Groups all of whose undirected Cayley graphs are integral

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A B S T R A C T

Let $G$ be a finite group, $S \subseteq G \setminus \{1\}$ be a set such that if $a \in S$, then $a^{-1} \in S$, where 1 denotes the identity element of $G$. The undirected Cayley graph $\text{Cay}(G, S)$ of $G$ over the set $S$ is the graph whose vertex set is $G$ and two vertices $a$ and $b$ are adjacent whenever $ab^{-1} \in S$. The adjacency spectrum of a graph is the multiset of all eigenvalues of the adjacency matrix of the graph. A graph is called integral whenever all adjacency spectrum elements are integers. Following Klotz and Sander, we call a group $G$ Cayley integral whenever all undirected Cayley graphs over $G$ are integral. Finite abelian Cayley integral groups are classified by Klotz and Sander as finite abelian groups of exponent dividing 4 or 6. Klotz and Sander have proposed the determination of all non-abelian Cayley integral groups. In this paper we complete the classification of finite Cayley integral groups by proving that finite non-abelian Cayley integral groups are the symmetric group $S_3$ of degree 3, $C_3 \times C_4$ and $Q_8 \times C_2^2$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8. © 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Let $G$ be a finite group and $S$ be a subset of $G \setminus \{1\}$ such that $S = S^{-1}$, where 1 is the identity element of $G$. The undirected Cayley graph $\text{Cay}(G, S)$ is the graph whose vertex set is $G$ and two vertices $a, b \in G$ are adjacent whenever $ab^{-1} \in S$. The adjacency spectrum of a graph is the multiset of all eigenvalues of the adjacency matrix of the graph. A graph is called integral whenever all adjacency spectrum elements are integers. The question of “which graphs are integral?” was first proposed by...
Harary and Schwenk [9]. Many research papers were written on integral graphs e.g. [3,5,14]. Cayley graphs which are integral were studied by many people (e.g. [1,2,4,7,10,11]). Following [10] we call a group $G$ Cayley integral whenever all undirected Cayley graphs over $G$ are integral. Finite abelian Cayley integral groups are classified in [10]: these are finite abelian groups of exponent dividing 4 or 6. In [10, p. 12, Problem 3] Klotz and Sander have proposed the problem of determination of all non-abelian Cayley integral groups. The non-abelian Cayley integral groups of order at most 12 found in [10, p. 12, Problem 3] are the symmetric group $S_3$ or $S_4$. In [10, p. 12, Problem 3] Cayley integral groups are classified in [10]: these are finite abelian groups of exponent dividing 4 or 6. In [10, p. 12, Problem 3] Klotz and Sander have proposed the problem of determination of all non-abelian Cayley integral groups. The non-abelian Cayley integral groups of order at most 12 found in [10, p. 12, Problem 3] are the symmetric group $S_3$ of degree 3, the quaternion group $Q_8$ of order 8, and the semidirect product $C_2$ by $C_4$ which is a non-abelian group of order 12. In this paper we complete the classification of finite Cayley integral groups by proving the following.

**Theorem 1.1.** A finite non-abelian group is Cayley integral if and only if it is isomorphic to one of the following groups:

1. the symmetric group $S_3$ of degree 3,
2. $C_3 	imes C_4 = \langle x, y \mid x^3 = y^4 = 1, xy = x^{-1} \rangle$,
3. $Q_8 \times C_2^n$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8.

So combining the above mentioned result of Klotz and Sander and Theorem 1.1, the classification of finite Cayley integral groups completes as follows.

**Theorem 1.2.** A finite group is Cayley integral if and only if it is isomorphic to one of the following groups:

1. an abelian group of exponent dividing 4 or 6,
2. the symmetric group $S_3$ of degree 3,
3. $C_3 \times C_4 = \langle x, y \mid x^3 = y^4 = 1, xy = x^{-1} \rangle$,
4. $Q_8 \times C_2^n$ for some integer $n \geq 0$, where $Q_8$ is the quaternion group of order 8.

Throughout we denote by $C_n$ the cyclic group of order $n$, the dihedral group of order $2n$ is denoted by $D_{2n}$, the alternating group of degree 4 is denoted by $A_4$; we denote by $S_3$ and $S_4$ the symmetric groups of degree 3 and 4, respectively, and $C_k^n$ denotes the direct product $C_{n_1} \times \cdots \times C_{n_k}$. The semidirect product of a group $G$ by a group $K$ is denoted by $G \rtimes K$ and $G \triangleright K = 1$ is a (not necessarily unique) group $G$ containing a normal subgroup $N_1$ isomorphic to $N$ and a subgroup $K_1$ isomorphic to $K$ such that $G = N_1 \rtimes K_1$ and $N_1 \cap K_1 = 1$. For any two elements $x, y$ of a group $G$ we denote by $[x, y]$ the commutator $x^{-1}y^{-1}xy$. For a free group $F$ generated by free generators $x_1, \ldots, x_n$ and elements $r_1, \ldots, r_m$, the factor group $G = \langle r_1, \ldots, r_m \rangle$ is denoted by

$$\langle x_1, \ldots, x_n \mid r_1 = \cdots = r_m = 1 \rangle. \quad (\ast)$$

where $\langle r_1, \ldots, r_m \rangle$ is the normal closure of the subgroup $\langle r_1, \ldots, r_m \rangle$ in $F$. The notation $\langle \ast \rangle$ is called the presentation of the group $G$ by generators $x_1, \ldots, x_n$ and relations $r_1, \ldots, r_m$.

2. Preliminaries

In this section we state some facts which we need in the sequel. The following result is the classification of all undirected connected cubic Cayley integral graphs.

**Theorem 2.1 (Theorem 1.1 of [2]).** There are exactly seven connected cubic integral Cayley graphs. In particular, for a finite group $G$ and a subset $1 \not\in S \subseteq S = S^{-1}$ with three elements, $\text{Cay}(G, S)$ is integral if and only if $G$ is isomorphic to one of the following groups:

$$C_2^2, C_4, C_6, S_3, C_2^2, \text{Cay}(C_2 \times C_4, D_8, C_2 \times C_6, D_{12}, A_4, S_4, D_8 \times C_3, S_3 \times C_4, A_4 \times C_2).$$

**Lemma 2.2.** Let $G$ be a finite Cayley integral group. Then every subgroup of $G$ is also Cayley integral.

**Proof.** Let $H$ be an arbitrary subgroup of $G$ and let $T \subseteq H \setminus \{1\}$ with $T = T^{-1}$. Consider the Cayley graph $\text{Cay}(H, T)$. Note that $\text{Cay}(H, T)$ is isomorphic to a disjoint union of some $\Gamma = \text{Cay}(T, T)$. Thus $\text{Cay}(H, T)$ is integral if and only if $\Gamma$ is integral. Now, since $\text{Cay}(G, T)$ is also a disjoint union of some $\Gamma$, it follows that $\Gamma$ is integral. Hence $H$ is a Cayley integral group. This completes the proof. \qed
Proposition 2.3 (See [8, 12], Proposition 6.3.1 of [6]). Let G be a finite group and S a subset that is inverse closed and invariant under conjugation. The graph $\text{Cay}(G, S)$ has eigenvalues $\theta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$ with multiplicity $\chi(1)^2$, where $\chi$ ranges over the irreducible characters of $G$.

Note that for every finite group $G$, since $\sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G| (\text{Irr}(G)$ is the set of all irreducible characters of $G$), the multiset $\{\theta_\chi \mid \chi \in \text{Irr}(G)\}$ is the spectrum of $\text{Cay}(G, S)$ for any $S$ as in Proposition 2.3.

The following result has its own interest and we do not use it in the sequel but we would like to state it here!

Proposition 2.4 (See Theorem 2 of [7]). Let $G$ be a finite group and $S$ be a member of the boolean algebra generated by the normal subgroups of $G$. Then the Cayley graph $\text{Cay}(G, S)$ is integral.

Proof. Since $S$ can be obtained by arbitrary finite intersections, unions, or complements of normal subgroups of $G$, $S$ is closed under conjugation as well as inverse. Thus by Proposition 2.3 eigenvalues of $\text{Cay}(G, S)$ are of the form $\theta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$ for $\chi \in \text{Irr}(G)$. Now, Corollary 4.2 of [4] shows that $\sum_{s \in S} \chi(s)$ is an algebraic integer for each $\chi \in \text{Irr}(G)$. Since each eigenvalue of any graph is an algebraic integer, it follows that $\theta_\chi$ is integer. This completes the proof. □

Lemma 2.5 (Lemma 11 of [10]). If $G$ is a Cayley integral group, then the order of every non-trivial element of $G$ belongs to $\{2, 3, 4, 6\}$.

Lemma 2.6. $D_{2n}$ is not a Cayley integral group for all integers $n \geq 4$.

Proof. It is well-known that $D_{2n} = \langle a, b | a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$. If $S = \{b, ba\}$, then $\text{Cay}(D_{2n}, S)$ is a cycle on $2n$ vertices which is not an integral graph. Therefore $D_{2n}$ is not an integral group. □

Lemma 2.7. The following groups are not Cayley integral.

1. The alternating group $A_4$ of degree 4,
2. $C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = 1, x^2 = x^{-1} \rangle$,
3. $S_3 \times C_3$,
4. the special linear group $\text{SL}(2, 3)$ of $2 \times 2$ matrices over the field of order 3,
5. $(C_4 \times C_3) \times C_2$, where $C_4 \times C_3 = \langle a, b \mid a^2 = b^4 = 1, b^{-1}ab = a^{-1} \rangle$,
6. $(C_4 \times C_2) \times C_4 = \langle x, y \mid x^4 = y^4 = [x, y]^2 = [x^2, y] = [x, y^2]^2 = 1 \rangle$,
7. the non-abelian group of exponent 3 and order 27: $\langle x, y \mid x^3 = y^3 = (xy)^3 = (xy^{-1})^3 = 1 \rangle$,
8. $(C_3 \times C_3) \times C_4 = \langle x, y, z \mid x^3 = y^3 = z^4 = [x, y] = 1, x^2 = x^{-1}, y^2 = y^{-1} \rangle$,

Proof. For each of groups (1)–(8) we have found an inverse closed subset $S$ such that $\text{Cay}(G, S)$ is not integral. We have used the following codes in GAP [16] to obtain the spectra of these graphs. The following code constructs the Cayley graph of $A_4$ on the set $S$. We use GRAPE package of Soicher [15].

LoadPackage("grape");
F:=FreeGroup(2);
x:=F.1; y:=F.2;
G:=F/[[x^3, y^2, (x*y)^3] ]; #G=A_4
a:=G.1;
b:=G.2;
S:=[a, a^-1, b, a*b, b^-1*a^-1];
C:=CayleyGraph(G,S);

By the function admat [2, p. 6] one can construct the adjacency matrix of a given graph by GRAPE [15].
A:=admat(C,12);
The following command computes the characteristic polynomial of the adjacency matrix of \( \text{Cay}(A_4, S) \) and the second command factorizes the polynomial into irreducible ones.

\[
P: = \text{CharacteristicPolynomial}(A); \\
FP: = \text{Factors}(P).
\]

It follows that the characteristic polynomial of \( \text{Cay}(A_4, S) \) is as follows:

\[
(x - 5)(x + 1)^3(x^2 - 5)^3,
\]

and so \( \text{Cay}(A_4, S) \) is not integral showing that \( A_4 \) is not a Cayley integral group. For the other groups \( G \) in (2)–(8) we give a presentation of \( G \) under which we introduce a subset \( S \) for which \( \text{Cay}(G, S) \) is not integral. Verifying that \( \text{Cay}(G, S) \) is not integral can be done as above by GAP. In each case, the factorized characteristic polynomial \( P(x) \) into irreducibles is exhibited.

\[
(2) \ G = \langle x, y \mid x^4 = y^4 = 1, x^9 = y^3 \rangle, \ S = \{ x, x^{-1}, y, y^{-1}, xy, y^{-1}x^{-1}, xy^2, y^{-2}x^{-1} \}
\]

and \( P(x) = (x - 8)x^6(x + 4)^2(x^2 - 8)^2. \)

\[
(3) \ G = \langle x, y, z \mid x^2 = y^3 = z^3 = [x, z] = [y, z] = 1, y^x = y^{-1}, \ S = \{ x, y, y^{-1}, z, z^{-1}, y^x, y^{-1}z^{-1} \}\]

and \( P(x) = (x - 7)(x - 5)(x - 1)^4(x + 1)^4 (x^2 + 3x - 1)^4. \)

\[
(4) \ G = \langle x, y \mid x^3 = y^4 = y^{-1}xyx^{-1}x = x^{-1}y^{-1}x(y^2) = (xy)^3 = 1, \ S = \langle x, y^x, y^{-1} \rangle \]

and \( P(x) = (x - 4)(x - 2)^2(x - 1)^2(x + 1)^2(x^2 - x - 4)^3. \)

\[
(5) \ G = \langle x, y, z \mid x^4 = y^3 = z^2 = [x, z] = [y, z] = 1, y^x = y^{-1}, \ S = \langle x, y^x, y^{-1}, z, xy, y^{-1}x^{-1}, xz, z^{-1}x^{-1} \rangle \]

and \( P(x) = (x - 9)(x - 3)(x - 1)^2x^6(x + 1)(x + 2)(x + 3)(x + 4)^2(x^2 - 12)^2. \)

**Lemma 2.8.** Let \( G \) be a finite Cayley integral group and \( x \) be any element of order 2 of \( G \) and \( y \) be an arbitrary element of \( G \). Then \( (x, y) \) is isomorphic to one of the groups: \( C_2, C_2^2, C_4, C_6, S_3, C_2 \times C_4, C_2 \times C_6. \)

**Proof.** If \( y \in \langle x \rangle \), then \( (x, y) \cong C_2 \). Suppose that \( y \not\in \langle x \rangle \). If \( o(y) = 2 \), then \( (x, y) \) is a dihedral group of order at most 6 by Lemma 2.6. Thus in the latter case, \( (x, y) \cong C_2 \times C_2 \) or \( S_3 \). Now, assume that \( o(y) > 2 \). Let \( S = \langle x, y, y^{-1} \rangle \). Then \( \text{Cay}(S, S) \) is a cubic integral Cayley graph. It follows from Theorem 2.1 that \( (S, S) \) is isomorphic to one of the following groups: \( C_4, C_6, S_3, C_2 \times C_4, C_2 \times C_6, D_{12}, A_4, S_4, D_8 \times C_3, S_3 \times C_4, A_4 \times C_2 \). It follows from Lemma 2.5 that \( (S, S) \) is isomorphic to one of the following groups: \( C_4, C_6, S_3, C_2 \times C_4, D_8 \times C_6, D_8 \times C_3, D_{12}, A_4, S_4, A_4 \times C_2 \). Since by Lemmas 2.6 and 2.7, \( G \) cannot have any subgroup isomorphic to \( D_{2n} (n \geq 4) \) or \( A_4 \), \( (S, S) \) is isomorphic to one of the following groups: \( C_4, C_6, C_2 \times C_4, C_2 \times C_6, S_3 \). This completes the proof. \( \square \)

**Lemma 2.9.** Let \( G \) be a finite Cayley integral group. Let \( x \) be any element of order 2 and \( y \in G \) is of order 4 or 6. Then \( xy = yx \).

**Proof.** It follows from Lemma 2.8 that the subgroup \( \langle x, y \rangle \) is abelian or isomorphic to \( S_3 \). Since \( S_3 \) has no element of order 4 or 6, we are done. \( \square \)

**Lemma 2.10.** Let \( G \) be a finite non-abelian Cayley integral group generated by three distinct elements of order 2. Then \( G \cong S_3 \).

**Proof.** Suppose that \( G = \langle x, y, z \rangle \), where \( x, y, z \) are all distinct and \( o(x) = o(y) = o(z) = 2 \). Consider the cubic Cayley graph \( \Gamma = \text{Cay}(G, \langle x, y, z \rangle) \). Since \( G \) is non-abelian and \( \Gamma \) is integral, it follows from Theorem 2.1 that \( G \) is isomorphic to one of the following groups: \( S_3, D_8, D_{12}, A_4, S_4, D_8 \times C_3, S_3 \times C_4, A_4 \times C_2 \).

The groups \( D_8, D_{12} \) and \( D_8 \times C_3 \) are ruled out by Lemma 2.6 and the groups \( A_4, S_4 \) and \( A_4 \times C_2 \) are not possible by Lemma 2.7. The group \( S_3 \times C_4 \) is not Cayley integral by Lemma 2.5. It follows that \( G \cong S_3 \). \( \square \)
Lemma 2.11. Let $G$ be a finite 3-group. Then $G$ is Cayley integral if and only if $G$ is elementary abelian.

Proof. If $G$ is an elementary abelian 3-group, then $G \cong C_3^k$ for some integer $k \geq 0$. Now, it follows from [10] that $G$ is Cayley integral.

Now, assume that $G$ is a finite Cayley integral 3-group. By Lemma 2.5 the exponent of $G$ is 3. Suppose, for a contradiction, that $G$ is non-abelian. Then $G$ has two non-commuting elements $x$ and $y$. Thus $\langle x, y \rangle$ is the group of order 27 and exponent 3. This contradicts Lemma 2.7. Thus $G$ is abelian of exponent 3 which means that $G$ is an elementary abelian 3-group.

Theorem 2.12. Let $G$ be a finite Cayley integral group. Then, there exist two non-commuting elements of order 2 in $G$ if and only if $G \cong S_3$.

Proof. Let $x, y \in G$ be two non-commuting elements of order 2. Then it follows from Lemma 2.8 that $\langle x, y \rangle \cong S_3$. Now, assume that $z$ is an element of order 2 such that $z \notin \langle x, y \rangle$. It follows from Lemma 2.10 that $\langle x, y, z \rangle \cong S_3$. Therefore $z \in \langle x, y \rangle$. This means that all elements of order 2 of $(x, y)$ are exactly all elements of order 2 of $G$. Thus $G$ has precisely three elements $x, y, z$ of order 2 and they are pairwise non-commuting. Now, assume that, if possible, $G$ has an element of order 4. Then by Lemma 2.9 $t$ commutes with all $x, y$ and $z$ and so $t^2 g = g t^2$ for all $g \in \{x, y, z\}$; this is a contradiction since $t^2 \in \{x, y, z\}$. Therefore $G$ has no element of order 4. Now, if $G$ has a subgroup $K$ of order 4, it must be isomorphic to $C_2 \times C_2$, a contradiction as all elements of order 2 of $G$ are pairwise non-commuting. Hence 4 does not divide $|G|$ and so $|G| = 2m$ for some odd integer $m$. It follows from Lemma 2.5 that $m$ is a power of 3. Let $M$ be a Sylow 3-subgroup of $G$. By Lemma 2.11 $M$ is elementary abelian. Assume that $|M| \geq 9$. Thus $M$ has two elements $b_1$ and $b_2$ such that $\langle b_1, b_2 \rangle = \langle b_1 \rangle \times \langle b_2 \rangle$. Note that if $a \in G$ and $b \in G$ such that $o(a) = 2$ and $o(b) = 3$, it follows from Lemma 2.8 that $b^a = b^o = b^{-1}$ since $\langle a, b \rangle$ is either abelian or isomorphic to $S_3$. Since $G$ has a subgroup isomorphic to $S_3$ (say $(x, y)$), we may assume that $b_1^2 = b_2^{-1}$. Thus $b_1^3 = b_2^3$ or $b_2^3 = b_1^{-3}$. If $b_2^3 = b_1^{-3}$, then $\langle x, b_1, b_2 \rangle \cong S_3 \times C_3$ which is not possible by Lemma 2.7. If $b_2^3 = b_1^{-3}$, then $\langle x, b_1, b_2 \rangle$ has 9 elements of order 2 which is a contradiction, since $G$ has only 3 elements of order 2. Thus $|M| = 3$ and so $G \cong S_3$.

The converse is easy to verify.

Theorem 2.13. Let $G$ be a finite non-abelian 2-group. Then $G$ is Cayley integral if and only if $G \cong Q_8 \times C_2^n$ for some integer $n \geq 0$.

Proof. Suppose that $G$ is a finite non-abelian Cayley integral 2-group. If we prove that every subgroup of $G$ is normal in $G$, it follows from a famous result of Dedekind–Baer (see Theorem 5.3.7 of [13]) that $G \cong Q_8 \times C_2^n$ for some integer $n \geq 0$.

To prove that every subgroup of $G$ is normal in $G$, it is enough to show that every cyclic subgroup of $G$ is normal. Since $G$ is a 2-group, every element of $G$ is of order 1, 2 or 4 by Lemma 2.5. Thus every cyclic subgroup of $G$ is either of order 1, 2 or 4. Every element of order 2 belongs to the center of $G$, this follows from Lemma 2.8 and so every (cyclic) subgroup of order 2 is normal in $G$. Therefore, it remains to prove that $\langle x \rangle \leq G$ for all elements $x$ of order 4.

Suppose in contrary that there exists an element $a$ of order 4 such that $\langle a \rangle$ is not normal in $G$. Thus there exists an element $g$ of $G$ such that $g^{-1}a g \notin \langle a \rangle$. Consider the subgroup $H = \langle a, g \rangle$ of $G$. Clearly $H$ is non-abelian. The order of $g$ is not 1 or 2, otherwise $a^g = a$ since elements of order 2 are central in $G$. Therefore the order of $g$ is 4. Now, we investigate group properties of $H$.

1. $H = \langle a, g \rangle$ is of exponent 4, $o(a) = o(g) = 4$ and $g^{-1}ag \notin \langle a \rangle$.
2. All elements of order 2 of $H$ are in the center $Z(H)$ of $H$. This follows from Lemma 2.8.
3. $H/Z(H)$ is of exponent 2. If $x \in H$, then $x$ is of order 1, 2 or 4 by the property (1). Thus $o(x^2) \in \{1, 2\}$ and so $x^2 \in Z(H)$ by the property (2). This proves that $H/Z(H)$ is of exponent 2.
4. $H$ is nilpotent of class 2. By the property (3), the derived subgroup $H' = Z(H)$ is contained in $Z(H)$ and so $H$ is nilpotent of class 2.
5. $H' = \langle (a, g) \rangle$ is of order 2. It is because that every commutator in $H$ is central by the property (4) and since $H$ is generated by two elements $a$ and $g$, we have $H' = \langle (a, g) \rangle$. Now, by the property (3), $[a, g]^2 = [a^2, g^2] = 1$ and so $|H'| = 2$. 


Lemma 2.7 implies that $H/H'$ is isomorphic to $C_4 \times C_4$, $C_2 \times C_4$, or $C_2 \times C_2$. This is because $H/H'$ is an abelian group generated by two elements $aH$ and $gH$ which are both of orders dividing 4 and we note that $H/H'$ cannot be cyclic otherwise $H$ is also cyclic.

It follows that the order of $H$ is 8, 16 or 32. If the order of $H$ is 8, then $H$ is isomorphic to $D_8$ or $Q_8$; since by Lemma 2.6, $D_8$ is not an integral group, $H \ncong D_8$. Every subgroup of $Q_8$ is normal and so $g^{-1}ag \in \langle a \rangle$ if $H \cong Q_8$. Therefore by the property (1), $H \ncong Q_8$.

Thus $|H| \in \{16, 32\}$. Now, suppose that the order of $H$ is 16. By the following code written in GAP [16], one can see what are groups $H$ of order 16 satisfying the properties (1)–(6).

a:=AllSmallGroups(16,IsAbelian,false);
b:=Filtered(a,i->Size(DerivedSubgroup(i))==2);
c:=Filtered(b,i->IdSmallGroup(FactorGroup(i,DerivedSubgroup(i)))=[8,2]);
d:=Filtered(c,i->Size(DerivedSubgroup(i))=2);
e:=Filtered(d,i->Exponent(i)=4);

The list e contains only one group isomorphic to $C_4 \times C_4 = \langle x, y | x^4 = y^4 = 1, xy^2 = x^{-1} \rangle$. Now, Lemma 2.7 implies that $H$ cannot be isomorphic to $C_4 \times C_4$ and so $|H| \neq 16$.

Now, assume that $|H| = 32$. By a similar code as above one can see that there is only one group $N$ satisfying the properties (1)–(6). The group $N$ is isomorphic to $(C_4 \times C_2) \times C_4 = \langle x, y | x^4 = y^4 = [x, y]^2 = [x^2, y] = [x, y^2] = 1 \rangle$ which is not a Cayley integral group by Lemma 2.7. Thus $|H| \neq 32$.

This completes the proof in this direction.

Now, let us prove that $T_n = Q_8 \times C_p^n$ is a Cayley integral group for all integers $n \geq 0$. We first prove that the conjugacy class $a^n = \{a^k | g \in T_n\}$ of any element $a \in T_n$ is equal to $\langle a \rangle$ or $\langle a, a^{-1} \rangle$. For, if $o(a) = 2$, then $a$ is central in $T_n$ and so $a^n = \{a\}$; and if $o(a) = 4$, then $a = xt$, with $x \in Q_8$ and $t \in C_2^n$. Let $g = ys$ be any element of $T_n$ such that $y \in Q_8$ and $s \in C_2^n$. We have $a^d = x^d t$ and since $x^d \in \langle x, x^{-1} \rangle$ (it is easy to check that the latter is valid in $Q_8$) it follows that $a^d \in \langle a, a^{-1} \rangle$.

It follows that if $S$ is an inverse closed subset of $T_n$ (not containing 1), $S$ is also closed under conjugation of elements of $T_n$. Now, consider the Cayley graph $Cay(T_n, S)$. Proposition 2.3 implies that for each irreducible character $\chi$ of $T_n$, $\frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$ is an eigenvalue $\theta_{\chi}$ of $Cay(T_n, S)$ of multiplicity $\chi(1)^2$. As we mentioned in the paragraph after the statement of Proposition 2.3 the multiset $\{\theta_{\chi} | \chi \in \text{Irr}(T_n)\}$ is the spectrum of $Cay(T_n, S)$. Now, since $\chi(g) \in \mathbb{Z}$ for all $g \in T_n$ (because all irreducible characters of $Q_8$ or $C_2$ have integer values and an irreducible character of $T_n$ is a tensor product of irreducible characters of $Q_8$ and $C_2$’s), $\sum_{s \in S} \chi(s) \in \mathbb{Z}$. Since $\chi(1)$ is an integer, it follows that $\theta_{\chi} \in \mathbb{Q}$ and so $\theta_{\chi} \in \mathbb{Z}$ as each eigenvalue of the adjacency matrix of a graph is an algebraic integer. Hence $Cay(T_n, S)$ is integral and so $T_n$ is a Cayley integral group.

It should be mentioned that the subsets $S$ of $Q_8 \times C_p^d$ (p a prime) for which the Cayley graph $Cay(Q_8 \times C_p^d, S)$ is integral are studied in the last section of [7]. We cannot derive the above result from discussions in [7].

3. Proof of the main theorem

In this section we prove the main theorem.

Proof of Theorem 1.1. Let $G$ be a finite non-abelian Cayley integral group. By Lemma 2.11 and Theorem 2.13, we may assume that 6 divides the order of $G$. By Theorem 2.12, we may assume that all elements of order 2 of $G$ pairwise commute. If $x$ is an arbitrary element of order 3, and $y$ is any element of order 2, the subgroup $\langle x, y \rangle$ must be abelian, otherwise it is isomorphic to $S_3$ by Lemma 2.8, and $S_3$ has non-commuting elements of order 2, a contradiction. Now, Lemmas 2.5 and 2.9 imply that elements of order 2 lie in the center of $G$. If $G$ has no elements of order 4, then $G = P \times Q$, where $P$ is the Sylow 2-subgroup of $G$ and $Q$ is the Sylow 3-subgroup of $G$; this is because $P$ is a central subgroup of $G$. Now, Lemma 2.11 implies that $G \cong C_2^k \times C_3^\ell$ for some positive integers $k$ and $\ell$ and so $G$ is abelian, a contradiction. Thus we may assume that $G$ has an element of order 4.
Now, let $a$ and $b$ be two elements of $G$ of order 4 and 3, respectively. By Lemma 2.5 $ab \neq ba$ and we claim that $b^a = b^{-1}$. To prove the latter, it is enough to show that $K = \langle a, b \rangle \cong C_4 \times C_2 = \langle x, y \mid x^4 = y^2 = 1, y^x = y^{-1} \rangle$. We need to note some group properties of $K$ as follows:

1. $K$ is a finite non-abelian Cayley integral group having two elements of orders 3 and 4.
2. The set of element orders of $K$ is contained in $\{1, 2, 3, 4, 6\}$.
3. All elements of order 2 of $K$ lie in the center of $K$.

By Von Dyck's theorem, there exists an epimorphism from some $G_{i,j}$ ($i, j \in \{2, 3, 4, 6\}$) onto $G$, where

$$
\begin{align*}
G_{i,j} &= \langle x, y \mid x^4 = y^2 = [x^2, y] = (xy)^j = [x, y]^j = [(xy)^i x^{-1}, x] \\
&= [(xy)^{\ell} x^{-1}, y] = [x, y]^{\ell} x, [x, y]^{\ell} y = 1, \\
\end{align*}
$$

and $\ell = \begin{cases} 1 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases}$. Therefore the group $K$ is isomorphic to a quotient of some $G_{i,j}$. All groups $G_{i,j}$ are finite and can be easily computed by GAP [16]. Hence we need to study quotients of $G_{i,j}$. Using the following code in GAP [16], one can find all possible quotients (satisfying properties (1)–(3) above) of $G_{i,j}$ which can be isomorphic to $K$. Note that the following code is for $G_{4,6}$.

```gap
f:=FreeGroup(2);;
x:=f.1; y:=f.2;;
G46:=f/[x^4,y^3,Comm(x^2,y), (x*y)^4, Comm(x,y)^6, Comm((x*y)^2,x),
Comm((x*y)^2, y),
Comm(Comm(x,y)^3,x), Comm(Comm(x,y)^3,y)];;
N:=NormalSubgroups(G46);;
T:=List(N,i->FactorGroup(G46,i));;
Tn:=Filtered(T,i->IsAbelian(i)=false);;
Tn1:=Filtered(Tn,i->exponent(i)=12);;
```

These possible quotients are $SL(2, 3), C_2 \times SL(2, 3)$ or $C_4 \times C_3$. Now, Lemma 2.7 implies that $K = C_4 \times C_3$ as we claimed.

Now, let $P$ be a Sylow 2-subgroup of $G$. Suppose for a contradiction that $P$ is non-abelian. Then Theorem 2.13 implies that $P$ has a subgroup $E$ isomorphic to the quaternion group $Q_8$ of order 8. Consider two elements $a$ and $a'$ of order 4 in $E$ such that $o(aa') = 4$ (take famous $i, j$ and $k$ in $Q_8$) and let $b$ be any element of order 3 in $G$. Since $b^a = b^{-1}$ and $b^{a'} = b^{-1}$, it follows that $b^{a0} = (b^a)^{a'} = b$ and since $o(aa') = 4$, $b^{a'} = b^{-1}$. Hence $b = b^{-1}$, a contradiction. Therefore Sylow 2-subgroups of $G$ are abelian.

Now, suppose for a contradiction that $P$ has an element $a$ of order 4 and an element $t$ of order 2 such that $(a, t) = (a) \times (t)$. Take any element $b$ of order 3 in $G$ and consider the group $M = \langle a, b, t \rangle$. The group $M$ is isomorphic to $(C_4 \times C_3) \times C_2$ which is listed in Lemma 2.7 as a group that is not Cayley integral. It follows that $P$ is cyclic of order 4. Now, let $Q$ be a Sylow 3-subgroup of $G$. Suppose for a contradiction that $|Q| \geq 9$. Then $Q$ has two elements $b$ and $b'$ of order 3 such that $\langle b, b' \rangle = \langle b \rangle \times \langle b' \rangle$. Since $b^a = b^{-1}$ and $b^{a'} = b^{-1}$, $L = \langle b, b', a \rangle = \langle a \rangle \times \langle b, b' \rangle$, it follows that the group $L$ is the group number (8) listed in Lemma 2.7 which is not Cayley integral. Therefore $Q$ is of order 3 and so $G \cong \langle x, y \mid x^4 = y^3 = y^2 = 1, x^y = x^{-1} \rangle$. This completes the proof. \qed

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