argument as above gives the result. □

### 3. Root systems arising from Lie algebra automorphisms

This section explains our motivation for considering the set $\pi(R)$, for the case of an EARS. We prove that $\pi(R)$ is an EARS, when it appears naturally in the study of *affinization* of extended affine Lie algebras (EALA’s). More exciting for us, we prove that $\pi(R)$, naturally appears as the root system of a *toral type extended affine Lie algebra*. We refer to a toral type extended affine Lie algebra as the algebra introduced in [You2] by axioms (T1)-(T6) which are a slight modification of those given in [AKY]. To keep this section as short as possible, we do not recall the axioms of an EALA and of a toral type extended affine Lie algebra and for the definitions we ask the reader to consult [AABGP] and [You2], respectively.

Now consider a fixed EALA $(\mathfrak{L}, \langle \cdot, \cdot \rangle, \mathfrak{h})$ with root system $R$. One knows (see [AABGP, Section 1]) how to transfer the form on $\mathfrak{h}$ to a form on $\mathfrak{h}^*$, using the non-deneracy of the form on $\mathfrak{h}$. Let $V := \text{span}_{\mathbb{Q}}(R) \subseteq \mathfrak{h}^*$ and $V^0$ be the radical of the form on $\mathfrak{h}^*$ restricted to $V$. We recall that $R$ is an
EARS in $V$, that is; $R$ is a reduced LEARS satisfying $\dim(V) < \infty$ and $\langle R \rangle$ is free of finite rank. Consider the corresponding root space decomposition $\mathcal{L} = \bigoplus_{\alpha \in R} \mathcal{L}_\alpha$. We Let $\sigma$ be an automorphism of $\mathcal{L}$ and denote by $\mathcal{L}(\sigma)$ the set of fixed points of $\mathcal{L}$ under $\sigma$. If $\sigma(h) = h$, we also use $h(\sigma)$ for the set of fixed points on $h$. Let $m \geq 1$ be an integer and let $\sigma$ satisfy:

(A1) $\sigma^m = 1$.
(A2) $\sigma(h) = h$.
(A3) $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in \mathcal{L}$.
(A4) $h(\sigma)$ is self-centralizing in $\mathcal{L}(\sigma)$ i.e., $C_{\mathcal{L}(\sigma)}(h(\sigma)) = h(\sigma)$.

Here we record some facts from [ABP, Section 3] about the automorphism $\sigma$ and the new gradings it imposes on $\mathcal{L}$. As one knows, $\sigma$ induces an automorphism of $h^*$, denoted again by $\sigma$, satisfying

$$\sigma^m = 1, \quad \sigma(\alpha) \in R \text{ for all } \alpha \in R.$$ 

Now let $\overline{j}$ denote the image of $j \in \mathbb{Z}$ in $\mathbb{Z}/m\mathbb{Z}$. From (A1) and (A2), we have

$$\mathcal{L}_j = \bigoplus_{i=0}^{m-1} \mathcal{L}_{\overline{i}}, \quad h(\overline{j}) = \bigoplus_{i=0}^{m-1} h_{\overline{i}},$$

where $\mathcal{L}_j$ and $h_j$ are the eigenspaces of $\mathcal{L}$ and $h$ respectively, corresponding to the $j$-th power of $\zeta := e^{2\pi i / m}$, namely

$$\mathcal{L}_j = \{ x \in \mathcal{L} | \sigma(x) = \zeta^j x \} \quad \text{and} \quad h_j := \mathcal{L}_j \cap h.$$

Then

$$\mathcal{L}(\sigma) = \mathcal{L}_0 \quad \text{and} \quad h(\sigma) = h_0.$$ 

Set $h(\alpha) := \sum_{i=1}^{m-1} h_{\overline{i}}$, then one can see that

$$\pi(\alpha)|_{h(\sigma)} = \alpha|_{h(\sigma)} \quad \text{and} \quad \pi(\alpha)|_{h(\sigma)} = \{0\}.$$ 

So without loss of generality, we may identify $\pi(\alpha), \alpha \in R$, as an element of $h(\sigma)^*$. Using this identification, one finds that $\mathcal{L}$ is an $h(\sigma)$-module admitting a weight space decomposition $\mathcal{L} = \bigoplus_{\overline{\alpha} \in \pi(R)} \mathcal{L}_{\overline{\alpha}}$ with respect to $h(\sigma)$ with $\mathcal{L}_{\overline{\alpha}} = \sum_{\beta \in R | \pi(\beta) = \overline{\alpha}} \mathcal{L}_\beta$. Moreover, since $\sigma(\mathcal{L}_{\pi(\alpha)}) = \mathcal{L}_{\pi(\alpha)}$, $\alpha \in R$, we have

$$\mathcal{L}_{\pi(\alpha)} = \bigoplus_{i \in \mathbb{Z}_m} \mathcal{L}_{i, \pi(\alpha)} \quad \text{with} \quad \mathcal{L}_{i, \pi(\alpha)} := \mathcal{L}_i \cap \mathcal{L}_{\pi(\alpha)}, \quad \alpha \in R, \overline{i} \in \mathbb{Z}_m.$$ 

Set

$$R_i := \{ \alpha \in R | \mathcal{L}_{\overline{i}, \pi(\alpha)} \neq \{0\} \}; \quad i \in \mathbb{Z},$$

$$\mathcal{L}_{\pi(\alpha)} = \bigoplus_{i \in \mathbb{Z}_m} \mathcal{L}_{i, \pi(\alpha)} \quad \text{with} \quad \mathcal{L}_{i, \pi(\alpha)} := \mathcal{L}_i \cap \mathcal{L}_{\pi(\alpha)}, \quad \alpha \in R, \overline{i} \in \mathbb{Z}_m.$$ 

Set

$$R_i := \{ \alpha \in R | \mathcal{L}_{\overline{i}, \pi(\alpha)} \neq \{0\} \}; \quad i \in \mathbb{Z},$$
then $R = \bigcup_{i=0}^{m-1} R_i$.

Here we record our main result of this section.

**Theorem 3.4.** Let $(\mathfrak{L}, \langle \cdot, \cdot \rangle, \mathfrak{h})$ be an EALA with root system $R$. Suppose that $\sigma$ is an automorphism of $\mathfrak{L}$ satisfying $(A1)$-$(A4)$ and assume that $\pi(R)^\times \neq \emptyset$. Then $(\mathfrak{L}, \langle \cdot, \cdot \rangle, \mathfrak{h}(\sigma))$ is a toral type extended affine Lie algebra with root system $\pi(R)$. In particular $\pi(R)$ is an EARS.

**Proof.** One knows that $\mathfrak{h}(\sigma)$ is a finite dimensional subalgebra of $\mathfrak{L}$ with respect to which $\mathfrak{L}$ has a weight space decomposition and that $\mathfrak{L}$ is equipped with the non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ whose restriction to $\mathfrak{h}(\sigma) \times \mathfrak{h}(\sigma)$ is non-degenerate. We also note that as $\sigma$ satisfies (3.1), parts (ii)-(iii) of Proposition 2.6 imply that $\pi(R)$, which is the root system of $\mathfrak{L}$ with respect to $\mathfrak{h}(\sigma)$, is indecomposable with respect to the form and that elements of $\pi(R)^0$ are non-isolated. Moreover applying the same argument as in [ABY, Proposition 2.34], one gets that weight vectors corresponding to non-isotropic roots act locally nilpotently on $\mathfrak{L}$ using the adjoint action. Next we observe that $\pi(R) \subseteq \frac{1}{m} \langle R \rangle$ and so as $\langle R \rangle$ is a free abelian group of finite rank, we conclude that $\langle \pi(R) \rangle$ is also a free abelian group of finite rank. These all together imply that axioms (T1), (T2) and (T4)-(T6) stated in [You2] hold for $(\mathfrak{L}, \langle \cdot, \cdot \rangle, \mathfrak{h}(\sigma))$. To complete the proof that $\mathfrak{L}$ is a toral type extended affine Lie algebra it is enough to show that for $\alpha \in R$,

\begin{equation}
[\mathfrak{L}_{\pi(\alpha)}, \mathfrak{L}_{-\pi(\alpha)}] = \mathbb{C} t_{\pi(\alpha)},
\end{equation}

(which implies (T3)). Let $\alpha \in R$. For any $\bar{i} \in \mathbb{Z}_m$ we have $[\mathfrak{L}_{\pi(\alpha)}, \mathfrak{L}_{-\pi(\alpha)}] \subseteq \mathfrak{L}(\sigma) \cap \mathfrak{L}_0 \subseteq C_{\mathfrak{L}(\sigma)}(\mathfrak{h}(\sigma))$. Thus by (A4), $[\mathfrak{L}_{\pi(\alpha)}, \mathfrak{L}_{-\pi(\alpha)}] \subseteq \mathfrak{h}(\sigma)$. Now let $x_\pm \in \mathfrak{L}_{\pi(\alpha)}$. Then $[x_+, x_-] \in \mathfrak{h}(\sigma)$ and for any $h \in \mathfrak{h}(\sigma)$,

$$
([x_+, x_-] - (x_+, x_-) t_{\pi(\alpha)}, h) = (x_+, [x_-, h]) - (x_+, x_-) (t_{\pi(\alpha)}, h) = (x_+, \pi(\alpha)(h) x_-) - \pi(\alpha)(h)(x_+, x_-) = 0.
$$

This together with the non-degeneracy of the form on $\mathfrak{h}(\sigma)$ gives $[x_+, x_-] = (x_+, x_-) t_{\pi(\alpha)}$. Since $\{0\} \neq \mathfrak{L}_{\alpha} \subseteq \mathfrak{L}_{\pi(\alpha)} = \bigoplus_{\bar{i} \in \mathbb{Z}_m} \mathfrak{L}_{\bar{i}, \pi(\alpha)}$, we have $\mathfrak{L}_{\bar{i}, \pi(\alpha)} \neq \{0\}$ for some $\bar{i} \in \mathbb{Z}_m$. Also from the non-degeneracy of the form on $\mathfrak{L}$, we conclude that the form restricted to $\mathfrak{L}_{\bar{i}, \pi(\alpha)} \oplus \mathfrak{L}_{-\bar{i}, \pi(\alpha)}$ is non-degenerate. These all together now give (3.5). Now by [You2, Theorem 1.14], $\pi(R)$ is an EARS.

**Corollary 3.6.** Let $R$ be a finite root system in a Euclidean space $\mathbb{V}$ and $\sigma$ be an automorphism of $\mathbb{V}$ satisfying (3.1). Define $\pi \in \text{End}(\mathbb{V})$ by $\pi(\alpha) :=$
(1/m) \sum_{i=1}^{m-1} \sigma^i(\alpha), \alpha \in \mathcal{V} and suppose that \( \pi(R^\times) = \pi(R)^\times \). Then \( \pi(R) \) is a finite root system in \( \pi(V) \).

**Proof.** Fix a base \( \{\alpha_1, \ldots, \alpha_\ell\} \) of \( R \). Let \( \mathfrak{h} := \sum_{i=1}^{\ell} \mathbb{C} x_i \), be an \( \ell \)-dimensional complex vector space with a non-degenerate form defined by \( (x_i, x_j) = (\alpha_i, \alpha_j) \). Define \( \hat{\alpha}_i \in \mathfrak{h}^* \) by \( \hat{\alpha}_i(x_j) = (\alpha_i, \alpha_j) \). Then the set of \( \hat{\alpha}_i \)'s forms a basis of \( \mathfrak{h}^* \). Indeed if \( \sum_{j=1}^{\ell} (k_j + ik'_j) \hat{\alpha}_j = 0 \) for \( k_j, k'_j \in \mathbb{R}, \) then acting both sides on \( x_i \)'s and using the fact that \( \hat{\alpha}_j(x_i) = (\alpha_j, \alpha_i) \in \mathbb{R}, \) we conclude that \( k_j = k'_j = 0 \) for all \( j \). Thus the assignment \( \alpha_i \mapsto \hat{\alpha}_i \) induces a linear isomorphism \( \hat{\cdot} \) from \( \mathcal{V} \) onto the real subspace \( \hat{\mathcal{V}} = \sum_{i=1}^{\ell} \mathbb{R} \hat{\alpha}_i \) of \( \mathfrak{h}^* \). We note that \( x_i \) is the unique element in \( \mathfrak{h} \) which represents \( \hat{\alpha}_i \) through the form on \( \mathfrak{h} \). Therefore if we transfer the form on \( \mathfrak{h} \) to \( \mathfrak{h}^* \) by \( (\hat{\alpha}_i, \hat{\alpha}_j) := (x_i, x_j) \), the map \( \hat{\cdot} \) becomes an isometry. Thus \( \hat{R} \), the image of \( R \) under \( \hat{\cdot} \) is a finite root system in \( \hat{\mathcal{V}} \) isomorphic to \( R \). Let \( \hat{\sigma} \) be the automorphism of \( \hat{R} \) defined by \( \hat{\sigma}(\hat{\alpha}) = \sigma(\alpha) \hat{\cdot} \) for \( \alpha \in \mathcal{V} \). If \( \hat{\pi} := (1/m) \sum_{i=1}^{m} \hat{\sigma}^i \), then we have \( \hat{\pi}(\hat{\alpha}) = \pi(\alpha) \hat{\cdot} \) for all \( \alpha \in \mathcal{V} \), so \( \hat{\pi}(\hat{R}) = \pi(R) \). Thus \( \hat{\pi} \) maps \( \pi(R) \) isometrically onto \( \hat{\pi}(\hat{R}) \).

In particular from the statement we have

\[
(3.7) \quad \hat{\alpha} \in \hat{R} \setminus \{0\} \implies \hat{\pi}(\hat{\alpha}) \in \hat{R} \setminus \{0\}.
\]

We also note that \( \hat{\sigma} \) remains an automorphism of \( \hat{R}_{\text{ind}} := (\hat{R} \setminus 2\hat{R}) \cup \{0\} \), the set of indivisible roots of \( \hat{R} \).

Next we consider \( \mathfrak{h} \) as a Cartan subalgebra of a complex finite dimensional simple Lie algebra \( \mathfrak{g} \) with root system \( \hat{R}_{\text{ind}} \). It is known that \( \hat{\sigma} \) can be lifted to an automorphism of \( \mathfrak{g} \) with \( \hat{\sigma}(\mathfrak{h}) = \mathfrak{h} \) (see [Hu, Theorem 14.2]). Let \( \mathfrak{g}(\hat{\sigma}) \) (resp. \( \mathfrak{h}(\hat{\sigma}) \)) be the fixed points of \( \mathfrak{g} \) (resp. \( \mathfrak{h} \)) under \( \hat{\sigma} \). We have \( \mathfrak{h}(\hat{\sigma}) \neq \{0\} \), as \( \hat{\pi}(\hat{R}) \subseteq (\mathfrak{h}^*)(\hat{\sigma}) \cong (\mathfrak{h}(\hat{\sigma}))^* \) and \( \hat{\pi}(\hat{R}) \cong \pi(R) \neq \{0\} \).

Next we show that \( C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma) = \mathfrak{h}^\sigma \). (The proof of this identity is adopted from [ABP, Proposition 3.25].) Let \( x \in C_{\mathfrak{g}^\sigma}(\mathfrak{h}^\sigma) \). Then \( x = \sum_{\hat{\alpha} \in \hat{R}} x_{\hat{\alpha}} \), \( x_{\hat{\alpha}} \in \mathfrak{g}_{\hat{\alpha}} \) for all \( \hat{\alpha} \in \hat{R} \) and so

\[
0 = [\mathfrak{h}^\sigma, x] = \sum_{\hat{\alpha} \in \hat{R}} \hat{\alpha}(\mathfrak{h}^\sigma)x_{\hat{\alpha}}.
\]

Thus if \( \hat{\alpha} \in \hat{R} \) and \( x_{\hat{\alpha}} \neq 0 \) then \( \hat{\alpha}(\mathfrak{h}^\sigma) = \{0\} \) and so \( \hat{\pi}(\hat{\alpha}) = 0 \). But then by (3.7), \( \hat{\alpha} = 0 \). Thus \( x = x_0 \in \mathfrak{h} \cap \mathfrak{g}^\sigma = \mathfrak{h}^\sigma \).

Now it follows from Theorem 3.4 that \( (\mathfrak{g}, \kappa(\cdot, \cdot), \mathfrak{h}^\sigma) \) is a finite dimensional simple toral type extended affine Lie algebra, with root system \( \hat{\pi}(\hat{R}_{\text{ind}}) \). (Here \( \kappa(\cdot, \cdot) \) is the Killing form on \( \mathfrak{g} \).) It follows that \( \hat{\pi}(\hat{R}) \) is a finite root system in \( \hat{\mathcal{V}} \). Since \( \pi(R) \) is isometrically isomorphic to \( \hat{\pi}(\hat{R}) \), we are done. \( \square \)
We know that if $\pi(R)$ is an EARS, $\pi(R)$ is a finite root system. The following result gives a criterion for $\pi(R)$ to be a finite root system without knowing the exact structure of $\pi(R)$.

**Proposition 3.8.** Let $\sigma$ be a finite order automorphism of an EARS $R$ satisfying $\pi(R^\times) = \pi(R)^\times$. Then $\overline{\pi(R)}$ is a finite root system in $\overline{V}^\sigma = \overline{V}$. In particular, $(\tilde{\alpha}, \tilde{\beta}^\vee) \in \mathbb{Z}$ for all $\tilde{\alpha}, \tilde{\beta} \in \pi(R)^\times$.

**Proof.** We know that $\overline{R}$ is a finite root system in $\overline{V}$. Moreover from the statement, we have that the induced automorphism $\overline{\sigma}$ on $\overline{V}$ satisfies $\overline{\sigma}(\overline{\alpha}) \neq 0$ for all $\overline{\alpha} \in \overline{R} \setminus \{0\}$. The first statement now follows from Corollary 3.6. The second statement is clear, as the form on $\overline{V}$ is induced from $V$. \hfill \square

4. Orbits of type B automorphisms

We keep the same notation as in Section 2. We fix a LEARS $R$ of type $B$ in the $\mathbb{Q}$-vector space $V$ equipped with a positive semi-definite bilinear form $(\cdot, \cdot)$ and an automorphism $\sigma$ of $V$ satisfying (AR1)-(AR2). We study some orbit theory which leads to the fact that the axioms (AR1)-(AR2) are not enough to guarantee $\pi(R)$ is a LEARS. As before we assume that $\pi(R)^\times \neq \emptyset$.

Since $R$ is a LEARS of type $B$, $R$ has the form

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L)$$

with

$$\dot{R}_{sh} = \{\pm \epsilon_i \mid i \in J\} \quad \text{and} \quad \dot{R}_{lg} = \{\pm (\epsilon_i \pm \epsilon_j) \mid i \neq j \in J\}$$

where $\mathcal{B} := \{\epsilon_i \mid i \in J\}$ is an orthogonal basis for $\dot{V} = \text{span}_{\mathbb{Q}} \dot{R}$ and $S$ and $L$ span $V^0$ and contain zero,

$$S \pm L \subseteq S \quad \text{and} \quad L \pm 2S \subseteq L,$$

where $L$ is a subgroup of $V^0$ if $|J| \geq 3$.

Note that, it follows from (4.1) that $S \pm 2S \subseteq S$ and $L \pm 2L \subseteq L$. We identify $\check{V} = V/V^0$ and $\dot{V}$ via the isometry $\check{\cdot} : V \to \dot{V}$, then $\dot{R}$ is also identified with $\check{R}$, the image of under $\check{\cdot}$. Using this identification and the fact that $\sigma$ and $\check{\sigma}$ preserve the forms on $V$ and $\dot{V}$, we have

$$\sigma(\mathcal{B}) \subseteq \mathcal{B}_\pm + S \quad \text{and} \quad \check{\sigma}(\mathcal{B}) \subseteq \mathcal{B}_\pm,$$

where

$$\mathcal{B}_\pm := \mathcal{B} \cup -\mathcal{B}.$$

Recall that the automorphism $\sigma$ induces an automorphism $\check{\sigma} \in \text{Aut}(\check{V})$ mapping $\check{R}$ onto $\check{R}$. That is, $\check{\sigma}$ is a period $m$ automorphism of the finite
root system $\tilde{R}$. We denote by $\langle \sigma \rangle$ (resp. $\langle \tilde{\sigma} \rangle$) the subgroup of $GL(V)$ (resp. $GL(\tilde{V})$) generated by $\sigma$ (resp. $\tilde{\sigma}$).

For $\alpha \in V$, let $\text{orb}(\alpha)$ (resp. $\overline{\text{orb}}(\tilde{\alpha})$) be the orbit of $\alpha$ (resp. $\tilde{\alpha}$) under the action of $\langle \sigma \rangle$ (resp. $\langle \tilde{\sigma} \rangle$), namely

$$\text{orb}(\alpha) := \{\sigma^k(\alpha) \mid k \geq 0\} \quad \text{and} \quad \overline{\text{orb}}(\tilde{\alpha}) := \{\tilde{\sigma}^k(\tilde{\alpha}) \mid k \geq 0\}.$$

Let

$$l(\alpha) := |\text{orb}(\alpha)| \quad \text{and} \quad \tilde{l}(\tilde{\alpha}) := |\overline{\text{orb}}(\tilde{\alpha})|.$$

Then $l(\alpha)$ and $\tilde{l}(\tilde{\alpha})$ are the smallest positive integers for which

$$\sigma^{l(\alpha)}(\alpha) = \alpha \quad \text{and} \quad \tilde{\sigma}^{\tilde{l}(\tilde{\alpha})}\tilde{\alpha} = \tilde{\alpha},$$

and so $\tilde{l}(\tilde{\alpha})$ divides $l(\alpha)$ and $l(\alpha)$ divides $m$. As in Section 2, let $\tilde{\pi}$ be the map on $\tilde{V} \equiv \tilde{V}$ induced from $\pi$.

**Lemma 4.3.** Let $\alpha, \beta \in \mathcal{B}_\pm$. Then

(i) $\tilde{\pi}(\alpha) = 0$ if and only if $\overline{\text{orb}}(\tilde{\alpha}) = \overline{\text{orb}}(-\alpha)$.

(ii) If $\tilde{\pi}(\alpha) \neq 0$, then

$$\overline{\text{orb}}(\tilde{\alpha}) = \overline{\text{orb}}(\tilde{\beta}) \iff \tilde{\pi}(\alpha) = \tilde{\pi}(\beta).$$

**Proof.** (i) If $\overline{\text{orb}}(\tilde{\alpha}) = \overline{\text{orb}}(-\alpha)$ then $\tilde{\pi}(\alpha) = \tilde{\pi}(-\alpha) = -\tilde{\pi}(\alpha)$ and so $\tilde{\pi}(\alpha) = 0$. Suppose next that $\tilde{\pi}(\alpha) = 0$ but $\overline{\text{orb}}(\tilde{\alpha}) \neq \overline{\text{orb}}(-\alpha)$. Then $\overline{\text{orb}}(\tilde{\alpha}) \cap \overline{\text{orb}}(-\alpha) = \emptyset$. So there exists no $\gamma \in \mathcal{B}$ with $\{\gamma, -\gamma\} \subseteq \overline{\text{orb}}(\alpha)$. Therefore $\overline{\text{orb}}(\alpha)$ is a linearly independent subset of $\mathcal{B}_\pm$. But

$$0 = \tilde{\pi}(\alpha) = \frac{1}{m} \sum_{i=0}^{m-1} \tilde{\sigma}^i(\alpha) = \sum_{\gamma \in \overline{\text{orb}}(\alpha)} \frac{1}{l(\alpha)} \gamma,$$

which is impossible. Hence $\overline{\text{orb}}(\tilde{\alpha}) = \overline{\text{orb}}(-\alpha)$.

(ii) Suppose $\tilde{\pi}(\alpha) \neq 0$. We only need to show that the equality $\tilde{\pi}(\alpha) = \tilde{\pi}(\beta)$ implies $\overline{\text{orb}}(\tilde{\alpha}) = \overline{\text{orb}}(\tilde{\beta})$. Suppose to the contrary that $\tilde{\pi}(\alpha) = \tilde{\pi}(\beta)$ but $\overline{\text{orb}}(\tilde{\alpha}) \cap \overline{\text{orb}}(\tilde{\beta}) = \emptyset$. If there exists $\gamma \in \overline{\text{orb}}(\alpha)$ such that $-\gamma \in \overline{\text{orb}}(\beta)$ then

$$\tilde{\pi}(\gamma) = \tilde{\pi}(\alpha) = \tilde{\pi}(\beta) = \tilde{\pi}(-\gamma),$$

and so $\tilde{\pi}(\alpha) = \tilde{\pi}(\beta) = 0$ which is absurd. Thus $\overline{\text{orb}}(\alpha)$ and $\overline{\text{orb}}(\beta)$ are two disjoint subsets of $\mathcal{B}_\pm$ with $\left(\pm \overline{\text{orb}}(\alpha)\right) \cap \overline{\text{orb}}(\beta) = \emptyset$. Therefore $\overline{\text{orb}}(\alpha) \cup \overline{\text{orb}}(\beta)$ is a linearly independent subset of $\mathcal{B}_\pm$. Now we have

$$\frac{1}{l(\alpha)} \sum_{\gamma \in \overline{\text{orb}}(\alpha)} \gamma - \frac{1}{l(\beta)} \sum_{\gamma' \in \overline{\text{orb}}(\beta)} \gamma' = \tilde{\pi}(\alpha) - \tilde{\pi}(\beta) = 0.$$

But this contradicts the fact that $\overline{\text{orb}}(\alpha) \cup \overline{\text{orb}}(\beta)$ is linearly independent. □
Note that since $\mathcal{B}$ is an orthogonal set, we have
\[(4.4) \quad (\alpha, \sum_{\gamma \in \text{orb}(\alpha)} \gamma) = (\alpha, \alpha) = 1, \quad \alpha \in \mathcal{B} \text{ with } \tilde{\pi}(\alpha) \neq 0.\]

Now let $x \in R^\times$, then $\tilde{x} \in \tilde{R} \setminus \{0\}$, so
\[\tilde{x} = \pm \alpha \quad \text{or} \quad \tilde{x} = \pm (\alpha \pm \beta), \quad (\alpha \neq \beta \in \mathcal{B}).\]
Thus
\[(4.5) \quad \tilde{\pi} (\tilde{R}) = \{ \pm \tilde{\pi}(\alpha), \pm (\tilde{\pi}(\alpha) \pm \tilde{\pi}(\beta)) \mid \alpha \neq \beta \in \mathcal{B}\}.

**Lemma 4.6.** $\pi(R)$ satisfies
\[(4.7) \quad (\tilde{\alpha}, \tilde{\beta}^\vee) \in \mathbb{Z} \text{ for all } \tilde{\alpha}, \tilde{\beta} \in \pi(R)^\times,\]
if and only if
\[(4.8) \quad \bar{l}(\alpha) = \bar{l}(\beta) \text{ for all } \alpha, \beta \in \mathcal{B} \text{ with } \tilde{\pi}(\alpha) \neq 0, \tilde{\pi}(\beta) \neq 0.

In particular, if (4.8) does not hold, then (LR3) is not satisfied for $\pi(R)$, and so $\pi(R)$ is not a LEARS.

**Proof.** Let $\alpha, \beta \in \mathcal{B}$. Without loss of generality, we may assume that $\tilde{\pi}(\alpha) \neq 0, \tilde{\pi}(\beta) \neq 0$ and $\tilde{\pi}(\alpha) \neq \pm \tilde{\pi}(\beta)$. From (2.4) and (4.4) we have
\[(4.9) \quad (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha)) = (\alpha, \tilde{\pi}(\alpha)) = \frac{1}{\bar{l}(\alpha)}(\alpha, \sum_{\gamma \in \text{orb}(\alpha)} \gamma) = \frac{1}{\bar{l}(\alpha)}(\alpha, \alpha) = \frac{1}{\bar{l}(\alpha)}.

Also since $\tilde{\pi}(\alpha) \neq \pm \tilde{\pi}(\beta)$, we have from Lemma 4.3 that $\overline{\text{orb}}(\alpha) \cap (\pm \overline{\text{orb}}(\beta)) = \emptyset$. In particular $(\tilde{\pi}(\alpha), \tilde{\pi}(\beta)) = 0$. This together with (4.9) now gives
\[(4.10) \quad (\tilde{\pi}(\alpha), \tilde{\pi}(\beta)) = 0 \quad \text{and} \quad (\tilde{\pi}(\alpha \pm \beta), \tilde{\pi}(\alpha \pm \beta)) = \frac{1}{\bar{l}(\alpha)} + \frac{1}{\bar{l}(\beta)}.

Therefore, if $\alpha, \beta, \gamma \in \mathcal{B}$ are such that $\pm \tilde{\pi}(\alpha), \pm \tilde{\pi}(\beta)$ and $\pm \tilde{\pi}(\gamma)$ are all non-zero and distinct, then it follows immediately from (4.9) and (4.10) that
\[(\tilde{\pi}(\alpha), \tilde{\pi}(\alpha \pm \beta)^\vee) = \frac{2\bar{l}(\beta)}{\bar{l}(\alpha) + \bar{l}(\beta)},
\[(4.11) \quad (\tilde{\pi}(\alpha \pm \beta), \tilde{\pi}(\alpha \pm \gamma)^\vee) = \frac{2\bar{l}(\gamma)}{\bar{l}(\alpha) + \bar{l}(\gamma)},
\[(\tilde{\pi}(\alpha + \beta), \tilde{\pi}(\alpha - \beta)^\vee) = \frac{2\bar{l}(\beta)-2\bar{l}(\alpha)}{\bar{l}(\beta)+\bar{l}(\alpha)}.

Now from (4.5) and (4.11) the claim in the statement follows. \qed

It is easy to construct automorphisms of $R$ satisfying (AR1)-(AR2), but not (4.8), see Example 7.1.
5. Isotropic orbits of type B automorphisms

We keep the same notations as in the previous section. Unlike the cases $A_1$ and $BC_1$, the situation here is much more complicated and the structure of $\pi(R)$ is much subtle to analyze. This is because in this case we might have non-isotropic roots which map, under $\pi$, to some isotropic elements. Therefore, in order to describe $\pi(R)$ in the form (1.7), one should consider such roots and their orbits (isotropic orbits). In this section we do this, and collect the isotropic elements of $\pi(R)$ in terms of their properties in three sets $S'$, $L'$ and $E'$, which eventually play the same roles as $S$, $L$ and $E$ for $\pi(R)$.

Considering Lemma 4.6, from now on we assume that the following three axioms hold:

- (AR1) $\sigma^m = 1$.
- (AR2) $\sigma(\alpha) \in R$ for all $\alpha \in R$.
- (AR3) $\bar{l}(\alpha) = \bar{l}(\beta)$ for all $\alpha, \beta \in \mathcal{B}$ with $\bar{\pi}(\alpha) \neq 0$, $\bar{\pi}(\beta) \neq 0$.

Set

$$\bar{l} := \bar{l}(\alpha), \quad \alpha \in \mathcal{B}, \quad \bar{\pi}(\alpha) \neq 0.$$  

**Lemma 5.1.** Suppose $\alpha, \beta \in \mathcal{B}_\pm$ with $\bar{\pi}(\alpha) = \bar{\pi}(\beta)$. Then

(i) $\bar{\pi}(\alpha) \neq 0 \implies \pi(\alpha - \beta + S) \subseteq \pi(S)$.

(ii) $\bar{\pi}(\alpha) = 0 \implies \pi(2\alpha + S) \subseteq \pi(S)$.

(iii) If $\bar{l} \geq 2$ and $\bar{\pi}(\alpha) \neq 0$, then $\pi(2\alpha) \in \pi(R_{\theta} + S)$.

**Proof.** (i) By Lemma 4.3(ii), $\text{orb}(\alpha) = \text{orb}(\beta)$ and so there exists $0 \leq k \leq m - 1$ such that $\bar{\sigma}^k(\alpha) = \beta$. Since $\sigma$ maps a root to a root of the same length, we have $\sigma^k(\alpha) = \beta + \delta$ for some $\delta \in S$. So $\pi(\alpha) = \pi(\beta + \delta)$ and

$$\sigma^k(\alpha + S) = \beta + \delta + \sigma^k(S) \quad \text{with} \quad \delta + \sigma^k(S) \subseteq S.$$  

Thus $\pi(\alpha - \beta + S) = \pi(\delta + S) \subseteq \pi(S)$.

(ii) By Lemma 4.3(i), we have $\text{orb}(\alpha) = \text{orb}(-\alpha)$. Now the same argument as in part (i) with $-\alpha$ in place of $\beta$ shows that there exists $\delta \in S$ such that $\pi(\alpha) = \pi(-\alpha + \delta)$ and $\pi(\delta + S) \subseteq \pi(S)$. Since $\delta \in S$, this gives $2\pi(\alpha) = \pi(\delta) \in \pi(S)$ and $\pi(2\alpha + S) = \pi(\delta + S) \subseteq \pi(S)$.

(iii) We have $\bar{\pi}(\alpha) \neq 0$ and so by Lemma 4.3(i), $-\alpha \notin \text{orb}(\alpha)$. Since $\bar{l} \geq 2$, there exists $\alpha' \in \text{orb}(\alpha)$ such that $\alpha' \neq \pm \alpha$ with $0 \neq \bar{\pi}(\alpha) = \bar{\pi}(\alpha')$. So by part (i), $\pi(\alpha) = \pi(\alpha' + \mu)$ for some $\mu \in S$. Therefore $\pi(2\alpha) = \pi(\alpha + \alpha' + \mu) \in \pi(R_{\theta} + S)$.

\[\square\]

Here we introduce some more notations which will be crucial for the rest of this work. Let $\text{orb}_0$ be the set of elements $\epsilon \in \mathcal{B}_\pm$ with $\bar{\pi}(\epsilon) = 0$. Since we have assumed $\pi(R)^\times \neq \emptyset$, there exists $\epsilon \in \mathcal{B}_\pm$ with $\bar{\pi}(\epsilon) \neq 0$. Let $\text{orb}_1$,
Since \((\bigcup_{\emptyset})\) is assumed to be empty.) Finally we set \(t\) to clarify the notation, we note that for a fixed \(\alpha\) of \(T\) of \(\alpha\)’s with \(\alpha_1\) (by assuming \(1 \in \mathfrak{S}\)). We have \(\bar{\pi}(\alpha_j) = \bar{\pi}(\epsilon) \neq 0\) for all \(\epsilon \in \overline{\operatorname{orb}}_{\mathfrak{S}}\).

Recall that \(\sigma^m = \text{id}\). Without loss of generality we may assume that \(m\) is the order of \(\sigma\). Let \(\alpha_0 \in \overline{\operatorname{orb}}_0\) and \(j \in \mathfrak{S}\). Set

\begin{equation}
\sigma^t(\alpha_j) := \alpha_j^t + \lambda_j^t; \quad 0 \leq t \leq m,
\end{equation}

where for \(0 \leq t \leq m\), \(\alpha_j^t \in \overline{\operatorname{orb}}_j\), \(\lambda_j^t \in S\) with

\begin{equation}
\lambda_j^0 := 0, \quad \lambda_j^m := 0 \quad \text{and} \quad \alpha_j^0 := \alpha_j.
\end{equation}

To clarify the notation, we note that for a fixed \(t\), the elements \(\lambda_j^t\) are fixed since \(\alpha_j\)’s are fixed for \(j \in \mathfrak{S}\). However the element \(\lambda_j^0\) depends on the choice of \(\alpha_0 \in \overline{\operatorname{orb}}_0\).

Set

\begin{align*}
S'_{\text{sh}} & \colon = \bigcup_{j \in \mathfrak{S}} \{ \pi(\epsilon - \alpha_j + \delta) \mid \epsilon \in \overline{\operatorname{orb}}_j, \delta \in S \} \\
& = \bigcup_{j \in \mathfrak{S}} \{ \pi(\lambda_j^t + S) \mid 0 \leq t \leq m \},
\end{align*}

and

\begin{align*}
S'_{\text{lg}} & \colon = \bigcup_{j \in \mathfrak{S}} \{ \pi(\epsilon - \alpha_j + \zeta + \lambda) \mid \epsilon \in \overline{\operatorname{orb}}_j, \zeta \in \overline{\operatorname{orb}}_0, \lambda \in L \} \\
& = \bigcup_{j \in \mathfrak{S}} \{ \pi(\overline{\operatorname{orb}}_0 + \lambda_j^t + L) \mid 0 \leq t \leq m \},
\end{align*}

and

\[ S' := S'_{\text{sh}} \cup S'_{\text{lg}}. \]

We note that \(\pi(S) \subseteq S'_{\text{sh}} \subseteq S'\). Also set

\[ L' := \bigcup_{i,j \in \mathfrak{S}} \{ \pi((\epsilon - \alpha_i) + (\epsilon' - \alpha_j) + \pi(\lambda)) \mid \epsilon \in \overline{\operatorname{orb}}_i, \epsilon' \in \overline{\operatorname{orb}}_j, \lambda \in L \} \]

where union runs over all \(i, j \in \mathfrak{S}\) with \(\overline{\operatorname{orb}}_i \cap (\pm \overline{\operatorname{orb}}_j) = \emptyset\). (Union over the empty set is assumed to be empty.) Finally we set

\[ E' := \bigcup_{j \in \mathfrak{S}} \{ \pi(\epsilon - \alpha_j) + \pi(\epsilon' - \alpha_j) + \pi(\lambda) \mid \epsilon \neq \epsilon' \in \overline{\operatorname{orb}}_j, \lambda \in L \}. \]

Since \(|\overline{\operatorname{orb}}_j| = \overline{l}\) for all \(j \in \mathfrak{S}\), we may write \(L'\) and \(E'\) according to the above notation in the forms

\begin{equation}
L' = \{ \pi(\lambda_j^t + \lambda_j^{t'}) \mid i, j \in \mathfrak{S} \text{ with } \overline{\operatorname{orb}}_i \cap (\pm \overline{\operatorname{orb}}_j) = \emptyset, 0 \leq t, t' < \overline{l} \} + \pi(L),
\end{equation}

and

\begin{equation}
E' = \{ \pi(\lambda_j^t + \lambda_j^{t'}) \mid j \in \mathfrak{S}, 0 \leq t \neq t' < \overline{l} \} + \pi(L).
\end{equation}
Remark 5.6. We note from definitions of $S'_i$, $L'$ and $E'$ that they might be empty sets, for example if $\text{orb}_0 = \emptyset$, then $S'_i = \emptyset$. Therefore when we discuss the properties of one of these sets, we conventionally assume that it is not an empty set. We also note that

\[(5.7) \quad L' \neq \emptyset \iff |\mathfrak{T}| \geq 4 \quad \text{and} \quad E' \neq \emptyset \iff \bar{l} \geq 2.\]

We also mention that if $L' \neq \emptyset$ and $\bar{l} \geq 2$, then we have $|\mathfrak{T}| \geq 4$ and so $L$ is a group.

Lemma 5.8. Let $\alpha_0 \in \overline{\text{orb}_0}$ and $\lambda'_j$'s be as above for $i, j \in \{0\} \cup \mathfrak{T}$ and $0 \leq t \leq m - 2$. Then we have

\[\pi(\lambda'_t + L) = \pi(\lambda'_t^{t+2} + L).\]

In particular, $\pi(\lambda'_t^{t} + L) = \pi(\lambda'_j + L)$ and

\[\pi(L) = \begin{cases} \pi(\lambda'_t^{2} + L) & \text{if } 0 \leq 2t \leq m \\ \pi(\lambda'_t^{t} + L) & \text{if } m \text{ is odd and } 0 \leq t \leq m. \end{cases}\]

Proof. First we note that the second claim in the statement is an immediate consequence of the first one as $\lambda'_0 = \lambda'_m = 0$.

Next, fix $j \in \{0\} \cup \mathfrak{T}$. Assume first that $\alpha'_t \neq \pm \alpha'_t$. Then we have

\[\sigma(\alpha'_j + \alpha'_t + L) = \alpha'_t + \lambda'_t + \sigma(\lambda'_j) + \alpha'_t^{t+2} + \lambda'_t^{t+2} - \sigma(\lambda'_t) + \sigma(L).\]

This implies that $\lambda'_t^{t+1} - \sigma(\lambda'_j) + \lambda'_t^{t+2} - \sigma(\lambda'_t) + \sigma(L) \subseteq L$. Applying $\pi$ to both sides of this inclusion, we obtain,

\[\pi(\lambda'_t^{t+2} - \lambda'_t^{t} + L) \subseteq \pi(L).\]

Thus

\[\pi(\lambda'_t + L) = \pi((\lambda'_t - \lambda'_t^{t+2}) + \lambda'_t^{t+2} + L) \subseteq \pi(\lambda'_t^{t+2} + L)\]

and

\[\pi(\lambda'_t^{t+2} + L) = \pi((\lambda'_t^{t+2} - \lambda'_j) + \lambda'_t^t + L) \subseteq \pi(\lambda'_t^t + L),\]

therefore we get (if $\alpha'_t \neq \pm \alpha'_t$),

\[(5.10) \quad \pi(\lambda'_t + L) = \pi(\lambda'_t^{t+2} + L); \quad 0 \leq t \leq m - 2, \quad j \in \{0\} \cup \mathfrak{T}.\]

We now show that (5.10) also holds if $\alpha'_t = \pm \alpha'_t$.

Assume first that $\alpha'_j = \alpha'_t^{t+1}$. It follows that $\alpha'_t^{t+2} = \alpha'_t + \lambda'_t$. Since $\sigma^{t}((\alpha'_j)) = \alpha'_t^{t} + \lambda'_t$, we have $\pi(\alpha'_j) = \pi(\alpha'_j + \lambda'_t)$. On the other hand $\sigma^{t+1}(\alpha'_j) = \alpha'_t^{t+1} + \lambda'_t^{t+1}$, which implies $\pi(\alpha'_j) = \pi(\alpha'_j^{t+1}) + \pi(\lambda'_t^{t+1})$. Therefore $\pi(\lambda'_j) = \pi(\lambda'_t^{t+1})$.

Repeating this argument starting from $\alpha'_t^{t+1} = \alpha'_t^{t+2}$, we get $\pi(\lambda'_t^{t+1}) = \ldots$
\(\pi(\lambda_j^{t+2})\), and so \(\pi(\lambda_j^t) = 0\) for all \(t\). Therefore (5.10) holds in this case too.

Next, suppose \(\alpha_j^t = -\alpha_j^{t+1}\). Then \(\pi(\alpha_j^t) = 0\) and so \(\alpha_j \in \overline{\text{orb}_0}\), implying that \(j = 0\). Now we have

\[
\alpha_0^{t+2} + \lambda_0^{t+2} = \sigma^{t+2}(\alpha_0) = \alpha_0^t - \lambda_0^{t+1} + \sigma(\lambda_0^t + \sigma(\lambda_0^{t+1})).
\]

Applying \(\pi\) on both sides gives, \(\pi(\lambda_0^{t+2}) = \pi(\lambda_0^t)\). Thus \(\pi(\lambda_0^t) = 0\) for all \(t\). This completes the proof that (5.10) holds in general.

We now complete the proof of lemma. Let \(i \neq j \in \{0\} \cup \mathcal{T}\). Then \(\alpha_i + \alpha_j + \lambda, \lambda \in L\), is a long root and so is its image \(\alpha_i + \alpha_j + \lambda_i + \lambda_j + \sigma'(\lambda)\) under \(\sigma'\). Thus for \(0 \leq t \leq m\), we have \(\pi(\lambda_i^t + \lambda_j^t + L) \subseteq \pi(L)\). This together with the fact that \(2S + L \subseteq L\) gives

\[
\pi(\lambda_i^t + L) = \pi(\lambda_i^t + \lambda_j^t - \lambda_j^t + L) \subseteq \pi(-\lambda_j^t + L) = \pi(\lambda_j^t + L).
\]

Replacing the roles of \(i\) and \(j\), we obtain

\[
\pi(\lambda_i^t + L) = \pi(\lambda_j^t + L), \quad i, j \in \{0\} \cup \mathcal{T}, \quad 0 \leq t \leq m.
\]

This now together with (5.10) completes the proof. \(\square\)

**Lemma 5.12.** If \(L' \neq \emptyset\), then \(L' = \pi(\lambda_i^t + L) \cup \pi(L)\) and if \(E' \neq \emptyset\) then

\[
E' = \begin{cases} 
\pi(\lambda_i^t + L) & \text{if } \bar{l} = 2 \\
\pi(\lambda_i^t + L) \cup \pi(L) & \text{if } \bar{l} \geq 3.
\end{cases}
\]

In particular if \(\bar{l} \geq 3\), then \(E' = L'\).

**Proof.** We note from (5.7) that \(L' \neq \emptyset\) only if \(|\mathcal{T}| \geq 4\), and \(E' \neq \emptyset\) only if \(\bar{l} \geq 2\). Now Let \(L' \neq \emptyset\), since \(\lambda_0^t = 0\) for all \(i \in \{0\} \cup \mathcal{T}\), we have \(\pi(L) \subseteq L'\). Since \(|\mathcal{T}| \geq 4\), there is \(i \in \mathcal{T}\) such that \(\overline{\text{orb}_i} \cap \pm \overline{\text{orb}_1} = \emptyset\). Then we have

\[
\pi(\lambda_i^t + L) = \pi(\lambda_i^t + \lambda_0^t + L) \subseteq L'.
\]

Thus \(\pi(\lambda_i^t + L) \cup \pi(L) \subseteq L'\). We now show the reverse inclusion. Let \(\eta = \pi(\lambda_i^t + \lambda_j^t + \lambda)\) for some \(0 \leq t, t' < \bar{l}, \lambda \in L\) and \(i, j \in \mathcal{T}\) with \(\overline{\text{orb}_i} \cap \pm \overline{\text{orb}_j} = \emptyset\). If \(t, t'\) are both even or both odd, then by Lemma 5.8 and the fact that \(2S + L \subseteq L\), we get \(\eta \in \pi(L)\) and so we are done. Therefore by symmetry we may assume that \(t\) is odd and \(t'\) is even. Now from Lemma 5.8 we have

\[
\eta \in \pi(\lambda_i^t + L) = \pi(\lambda_i^t + L).
\]

This completes the proof of the first statement.

For the second statement, note that with the same argument as before, for \(i \in \mathcal{T}\) and \(0 \leq t, t' \leq \bar{l}\), we have

\[
\pi(\lambda_i^t + \lambda_i^{t'} + L) = \begin{cases} 
\pi(L) & \text{if } t, t' \text{ are both even or both odd} \\
\pi(\lambda_i^t + L) & \text{otherwise}.
\end{cases}
\]
Lemma 5.13. (i) \( S'_{sh} = \pi(S) \).
(ii) \( S'_{tg} = \pi(\overline{orb}_0 + L) \), in particular, \( \pi(\overline{orb}_0 + \lambda'_j + L) = \pi(\overline{orb}_0 + L) \) for all \( j \in \{0\} \cup \mathfrak{T} \), \( 1 \leq t \leq m \).
(iii) \( S' \subseteq \pi(R)^0 \).

Proof. (i) Clearly we have \( \pi(S) \subseteq S'_{sh} \). Since for any \( j \in \mathfrak{T} \) and \( \epsilon \in \overline{orb}_j \), \( \pi(\epsilon) = \pi(\alpha_j) \), we have from Lemma 5.1 that \( S'_{sh} \subseteq \pi(S) \).

(ii) Let \( \zeta_0 := \alpha_0 + \alpha'_j - \alpha_j \), where \( j \in \mathfrak{T} \), \( 1 \leq t \leq \bar{l} \) and \( \alpha_0 \in \overline{orb}_0 \). We have \( \sigma(\alpha_0) = \alpha'_0 + \lambda'_0 \) and \( \sigma'(\alpha_j) = \alpha'_j + \lambda'_j \). Thus
\[
\pi(\zeta_0 + L) = \pi(\alpha_0 - \lambda'_j + L) = \pi(\alpha'_0 + \lambda'_0 - \lambda'_j + L).
\]
Now we conclude from this and Lemma 5.8 that either \( \pi(\zeta_0 + L) = \pi(\alpha_0 + L) \) (if \( \bar{l} \) is even), or \( \pi(\zeta_0 + L) = \pi(\alpha'_0 + L) \) (if \( \bar{l} \) is odd). We note that \( \pi(\alpha'_0) = \pi(\alpha_0) = 0 \) and so \( \alpha'_0 \in \overline{orb}_0 \). This proves that \( S'_{tg} \) is a subset of the right hand set. The reverse inclusion is immediate to see.

(iii) We have
\[
S' = S'_{sh} \cup S'_{tg} = \pi(S \cup (\overline{orb}_0 + L)) \subseteq \pi(S \cup (\overline{orb}_0 + S)) \subseteq \pi(R) \cap \mathcal{V}^0 = \pi(R)^0.
\]

6. Description of \( \pi(R) \) for type B

We keep the notations as in the two previous sections. We fully describe the structure of \( \pi(R) \) under an automorphism \( \sigma \) of \( R \) satisfying (AR1)-(AR3).

Proposition 6.1. (i) \( S' + L' \subseteq S' \), \( 2S' + L' \subseteq L' \), \( L' + E' \subseteq L' \), \( 2L' + E' \subseteq E' \), \( S' + E' \subseteq S' \) and \( 4S' + E' \subseteq E' \). Moreover 0 \( \in S' \) and 0 \( \in L' \). Also 0 \( \in E' \) if \( \bar{l} \geq 3 \).

(ii) \( S', L' \) and \( E' \) (if nonempty) are spanning subsets of \( \pi(\mathcal{V})^0 \), satisfying \( F \pm 2F \subseteq F \) for \( F \in \{S', L', E'\} \).

(iii) Suppose that \( L' \) is a group. Then \( L' \) is a group (if \( L' \neq \emptyset \)) and \( E' \) is a group if \( \bar{l} \geq 3 \). In particular if \( |\mathfrak{T}| \geq 6 \) and \( \bar{l} \geq 3 \), then \( L' \) and \( E' \) are groups.

Proof. (i) We first prove that \( S' + L' \subseteq S' \). To prove, we need to show \( S'_{sh} + L' \subseteq S' \) and \( S'_{tg} + L' \subseteq S' \) if \( S'_{tg} \neq \emptyset \). For the first one, using Lemma 5.13(i), we have \( S'_{sh} + \pi(L) = \pi(S + L) \subseteq \pi(S) \subseteq S' \). Also using Lemma 5.1, we have \( \pi(\lambda'_1 + S) \subseteq \pi(S) \) and so \( S'_{sh} + \pi(\lambda'_1 + L) = \pi(L + S + \lambda'_1) \subseteq \pi(L + S) \subseteq \pi(S) \subseteq S' \). These together with Lemma 5.12 show that \( S'_{sh} + L' \subseteq S' \). Now let \( S'_{tg} \neq \emptyset \), then since \( L' \) is a nonempty set, using Lemma 5.8, we have...
that $L$ is a group. In this case we have $S'_h + \pi(L) = \pi(\overline{orb}_0 + L + L) = \pi(\overline{orb}_0 + L) \subseteq S'_h$, also for any $\alpha_0 \in \overline{orb}_0$ we have

$$\pi(\alpha_0 + L) + \pi(\lambda^1 + L) = \pi(\alpha_0 + \lambda^1 + L + L) = \pi(\alpha_0 - \lambda^1 + L) = \pi(\alpha_0 + L) \subseteq S'_h$$

which together with Lemmas 5.13 and 5.12 complete the proof of $S' + L' \subseteq S'$.

Next we prove that $2S' + L' \subseteq L'$. Using Lemma 5.13, we have $2S'_sh + \pi(L) = \pi(2S + L) = \pi(L) \subseteq L'$ and $2S'_sh + \pi(\lambda^1 + L) = \pi(\lambda^1 + 2S + L) = \pi(\lambda^1 + L) \subseteq L'$. So using Lemma 5.12, it remains to prove that $2S'_h + L' \subseteq L'$. To prove this, by Lemmas 5.13 and 5.12, it is enough to show $2\pi(\alpha_0 + L) + \pi(\lambda^1 + L) \subseteq L'$ and $2\pi(\alpha_0 + L) + \pi(L) \subseteq L'$ for all $\alpha_0 \in \overline{orb}_0$. So let $\alpha_0 \in \overline{orb}_0$, then $\sigma'(\alpha_0) = -\alpha_0 + \lambda^r_0$ for some $1 \leq r \leq m$ and some $\lambda^r_0 \in S$. Then $\pi(2\alpha_0 + 2L) + \pi(L) = \pi(\lambda^r_0 + L)$. But using Lemma 5.8, we see that $\pi(\lambda^r_0 + L) = \pi(L) \subseteq L'$ if $r$ is even, and $\pi(\lambda^r_0 + L) = \pi(\lambda^1 + L) \subseteq L'$ if $r$ is odd. Similarly we have $2\pi(\alpha_0 + L) + \pi(\lambda^1 + L) = \pi(\lambda^1 + L + L)$. As above if $r$ is even, we have $\pi(\lambda^r_0 + \lambda^1 + L) = \pi(\lambda^1 + L) \subseteq L'$ and if $r$ is odd, we get $\pi(\lambda^r_0 + \lambda^1 + L) = \pi(2\lambda^1 + L) \subseteq \pi(2S + L) \subseteq \pi(L) \subseteq L'$. This completes the proof of $2S' + L' \subseteq L'$.

Now we prove that $E' + L' \subseteq L'$ and $E' + 2L' \subseteq E'$. Contemplating Remark 5.6, one gets that $L$ is a group. Thus we are done using Lemma 5.12 together with the facts that $\lambda^1_1 \subseteq S$ and $2S + L \subseteq L$.

It is clear from Lemma 5.12 and the arguments we used to show $S' + L' \subseteq S'$ and $2L' + E' \subseteq E'$ that $S' + E' \subseteq S'$ and $4S' + E' \subseteq E'$.

Finally since $S$ and $L$ contain zero, Lemmas 5.12 and 5.13 show that the last two claims in (i) hold.

(ii) Since $S$ spans $R^0$, $\pi(S)$ spans $\pi(R^0)$. Now as $\pi(S) \subseteq S'$ (Lemma 5.13), we get from Proposition 2.6(vi) that $S'$ spans $\pi(R)^0$. It now follows from this and part (i) that $L'$ and $E'$ (if non-empty) also span $\pi(R)^0$. We now consider the next claim. First we notice that to prove relations containing $L'$ or $E'$ in part (i), we used their equivalent expressions from Lemma 5.12. Since these expressions are non-empty, the relations in part (i) are always hold considering the mentioned expressions of $L'$ and $E'$. Now, keeping this in mind, the relation $F \pm 2F \subseteq F$ for $F \in \{S', L', E'\}$ easily follows from the relations in part(i) and the fact that $S'$ and $L'$ contain zero.

From part (i), it is now clear that $L'$ and $E'$ also span $\pi(R)^0$. The next claim in the statement also follows from part (i), if $L' \neq \emptyset$ and $E' \neq \emptyset$.

(iii) If $L$ is a group, then $\pi(\lambda^1 + L) \cup \pi(L)$ is clearly closed under addition and so $L'$ is a group. We also note that if $|\Sigma| \geq 6$ then $|\mathfrak{g}| \geq 3$ and so $L$ is a group. Now every thing is clear from Lemma 5.12.

To state our main theorem, we set

$$\mathfrak{K}_sh := \{\pi(\alpha_j) \mid j \in \mathfrak{g}\},$$
\hat{\mathcal{R}}_{tg} := \{ \pi(\alpha_i + \alpha_j) \mid i, j \in \Im, \pi(\alpha_i) \neq \pm \pi(\alpha_j) \},
\hat{\mathcal{R}}_{ex} := 2\hat{\mathcal{R}}_{sh},
\text{and}
\hat{\mathcal{R}} := \{0\} \cup \hat{\mathcal{R}}_{sh} \cup \hat{\mathcal{R}}_{tg} \cup \hat{\mathcal{R}}_{ex}.

**Theorem 6.2.** Let $R$ be a LEARS of type $B$ and $\sigma$ be an automorphism of $R$ satisfying (AR1)-(AR3). If $\pi(R)^{\times} \neq \emptyset$, then $\pi(R)$ is a LEARS of the form
\begin{equation}
\pi(R) = (S' + S') \cup (\hat{\mathcal{R}}_{sh} + S') \cup (\hat{\mathcal{R}}_{tg} + L') \cup (\hat{\mathcal{R}}_{ex} + E')
\end{equation}
where
\[S' = \pi(S) \cup \pi(\overline{\text{orb}}_{0} + L), \quad L' = \begin{cases} \pi(\lambda_1^1 + L) \cup \pi(L) & \text{if } |\Im| \geq 4 \\
\emptyset & \text{otherwise}, \end{cases}\]
and
\[E' = \begin{cases} \pi(\lambda_1^1 + L) & \text{if } \bar{l} = 2 \\
\pi(\lambda_1^1 + L) \cup \pi(L) & \text{if } \bar{l} \geq 3 \\
\emptyset & \text{otherwise}. \end{cases}\]
Moreover, $\pi(R)$ is of type
\[
\begin{cases} 
A_1 & \text{if } \bar{l} = 1, \quad |\Im| = 1 \\
B & \text{if } \bar{l} = 1, \quad |\Im| \geq 2 \\
BC & \text{otherwise.} 
\end{cases}
\]
Furthermore, if $L$ is a group and $L' \neq \emptyset$, then $L'$ is also a group and if $\bar{l} \geq 3$, then $E'$ is a group. Finally if $\bar{l} \geq 3$, then $\pi(R)$ is not reduced.

**Proof.** We have $R = R^0 \cup R_{sh} \cup R_{tg}$ with $R_{sh} := \hat{\mathcal{R}}_{sh} + S$ and $R_{tg} := \hat{\mathcal{R}}_{tg} + L$. We first show that $\pi(R)$ is a subset of the set on the right hand side of equation (6.3), where we denote the right hand side by $\Psi$. We know that $\pi(S) \subseteq S'$ and so $\pi(R^0) = \pi(S + S) \subseteq S' + S' \subseteq \Psi$. Next we show $\pi(\hat{\mathcal{R}}_{sh} + S) \subseteq \Psi$. Let $\epsilon \in \mathfrak{B}_\pm$ and $\delta \in S$. If $\bar{\pi}(\epsilon) = 0$, then
\begin{equation}
\pi(\epsilon) = \pi(\alpha_1) - \pi(\alpha_1) + \pi(\epsilon) \in S'_{tg} \subseteq S'
\end{equation}
and so $\pi(\epsilon + \delta) \in S' + S' \subseteq \Psi$. If $\bar{\pi}(\epsilon) \neq 0$, then $\epsilon \in \overline{\text{orb}}_{j}$ for some $j \in \Im$. Then
\[\pi(\epsilon + \delta) = \pi(\alpha_j) + (\pi(\epsilon) - \pi(\alpha_j)) + \pi(\delta) \in \pi(\alpha_j) + S'_{sh} \subseteq \hat{\mathcal{R}}_{sh} + S' \subseteq \Psi.
\]
Next we show $\pi(R_{tg}) \subseteq \Psi$. Let $\epsilon, \epsilon' \in \mathfrak{B}_\pm$ with $\epsilon \neq \pm \epsilon'$ and let $\lambda \in L$. We must show $\pi(\epsilon + \epsilon' + \lambda) \in \Psi$. By symmetry it is enough to consider the following possibilities:

1. $\bar{\pi}(\epsilon) = \bar{\pi}(\epsilon') = 0$,
2. $\bar{\pi}(\epsilon) \neq 0, \bar{\pi}(\epsilon') = 0$,
(3) \( \bar{T}(\epsilon) \neq 0, \bar{T}(\epsilon') \neq 0, \bar{T}(\epsilon) \neq \pm \bar{T}(\epsilon') \).

(4) \( \bar{T}(\epsilon) \neq 0, \bar{T}(\epsilon') \neq 0, \bar{T}(\epsilon) = \pm \bar{T}(\epsilon') \).

(1) By Lemma 5.13, \( \pi(\zeta + L) \subseteq S_{tg}' \) for any \( \zeta \in \mathfrak{B}_+ \) with \( \bar{T}(\zeta) = 0 \). Thus

\[
\pi(\epsilon + \epsilon' + \lambda) \in S_{tg}' + S_{tg}' \subseteq S' + S' \subseteq \Psi.
\]

(2) By assumption \( \epsilon \in \overline{\text{orb}}_j \) for some \( j \in \mathfrak{T} \). Then

\[
\pi(\epsilon + \epsilon' + \lambda) = \pi(\zeta_j + (\pi(\epsilon) - \pi(\zeta_j)) + \pi(\epsilon') + \pi(\lambda)) = \pi(\alpha_j) + S_{tg}' \subseteq \Psi.
\]

(3) By assumption \( \epsilon \in \overline{\text{orb}}_i \) and \( \epsilon' \in \overline{\text{orb}}_j \) for some \( i, j \in \mathfrak{T} \) with \( \overline{\text{orb}}_i \cap (\pm \overline{\text{orb}}_j) = \emptyset \). Then

\[
\pi(\epsilon + \epsilon' + \lambda) = \pi(\alpha_i + \alpha_j + \epsilon - \alpha_i + \epsilon' - \alpha_j + \lambda) \subseteq \pi(\alpha_i + \alpha_j + \lambda') \subseteq \Psi.
\]

(4) We first suppose \( \bar{T}(\epsilon) = \bar{T}(\epsilon') \), then by Lemma 4.3(ii), \( \epsilon \) and \( \epsilon' \) belong to the same orbit of \( \bar{\sigma} \), say \( \overline{\text{orb}}_j \). Then since \( \epsilon \neq \epsilon' \), we have

\[
\pi(\epsilon + \epsilon' + \lambda) = 2\pi(\alpha_j) + \pi(\epsilon - \alpha_j) + \pi(\epsilon' - \alpha_j) + \pi(\lambda) \subseteq 2\pi(\alpha_j) + \pi(\lambda) + E' \subseteq \Psi.
\]

Next we suppose \( \bar{T}(\epsilon) = -\bar{T}(\epsilon') \), then \( 0 \neq \bar{T}(\epsilon) = \bar{T}(\epsilon') \) and so by Lemma 5.1(i), we have

\[
\pi(\epsilon + \epsilon' + \lambda) \subseteq \pi(\epsilon + \epsilon' + L) \subseteq \pi(\epsilon + \epsilon' + S) \subseteq \pi(S) \subseteq S' \subseteq \Psi.
\]

This completes the proof of \( \pi(R) \subseteq \Psi \).

Now we show the reverse inclusion. First we show that \( S' + S' \subseteq \pi(R) \).

Using Lemma 5.13(i), we have \( S_{sh}' \subseteq S_{tg}' \subseteq \pi(S + S) \subseteq \pi(R) \). To see

\[
S_{tg}' + S_{tg}' \subseteq \pi(R), \quad \eta, \eta' \subseteq S_{tg}'.
\]

By Lemma 5.13, \( \eta = \pi(\zeta + \lambda) \) and \( \eta' = \pi(\zeta' + \lambda') \) for some \( \zeta, \zeta' \in \overline{\text{orb}}_0 \) and \( \lambda, \lambda' \in L \). If \( \zeta = -\zeta' \), then \( \eta + \eta' \in \pi(L + L) \subseteq \pi(S) \subseteq \pi(R) \).

Thus we have

\[
\eta + \eta' = \pi(\zeta + \zeta' + \lambda + \lambda') \subseteq \pi(\zeta + \lambda) \subseteq \pi(S) \subseteq \pi(R).
\]

So we may assume that \( \zeta \neq \pm \zeta' \). Therefore \( |\overline{\text{orb}}_0| \geq 3 \). Since \( \overline{\text{orb}}_j \neq \emptyset \) for some \( j \in \mathfrak{T} \), we conclude that we are in the case \( |\mathfrak{T}| \geq 3 \), and so \( L \) is a group.

Thus we have

\[
\eta + \eta' \subseteq \pi(\zeta + \zeta' + L) \subseteq \pi(R_{tg}) \subseteq \pi(R).
\]

To see \( S_{tg}' + S_{sh}' \subseteq \pi(R) \), let \( \eta \in S_{tg}' \). Then by Lemma 5.13, \( S_{sh}' = \pi(S) \) and \( \eta \in \pi(\zeta + L) \), then using the fact that \( S + L \subseteq S \), we have

\[
\eta + S_{sh}' \subseteq \pi(\zeta + L + S) \subseteq \pi(\zeta + S) \subseteq \pi(R_{sh}) \subseteq \pi(R).
\]

This completes the proof of \( S' + S' \subseteq \pi(R) \).

Next for fix \( j \in \mathfrak{T} \) we show that \( \pi(\alpha_j) + S' = \pi(\alpha_j) + (S_{sh}' \cup S_{tg}') \subseteq \pi(R) \).

Using Lemma 5.13, we have \( \pi(\alpha_j) + S_{sh}' = \pi(\alpha_j + S) \subseteq \pi(R_{sh}). \) Also if
\( \zeta \in \text{orb}_0 \), then \( \pi(\alpha_j) + \pi(\zeta + L) = \pi(\alpha_j + \zeta + L) \subseteq \pi(R_{1g}) \) which by Lemma 5.13 means that \( \pi(\alpha_j) + S'_{1g} \subseteq \pi(R_{1g}) \subseteq \pi(R) \).

Now we fix \( i, j \in \mathfrak{X} \) with \( \text{orb}_i \cap (\pm \text{orb}_j) = \emptyset \) and show that \( \pi(\alpha_i + \alpha_j) + L' \subseteq \pi(R) \). By Lemma 5.12, \( L' = \pi(\lambda_i^1 + L) \cup \pi(L) \). Clearly, we have \( \pi(\alpha_i + \alpha_j + L) \subseteq \pi(R_{ig}) \). Now from (5.2) we have \( \sigma(\alpha_j) = \alpha_j^1 + \lambda_j^1 \) and so \( \pi(\alpha_j) = \pi(\alpha_j^1 + \lambda_j^1) \). Therefore, using Lemma 5.12 together with the fact that \( 2S + L \subseteq L \), we have

\[
\pi(\alpha_i + \alpha_j + \lambda_j^1 + L) = \pi(\alpha_i + \alpha_j^1 + \lambda_j^1 + \lambda_i^1 + L) = \pi(\alpha_i + \alpha_j^1 + L) \subseteq \pi(R).
\]

Finally we show that for any fix \( j \in \mathfrak{X} \), \( 2\pi(\alpha_j) + E' \subseteq \pi(R) \), where we use the description of \( E' \) given by Lemma 5.12. We know that \( \sigma(\alpha_j) = \alpha_j^1 + \lambda_j^1 \) which implies that \( \pi(\alpha_j) = \pi(\alpha_j^1 + \lambda_j^1) \). Note that as we have already assumed that \( E' \neq \emptyset \), by (5.7), \( \ell \geq 2 \). This together with Lemma 4.3(i) implies that \( \alpha_j \neq \pm \alpha_j^1 \) and so by Lemma 5.8 we have

\[
\pi(2\alpha_j + \lambda_j^1 + L) = \pi(2\alpha_j + \lambda_j^1 + L) \subseteq \pi(\alpha_j + \alpha_j^1 + 2\lambda_j^1 + L) \subseteq \pi(R_{1g} + L) \subseteq \pi(R).
\]

Using Lemma 5.12, the proof of this part will be completed if we show that in the case \( \tilde{l} \geq 3 \), \( \pi(2\alpha_j + L) \subseteq \pi(R) \), \( j \in \mathfrak{X} \). Fix \( j \in \mathfrak{X} \), since \( \tilde{l} \geq 3 \), \( \sigma^2(\alpha_j) = \alpha_j^2 + \lambda_j^2 \) with \( \alpha_j \neq \pm \alpha_j^2 \) and so using Lemma 5.8 we have

\[
\pi(2\alpha_j + L) = \pi(\alpha_j + \alpha_j^2 - \lambda_j^2 + L) = \pi(\alpha_j + \alpha_j^2 + L) \subseteq \pi(R).
\]

The remaining parts of the statement follow from (1.7), Proposition 6.1, Remark 5.6 and Lemmas 5.13 and 5.12. \( \square \)

We close this section with an application of Theorem 6.2 for affinization of extended affine Lie algebras. We recall from Section 3 that if \( \sigma \) is an automorphism of an extended affine Lie algebra \((\mathfrak{L}, \langle \cdot, \cdot \rangle, \mathfrak{h})\) with root system \( R \) satisfying (A1)-(A4), then \( \sigma \) induces an automorphism of \( \mathcal{V} \), the \( \mathbb{Q} \)-span of \( R \), denoted again by \( \sigma \). Also for \( \alpha \in \mathcal{V} \), \( \pi(\alpha) \) is defined to be \( (1/m) \sum_{i=0}^{m-1} \sigma^i(\alpha) \) and \( \overline{\pi(\alpha)} \) is defined to be the image of \( \pi(\alpha) \) under the canonical projection map from \( \mathcal{V} \) to the quotient space \( \mathcal{V} \) over \( \mathcal{V}^0 \), the radical of the induced form on \( \mathcal{V} \).

**Corollary 6.5.** Starting from an extended affine Lie algebra \( \mathfrak{L} \) of type \( A_1 \), \( BC_1 \) or \( B \) and an automorphism \( \sigma \) of \( \mathfrak{L} \) satisfying conditions (A1)-(A4) (see Section 3), the possible affinizations we might get are of types given in the following table and no other type arises this way:

\[
\begin{array}{c|c|c|c}
\mathfrak{L} & A_1 & BC_1 & B \\
\text{Aff(\mathfrak{L})} & A_1 & BC_1 & A_1, B \text{ or } BC \\
\end{array}
\]
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Proof. Let \( \mathfrak{L} \) and \( \sigma \) be as in the statement. Let \( R \) be the root system of \( \mathfrak{L} \) and \( \pi(R)^{\times} \neq \emptyset \). According to [ABP, Thm. 3.63], \( \text{Aff}(\mathfrak{L}) \) is an extended affine Lie algebra. By [ABP, (3.48)], the type of \( \text{Aff}(\mathfrak{L}) \) is the same as the type of the finite root system \( \pi(R) \) (see Theorem 3.4). The result now follows from Propositions 2.7 and Theorem 6.2. \( \square \)

7. Examples

In this section we give several examples of automorphisms of locally extended affine root systems of type \( B \). We start with one which satisfies (AR1)-(AR2) but not (AR3), and we go on with constructing automorphisms which satisfy (AR1)-(AR3). Then we apply Theorem 6.2 to describe \( \pi(R) \).

Let \( R = (S + S) \cup (\hat{R}_{sh} + S) \cup (\hat{R}_{lg} + L) \) be a LEARS of type \( B \) in \( V = \hat{V} \oplus V^0 \) as in Section 1 where \( \hat{R}_{sh} = \{ \pm \epsilon_i \mid i \in J \} \), \( \hat{R}_{lg} = \{ \pm (\epsilon_i \pm \epsilon_j) \mid i \neq j \in J \} \), and \( S \) and \( L \) are subsets \( V^0 \) satisfying (4.1). We call \( |J| \) the rank of \( R \) if \( |J| < \infty \). Let \( \Lambda = \langle S \rangle \).

We always assume that \( \Lambda \) is a free abelian group with a basis \( \{ \delta_i \mid i \in I \} \). If \( |I| < \infty \), we call \( \nu := |I| \) the nullity of \( R \). (The notions of rank and nullity for other types are defined in a similar manner (see [Yos]).) If \( |J| = n < \infty \), we assume \( J = \{1, \ldots, n\} \).

In each example we first specify rank (if \( J \) is finite), \( \nu \), \( S \) and \( L \). Then we construct an automorphism \( \sigma \in \text{GL}(V) \) satisfying \( \sigma(R) = R \) by specifying \( \sigma \) on \( \epsilon_i \)'s and \( \delta_i \)'s. Next we specify \( \bar{l} \). We recall that \( \mathfrak{B}_\pm = \{ \pm \epsilon_i \mid i \in J \} \) and \( |\mathfrak{T}| \) is the cardinal of distinct orbits of elements \( \epsilon \in \mathfrak{B}_\pm \) with \( \pi(\epsilon) \neq 0 \).

Example 7.1. Let \( |J| \geq 3 \) and fix \( i \neq j \in J \). Define \( \sigma(\epsilon_i) = \epsilon_j \), \( \sigma(\epsilon_j) = \epsilon_i \) and \( \sigma(\epsilon_k) = \epsilon_k \) for all \( k \neq i, j \). Also define \( \sigma \) as identity on \( V^0 \). Then \( \sigma \) is an automorphism of a locally finite root system satisfying (AR1)-(AR2), but not (AR3).

Example 7.2. Suppose that \( \delta \) is a fixed element of \( \Lambda \).

\[
\begin{array}{|c|c|c|}
\hline
R & \text{rank} & \text{nullity} \\
\hline
6 & \nu & \Lambda \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\sigma & \epsilon_1 \mapsto \epsilon_2 + \delta, \ \epsilon_2 \mapsto \epsilon_1 - \delta, \ \epsilon_3 \mapsto \epsilon_4 + \delta, \ \epsilon_4 \mapsto \epsilon_3 - \delta, \ \epsilon_5 \mapsto \epsilon_6 + \delta, \ \epsilon_6 \mapsto \epsilon_5 - \delta, \ \lambda \mapsto \lambda \ \lambda \in \Lambda. \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{order of } \sigma & l \mid \mathfrak{T} \\
\hline
2 & 2 \ 6 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\pi(R) & \text{type} & \text{rank} & \text{nullity} & S' & L' & E' \\
\hline
BC & 3 & \nu & \Lambda & 2\Lambda \cup (\delta + 2\Lambda) & \delta + 2\Lambda \\
\hline
\end{array}
\]
Details:

\[
\pi(e_{2i}) = \frac{1}{2}(e_{2i} + e_{2i-1} - \delta), \quad \pi(e_{2i-1}) = \frac{1}{2}(e_{2i-1} + e_{2i} + \delta); \quad 1 \leq i \leq 3,
\]

\[
\pi(\lambda) = \lambda; \quad \lambda \in \Lambda,
\]

\[
\text{orb}_0 = \emptyset, \quad \text{orb}_1 = \{e_1, e_2\}, \quad \text{orb}_2 = \{-e_1, -e_2\}, \quad \text{orb}_3 = \{e_3, e_4\},
\]

\[
\text{orb}_4 = \{-e_3, -e_4\}, \quad \text{orb}_5 = \{e_5, e_6\}, \quad \text{orb}_6 = \{-e_5, -e_6\},
\]

\[
\tilde{\mathcal{R}} := \{\pm \pi(e_{2i-1}), \pm (\pi(e_{2i-1}) \pm \pi(e_{2j-1})) \mid 1 \leq i, j \leq 3\},
\]

\[
\pi(R) \text{ is not reduced } \iff \delta \in 2\Lambda.
\]

Example 7.3. Let \( \Lambda = \mathbb{Z} \delta_1 \oplus \mathbb{Z} \delta_2 \) and \( \delta = \delta_1 + 2\delta_2 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
R &: & \text{rank} & \text{nullity} & S & L \\
\hline
& 6 & 2 & 2\Lambda \cup (\delta_1 + 2\Lambda) \cup (\delta_2 + 2\Lambda) & 2\Lambda \\
\hline
\end{array}
\]

\[
\sigma : e_1 \mapsto e_2 + 2\delta, \quad e_2 \mapsto e_3 - 2\delta_1, \quad e_3 \mapsto e_4 + 2\delta, \quad e_4 \mapsto e_1 - 2\delta_1,
\]

\[
e_5 \mapsto -e_6 + 2\delta, \quad e_6 \mapsto e_5 + 2\delta_1, \quad \delta_1 \mapsto \delta, \quad \delta_2 \mapsto -\delta_2.
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{order of } \sigma & l & |\Sigma| \\
\hline
4 & 4 & 2 \\
\hline
\end{array}
\]

\[
\pi(R) :
\begin{array}{|c|c|c|c|c|}
\hline
\text{type} & \text{rank} & \text{nullity} & S' & L' & E' \\
\hline
BC & 1 & 1 & (\delta_1 + 2\delta_2) \cup 2\Lambda & \emptyset & 2\Lambda \\
\hline
\end{array}
\]

where \( \hat{\Lambda} = \pi(\Lambda) = \mathbb{Z}(\delta_1 + \delta_2)/2 \).

Details:

\[
\pi(e_3) = \pi(e_1) = \frac{1}{4}(e_1 + e_2 + e_3 + e_4) + \delta,
\]

\[
\pi(e_4) = \pi(e_2) = \frac{1}{4}(e_1 + e_2 + e_3 + e_4) + \delta_1,
\]

\[
\pi(e_5) = 0, \quad \pi(e_6) = \delta_1 + \delta_2, \quad \pi(\delta) = \pi(\delta_1) = (\delta_1 + \delta_2)/2, \quad \pi(\delta_2) = 0,
\]

\[
\text{orb}_0 = \{\pm e_5, \pm e_6\}, \quad \text{orb}_1 = \{e_1, e_2, e_3, e_4\}, \quad \text{orb}_2 = \{-e_1, -e_2, -e_3, -e_4\},
\]

\[
\tilde{\mathcal{R}} := \{\pm \pi(e_1), \pm 2\pi(e_1)\},
\]

\[
S'_{sh} = \pi(S) \quad \text{and} \quad S'_{lg} = \pi(\text{orb}_0 + L) = \pi(L),
\]

\[
\pi(R) \text{ is not reduced.}
\]

Example 7.4. Suppose that \( 3 \leq \nu < \infty \) and \( p, q \) are two positive integers.

Let \( \delta_0 = 0 \). Fix \( 1 \leq i \leq \nu \), \( 0 \leq j < \nu \). Set \( \delta := 2\delta_i \) and \( \lambda := \delta + 4\delta_j \). Also for \( 1 \leq t \leq q \), fix \( \mu_t \in \{2\delta_s \mid 1 \leq s \leq \nu, s \neq i, j\} \) and set \( \lambda_t := \delta + \mu_t \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
R &: & \text{rank} & \text{nullity} & S & L \\
\hline
& 2p + q & \nu & \cup_{t=0}^{\nu} (\delta_1 + 2\Lambda) & 2\Lambda \\
\hline
\end{array}
\]
\[ \sigma : \begin{array}{ccc}
\delta_i &\mapsto & \delta_i + 2\delta_j, \\
\delta_j &\mapsto & -\delta_j, \\
\delta_s &\mapsto & \delta_s - 2\delta_j,
\end{array} \]
for \( 1 \leq r \leq p, 1 \leq t \leq q, 1 \leq s \leq \nu, s \neq i, j. \)

| order of \( \sigma \) | \( l \) | \( |\Sigma| \) |
|-----------------|-----|-----|
| 2               | 2p  |      |

\[ \pi(R) : \]

<table>
<thead>
<tr>
<th>type</th>
<th>rank</th>
<th>nullity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BC )</td>
<td>( p = 1 )</td>
<td>( \nu - 1 )</td>
</tr>
<tr>
<td>( BC )</td>
<td>( p &gt; 1 )</td>
<td>( \nu - 1 )</td>
</tr>
</tbody>
</table>

where

\[ \tilde{\Lambda} = \pi(\Lambda) = \mathbb{Z}(\delta_i + \delta_j) \oplus \bigoplus_{i,j \neq s = 1}^{\nu} \mathbb{Z}(\delta_s - \delta_j), \]

\[ \pi(S) = 2\tilde{\Lambda} \cup (\delta_i + \delta_j + 2\tilde{\Lambda}) \cup \bigcup_{i,j \neq s = 1}^{\nu} (\delta_s - \delta_j + 2\tilde{\Lambda}). \]

Details:

\[ \pi(\epsilon_r) = \frac{1}{2}(\epsilon_r + \epsilon_{p+r} + \delta), \quad \pi(\epsilon_{p+r}) = \frac{1}{2}(\epsilon_{p+r} + \epsilon_r - \lambda); \quad 1 \leq r \leq p, \]

\[ \pi(\epsilon_{2p+t}) = \frac{1}{2}\lambda_t; \quad 1 \leq t \leq q, \]

\[ \pi(\delta_j) = 0, \quad \pi(\delta_i) = \delta_i + \delta_j, \quad \pi(\delta_s) = \delta_s - \delta_j; \quad 1 \leq s \leq \nu, s \neq i, j, \]

\[ \overline{\text{orb}}_0 = \{ \pm \epsilon_{2p+t} \mid 1 \leq t \leq q \}, \]

\[ \overline{\text{orb}}_r = \{ \epsilon_r, \epsilon_{p+r} \}, \quad \overline{\text{orb}}_{p+r} = \{ -\epsilon_r, -\epsilon_{p+r} \}; \quad 1 \leq r \leq p, \]

\[ \hat{\mathcal{R}} := \{ \pm \pi(\epsilon_r), \pm(\pi(\epsilon_r) \pm \pi(\epsilon_s)) \mid 1 \leq r, s \leq p \}, \]

\[ \pi(R) \text{ is not reduced.} \]

**Example 7.5.** Suppose that \( \nu < \infty \) and \( p \) is a positive integer. Fix \( 1 \leq i \leq \nu, 0 \leq j \leq \nu \) and set \( \delta := 2\delta_i \) and \( \lambda := \delta + 4\delta_j. \)

<table>
<thead>
<tr>
<th>rank</th>
<th>nullity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2p )</td>
<td>( \nu )</td>
</tr>
</tbody>
</table>

\[ \pi(S) = \bigcup_{i=0}^{\nu}(\delta_i + 2\Lambda) \]

\[ \sigma : \begin{array}{ccc}
\epsilon_r &\mapsto & \epsilon_{p+r} + \delta, \\
\epsilon_{p+r} &\mapsto & \epsilon_r - \lambda, \\
\delta_i &\mapsto & \delta_i + 2\delta_j, \\
\delta_j &\mapsto & -\delta_j, \\
\delta_s &\mapsto & \delta_s,
\end{array} \]
for \( 1 \leq r \leq p, 1 \leq s \leq \nu, s \neq i, j. \)

| order of \( \sigma \) | \( l \) | \( |\Sigma| \) |
|-----------------|-----|-----|
| 2               | 2p  |      |

\[ \pi(R) : \]

<table>
<thead>
<tr>
<th>type</th>
<th>rank</th>
<th>nullity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BC )</td>
<td>( p = 1 )</td>
<td>( \nu - 1 )</td>
</tr>
<tr>
<td>( BC )</td>
<td>( p &gt; 1 )</td>
<td>( \nu - 1 )</td>
</tr>
</tbody>
</table>

where

\[ \pi(S) = 2\Lambda \]

\[ \pi(S) = 2\Lambda \]
where
\[ \tilde{\Lambda} = \pi(\Lambda) = \mathbb{Z}(\delta_i + \delta_j) \oplus \bigoplus_{i,j \neq s=1} \mathbb{Z}\delta_s, \]
\[ \pi(S) = 2\tilde{\Lambda} \cup (\delta_i + \delta_j + 2\tilde{\Lambda}) \cup \bigcup_{i,j \neq s=1} (\delta_s + 2\tilde{\Lambda}) \]

Details:
\[ \pi(\epsilon_r) = 12(\epsilon_r + \epsilon_{p+r} + \delta), \quad \pi(\epsilon_{p+r}) = 12(\epsilon_{p+r} + \epsilon_r - \lambda); \quad 1 \leq r \leq p, \]
\[ \pi(\delta_i) = \delta_i + \delta_j, \quad \pi(\delta_j) = 0, \quad \pi(\delta_s) = \delta_s; \quad 1 \leq s \leq \nu, \; s \neq i, j, \]
\[ \overline{\text{orb}}_0 = \emptyset, \quad \overline{\text{orb}}_r = \{\epsilon_r, \epsilon_{p+r}\}, \quad \overline{\text{orb}}_{p+r} = \{-\epsilon_r, -\epsilon_{p+r}\}; \quad 1 \leq r \leq p, \]
\[ \hat{R} := \{\pm \pi(\epsilon_r), \pm (\pi(\epsilon_r) \pm \pi(\epsilon_s)) \mid 1 \leq r, s \leq p\}, \]
\[ \pi(R) \text{ is not reduced.} \]

**Example 7.6.** Suppose that \( m \) is a nonnegative integer and \( \nu = m + 4 \). Set \( \delta := 2\delta_{m+1}, \theta := 2\delta_{m+2}, \mu := 2\delta_{m+3}, \lambda := 2\delta_{m+4} \).

\[
R:
\begin{array}{|c|c|c|c|}
\hline
\text{rank} & \text{nullity} & S & L \\
\hline
4 & m+4 & \bigcup_{i=0}^{m+4} (\delta_i + 2\Lambda) & 2\Lambda \\
\hline
\end{array}
\]

\[
\begin{align*}
\epsilon_1 & \mapsto \epsilon_2 + \delta, \quad \epsilon_2 & \mapsto \epsilon_1 + \theta, \quad \epsilon_3 & \mapsto \epsilon_4 + \delta, \quad \epsilon_4 & \mapsto \epsilon_3 + \theta, \\
\sigma: \quad \delta & \mapsto \mu, \quad \mu & \mapsto -\delta, \quad \theta & \mapsto \lambda, \quad \lambda & \mapsto -\theta, \quad \delta_i & \mapsto \delta_i \quad \text{if } 0 \leq i \leq m.
\end{align*}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{order of } \sigma & l & |\Sigma| \\
\hline
4 & 2 & 4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{type} & \text{rank} & \text{nullity} & S' & L' & E' \\
\hline
BC & 2 & m & \bigcup_{j=0}^{m} (\delta_j + \sum_{0 \leq i \leq m} 2\mathbb{Z}\delta_i) & \sum_{0 \leq i \leq m} 2\mathbb{Z}\delta_i & \sum_{0 \leq i \leq m} 2\mathbb{Z}\delta_i \\
\hline
\end{array}
\]

Details:
\[ \pi(\epsilon_2) = \pi(\epsilon_1) = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{4}(\delta + \lambda + \theta + \mu), \]
\[ \pi(\epsilon_4) = \pi(\epsilon_3) = \frac{1}{2}(\epsilon_3 + \epsilon_4) + \frac{1}{4}(\delta + \lambda + \theta + \mu), \]
\[ \pi(\delta) = \pi(\mu) = \pi(\theta) = \pi(\lambda) = 0, \quad \pi(\delta_i) = \delta_i; \quad 0 \leq i \leq m, \]
\[ \overline{\text{orb}}_0 = \emptyset, \]
\[ \overline{\text{orb}}_1 = \{\epsilon_1, \epsilon_2\}, \quad \overline{\text{orb}}_2 = \{-\epsilon_1, -\epsilon_2\}, \quad \overline{\text{orb}}_3 = \{\epsilon_3, \epsilon_4\}, \quad \overline{\text{orb}}_4 = \{-\epsilon_3, -\epsilon_4\}, \]
\[ S' = \pi(S), \quad L' = \pi(\delta + L) \cup \pi(L) = \pi(L), \quad E' = \pi(\delta + L) = \pi(L), \]
\[ \hat{R} := \{\pm \pi(\epsilon_1), \pm \pi(\epsilon_3), \pm (\pi(\epsilon_1) \pm \pi(\epsilon_3)), \pm 2\pi(\epsilon_1), \pm 2\pi(\epsilon_3)\}, \]
\[ \pi(R) \text{ is not reduced.} \]
Example 7.7. Suppose that \( m \) is a nonnegative integer and \( \nu = m + 3 \). Set \( \delta := 2\delta_{m+1}, \lambda := 2\delta_{m+2}, \mu = 2\delta_{m+3} \).

\[
\begin{array}{|c|c|c|}
\hline
& \text{rank} & \text{nullity} \\
\hline R & 3 & m + 3 \\
\hline S & \cup_{i=0}^{m+3} (\delta_i + 2\Lambda) & 2\Lambda \\
\hline L & & \\
\hline
\end{array}
\]

\[
\sigma : \begin{array}{rcccc}
\epsilon_1 & \mapsto & \epsilon_2 - \delta, & \epsilon_2 & \mapsto & \epsilon_3, & \epsilon_3 & \mapsto & \epsilon_1 + \mu, & \delta_i & \mapsto & \delta_i; 0 \leq i \leq m. \\
\delta & \mapsto & \lambda, & \lambda & \mapsto & \mu, & \mu & \mapsto & \delta, & \delta_i & \mapsto & \delta_i; 0 \leq i \leq m.
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{order of } \sigma & l & |\Xi| \\
\hline 3 & 3 & 2 \\
\hline
\end{array}
\]

where \( \hat{\Lambda} = \Lambda(\delta + \lambda + \mu) + \sum_{i=0}^{m} 2\Xi \delta_i \).

Details:

\[
\pi(\epsilon_1) = \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta - \lambda), \quad \pi(\epsilon_3) = \pi(\epsilon_2) = \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \mu), \\
\pi(\mu) = \pi(\lambda) = \pi(\delta) = \frac{1}{3}(\delta + \lambda + \mu), \quad \pi(\delta_i) = \delta_i; 0 \leq i \leq m, \\
\overline{\text{orb}}_0 = \emptyset, \quad \overline{\text{orb}}_1 = \{\epsilon_1, \epsilon_2, \epsilon_3\}, \quad \overline{\text{orb}}_2 = \{-\epsilon_1, -\epsilon_2, -\epsilon_3\}, \\
S' = \pi(S), \quad E' = \pi(\delta + L) = \pi(L) = 2\hat{\Lambda} = \Xi(\delta + \lambda + \mu) + \sum_{i=0}^{m} 2\Xi \delta_i, \\
\hat{R} := \{\pm \pi(\epsilon_1), \pm 2\pi(\epsilon_1)\}, \\
\pi(R) \text{ is not reduced.}
\]

Example 7.8. Let \( \hat{\sigma} \) be a finite order automorphism of the locally finite root system \( \hat{R} \) and \( \sigma^0 \) be a finite order automorphism of \( \mathfrak{r}^0 \) mapping \( S \) to \( S \) and \( L \) to \( L \). Then \( \sigma := \hat{\sigma} \circ \sigma^0 \) is a finite order automorphism of \( R \). Since \( \sigma(\hat{R}) \subseteq \hat{R} \), we have \( \pi(\hat{R}) \subseteq \hat{V} \). So \( \overline{\text{orb}}_0 = \{\hat{\alpha} \in \mathfrak{B}_\pm \mid \hat{\pi}(\hat{\alpha}) = 0\} = \{\hat{\alpha} \in \mathfrak{B}_\pm \mid \pi(\hat{\alpha}) = 0\} \) and consequently \( \pi(\overline{\text{orb}}_0) = \{0\} \). Also, using the fact that \( \sigma(\hat{R}) \subseteq \hat{R} \), one gets that \( \lambda_1 = 0 \). Therefore Lemmas 5.13 and 5.12 imply that \( S' = \pi(S) \cup \pi(L) = \pi(S), \quad L' = \pi(L) \) (if \( L' \neq \emptyset \)) and \( E' = \pi(L) \) (if \( E' \neq \emptyset \)). Now using Theorem 6.2, one finds that if \( \hat{\sigma} \) is the identity map then \( \pi(R) \) is a LEARS of type \( B \), and otherwise, it is a LEARS of type \( BC \), which is not reduced.
ROOT SYSTEMS ARISING FROM AUTOMORPHISMS

References


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