Endpoints of set-valued asymptotic contractions in metric spaces

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By introducing a new concept called “set-valued asymptotic contraction” in metric spaces, the existence and uniqueness of endpoints for a set-valued asymptotic contraction which has the approximate endpoint property have been established.

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1. Introduction and preliminaries

Let \( X \) be a nonempty set and \( T : X \to 2^X \) be a set-valued map with nonempty values. Then a point \( x \in X \) is said to be an endpoint (or stationary point or strict fixed point) of \( T \) if \( T(x) = \{x\} \). The set of all endpoints of \( T \) is denoted by \( \text{End}(T) \). We say that \( T \) has the approximate endpoint property if \( \inf_{x \in X} \sup_{y \in T(x)} d(x, y) = 0 \). Equivalently, if there exists a sequence \( (x_n) \) such that \( H(\{x_n\}, T x_n) \to 0 \).

In recent years many authors studied the existence and uniqueness of endpoints for a set-valued map in metric spaces and topological spaces; see [1–11] and references therein. Recently Amini [1] extended the Boyd–Wong contraction [12] to set-valued maps and he proved the existence and uniqueness of endpoints for these contractions which have the approximate endpoint property. Also, Kirk [13] introduced the notion of asymptotic contractions on a metric space and obtained an asymptotic version of the Boyd–Wong fixed point theorem. Motivated by Kirk’s asymptotic contraction we introduce the concept of the set-valued asymptotic contraction in metric spaces; then the existence and uniqueness of endpoints for a set-valued asymptotic contraction which has the approximate endpoint property are proved.

Let \( (X, d) \) be a metric space and \( CB(X) \) denote the family of all nonempty closed and bounded subsets of \( X \). Then the Hausdorff metric on \( CB(X) \) is given by

\[
H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}
\]

where \( d(a, B) = \inf_{b \in B} d(a, b) \) is the distance of the point \( a \) from the set \( B \). Let \( T : X \to CB(X) \) be a set-valued map. Then the fixed point problem is well posed (see [4]) for \( T \) with respect to \( H \) if:

(i) \( \text{End}(T) = \{x^*\} \),
(ii) if \( x_n \in X, n \in \mathbb{N} \) and \( H(\{x_n\}, T x_n) \to 0 \) as \( n \to \infty \), then \( x_n \to x^* \) as \( n \to \infty \).

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Definition 1.1. Let \((X, d)\) be a metric space. A set-valued map \(T : X \to CB(X)\) is said to be a set-valued asymptotic contraction if there exist a sequence \((\phi_n)_{n=1}^{\infty}, \phi : \mathbb{R}_+ \to \mathbb{R}_+\), and \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\[
H(T^n(x), T^n(y)) \leq \phi_n(d(x, y)) \quad \text{for all } x, y \in X \text{ and } n \in \mathbb{N},
\] (1)
\[
\phi_n \to \phi \text{ uniformly on } \mathbb{R}_+, \phi(t) < t \text{ for all } t > 0 \text{ and } \phi \text{ is continuous}.
\]

Given a function \(\phi\) (not necessarily continuous) satisfying the last inequality, we say that \(T\) is a set-valued asymptotic \(\phi\)-contraction if there exists a sequence \((\phi_n)_{n=1}^{\infty}\) as in the above definition. If \(T\) is a single-valued map, then we say that \(T\) is an asymptotic \(\phi\)-contraction. Notice that the above definition for single-valued maps was introduced by Kirk [13].

2. Main results

Throughout this section we assume that \((X, d)\) is a complete metric space, \(T : X \to CB(X)\), \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) is upper semicontinuous such that \(\phi(t) < t \text{ for all } t > 0 \text{ and satisfies the following limit condition:}
\[
\liminf_{t \to \infty} (t - \phi(t)) > 0.
\] (2)

We set
\[
C_n := \left\{ x \in X : H([x], Tx) \leq \frac{1}{n} \right\} \text{ for all } n \in \mathbb{N}.
\]
Clearly End\((T) = \bigcap_{n \in \mathbb{N}} C_n\) and \(C_n\) is nonempty for all \(n \in \mathbb{N}\) if and only if \(T\) has the approximate endpoint property. The following lemmas will be used in the proof of the main result. The first one is an extension [14, Lemma 5].

Lemma 2.1. Let \(T\) be uniformly continuous and \((x_n)_{n=1}^{\infty}\) be a sequence in \(X\) such that \(H([x_n], Tx_n) \to 0\). Then
\[
\lim_{n \to \infty} H([x_n], T^kx_n) = 0 \quad \text{for all } k \in \mathbb{N}.
\] (3)

Proof. We apply the induction principle. By assumption, (3) is satisfied for \(k = 1\). Assume that (3) holds for some \(k \in \mathbb{N}\). Since \(T\) is uniformly continuous, then we have
\[
\forall \epsilon > 0 \quad \exists \delta > 0; \quad \forall x, y \in X, \quad d(x, y) < \delta \Rightarrow H(Tx, Ty) < \epsilon.
\] (4)

Fix \(\epsilon > 0\) and let \(\delta\) be as in (4). Since \(H([x_1], T^kx_n) \to 0\), then there exists \(N \in \mathbb{N}\) such that
\[
H([x_1], T^kx_n) = \sup_{y \in T^kx_n} d(x_1, y) < \delta \quad \forall n \geq N.
\] (5)

By (4), (5) we have
\[
H(Tx_n, Ty) \leq \epsilon \quad \forall y \in T^kx_n, \quad \forall n \geq N.
\]
Furthermore,
\[
\sup_{z \in Tx_n} d(z, T(T^kx_n)) \leq \sup_{z \in Tx_n} d(z, Ty) \leq H(Tx_n, Ty) \leq \epsilon \quad \text{for all } y \in T^kx_n
\]
and
\[
\sup_{z \in T(T^kx_n)} d(z, Tx_n) \leq \sup_{y \in Tx_n} \sup_{z \in Ty} d(z, Tx_n) \leq \sup_{y \in Tx_n} H(Ty, Tx_n) \leq \epsilon.
\]

Hence, by definition of the Hausdorff metric we have
\[
H(Tx_n, T^kx_n) < \epsilon \quad \forall n \geq N.
\]
Therefore, \(H(Tx_n, T^kx_n) \to 0\). Hence,
\[
H([x_n], T^{k+1}x_n) \leq H([x_n], Tx_n) + H(Tx_n, T^kx_n) \to 0.
\]

The proof of the following result is similar to that [14, Lemma 2]; hence it is omitted.

Lemma 2.2. \(\text{diam}C_n \to 0\) if and only if given sequences \((x_n)_{n=1}^{\infty}\) and \((y_n)_{n=1}^{\infty}\), conditions \(H([x_n], Tx_n) \to 0\) and \(H([y_n], Ty_n) \to 0\) imply that \(d(x_n, y_n) \to 0\).

Theorem 2.3. Assume that \(T\) is a uniformly continuous set-valued asymptotic \(\phi\)-contraction. Then \(T\) has a unique endpoint if and only if \(T\) has the approximate endpoint property. Furthermore, the fixed point problem is well posed for \(T\) with respect to \(H\).
Proof. It is clear that if $T$ has an endpoint, then $T$ has the approximate endpoint property. Conversely, suppose that $T$ has the approximate endpoint property; then $C_n$ is nonempty for all $n \in \mathbb{N}$. Also, $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$ and since $T$ is continuous we have that $C_n$ is closed. Now, we show that $\text{diam } C_n \to 0$. According to Lemma 2.2, it is enough to show that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are two sequences such that $H((x_n), T x_n) \to 0$ and $H((y_n), T y_n) \to 0$, then $d(x_n, y_n) \to 0$. To show this assertion, we repeat with some minor changes an argument given in the proof of [14, Lemma 6]. Let $(\phi_n)_{n=1}^{\infty}$ be a sequence satisfying (1) and uniformly convergent to $\phi$. Then for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that
\[
\phi_k(t) < \phi(t) + \epsilon \quad \forall t \in \mathbb{R}_+.
\]
If $a_n := d(x_n, y_n)$, then from (1) and the triangle inequality we have
\[
a_n \leq H((x_n), T^k x_n) + H((y_n), T^k y_n) + \phi_k(a_n).
\]
Hence, by Lemma 2.1, (6) and (7) we obtain
\[
a_n - \phi(a_n) - \epsilon < a_n - \phi_k(a_n) \leq H((x_n), T^k x_n) + H((y_n), T^k y_n) \to 0.
\]
Since $\epsilon$ was arbitrary and $a_n - \phi(a_n) \geq 0$ for all $n \in \mathbb{N}$ we conclude that
\[
\lim_{n \to \infty} (a_n - \phi(a_n)) = 0.
\]
Therefore, the limit condition (2) for $\phi$ implies that the sequence $(a_n)_{n=1}^{\infty}$ is bounded. In contrast, if we assume that $a_n \to 0$, then there is a subsequence $(a_{n_k})$ such that $a_{n_k} \to a$ for some $a > 0$. Now by (8) and the upper semicontinuity of $\phi$ we have
\[
a = \lim_{k \to \infty} \phi(a_{n_k}) \leq \limsup_{t \to a} \phi(t) \leq \phi(a),
\]
which is a contradiction. Therefore, $\text{diam } C_n \to 0$. Hence, by the Cantor intersection theorem there exists $x^* \in X$ such that $\text{End}(T) = \bigcap_{n \in \mathbb{N}} C_n = \{x^*\}$. Assume that $x_n \in X$ and $H((x_n), T x_n) \to 0$ as $n \to \infty$. Set $y_n := x_n$ for all $n \in \mathbb{N}$. Then, by Lemma 2.2 we get $x_n \to x$ as $n \to \infty$, i.e., the fixed point problem is well posed for $T$ with respect to $H$.

As a consequence of the above theorem we obtain the following result.

**Theorem 2.4.** Let $(X, d)$ be a complete metric space. Let $T : X \to CB(X)$ be a set-valued map with compact values such that
\[
H(Tx, Ty) \leq \psi(d(x, y)), \quad \text{for each } x, y \in X,
\]
where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semicontinuous, $\psi(t) < t$ for each $t > 0$, and satisfies $\liminf_{t \to \infty} (t - \psi(t)) > 0$. Then $T$ has a unique endpoint if and only if $T$ has the approximate endpoint property. Furthermore, the fixed point problem is well posed for $T$ with respect to $H$.

**Proof.** We show that
\[
H(T^n x, T^n y) \leq \psi(d(x, y)), \quad \text{for each } x, y \in X \text{ and } n \in \mathbb{N}.
\]
We apply the induction principle. By hypothesis, (9) is satisfied for $n = 1$. Assume that (9) holds for some $k \in \mathbb{N}$. Since $T$ is continuous with respect to the Hausdorff metric and $T(x)$ is compact for all $x \in X$, then $T$ is upper semicontinuous; see [15]. Therefore, $T^k x$ is compact for all $x \in X$ and by applying the induction principle, we have that $T^n x$ is compact for any $x \in X$ and $n \in \mathbb{N}$. Assume that $z \in T^{k+1} x$; then there exists $w \in T^k x$ such that $z \in T w$. Since $T^k y$ is compact, then there is a $w' \in T^k y$ such that $d(w, w') = d(w, T^k y)$. Therefore,
\[
d(z, T^{k+1} y) \leq d(z, T w') \\
\leq d(z, T w) + H(T w, T w') \\
\leq \psi(d(w, w')) \leq d(w, w') = d(w, T^k y).
\]
Consequently,
\[
d(z, T^{k+1} y) \leq d(w, T^k y) \leq H(T^k x, T^k y) \leq \psi(d(x, y)) \quad \text{for all } z \in T^{k+1} x.
\]
Similarly
\[
d(u, T^{k+1} x) \leq H(T^k x, T^k y) \leq \psi(d(x, y)) \quad \text{for all } u \in T^{k+1} y.
\]
Hence, by (10) and (11) we obtain (9) as asserted. If we set $\phi_n = \psi$ for all $n \in \mathbb{N}$, then by (9) $T$ is a set-valued asymptotic $\psi$-contraction. On using Theorem 2.3, the proof is complete. □

The above result has been established by Amini [1] without the compactness condition on values of $T$.

As another application of Theorem 2.3 we obtain Kirk’s fixed point theorem; [14, Theorem 2].
Theorem 2.5. Let \((X, d)\) be a complete metric space and \(f : X \to X\) be a uniformly continuous asymptotic \(\phi\)-contraction, where \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) is upper semicontinuous, and \(\phi(t) < t\) for each \(t > 0\) and satisfies \(\lim_{t \to \infty} (t - \phi(t)) = \infty\). Then \(f\) has a unique fixed point.

Proof. [14, Lemma 4], \(d(f^n(x), f^n(y)) \to 0\) for all \(x, y \in X\). Therefore, for every \(x \in X\) we have \(d(f^{n+1}(x), f^n(x)) \to 0\). Hence, 
\[
\inf_{x \in X} d(x, f(x)) = 0
\]
and so \(f\) has the approximate endpoint property. Therefore, by Theorem 2.3, \(f\) has a unique endpoint, i.e., \(f\) has a unique fixed point. □

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