ON SEMIPRIME RIGHT GOLDIE MCCOY RINGS

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Abstract. In this note we first show that for a right (resp. left) Ore ring \( R \) and an automorphism \( \sigma \) of \( R \), if \( R \) is \( \sigma \)-skew McCoy then the classical right (resp. left) quotient ring \( Q(R) \) of \( R \) is \( \sigma \)-skew McCoy. This gives a positive answer to the question posed in Başer et al. [1]. We also characterize semiprime right Goldie (von Neumann regular) McCoy (\( \sigma \)-skew McCoy) rings.

Keywords: McCoy ring, Classical quotient ring, Semiprime right Goldie ring.
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1. Introduction

Throughout this note \( R \) denotes an associative ring with unity and \( \sigma \) is an automorphism of \( R \). We denote \( R[x; \sigma] \) the Ore extension (skew polynomial ring) whose elements are the polynomials over \( R \), the addition is defined as usual and the multiplication subject to the relation \( xa = \sigma(a)x \) for any \( a \in R \). According to the Nielsen [4], a ring \( R \) is called right (resp. left) McCoy if for any nonzero polynomials \( f(x), g(x) \in R[x] \), \( f(x)g(x) = 0 \) implies that \( f(x)r = 0 \) (resp. \( rf(x) = 0 \)), for some \( 0 \neq r \in R \). Başer et al. in [1], introduced a natural generalization of McCoy rings. Namely, a ring \( R \) with the endomorphism \( \sigma \) is called \( \sigma \)-skew McCoy if for any nonzero polynomials \( f(x), g(x) \in R[x; \sigma] \), \( f(x)g(x) = 0 \) implies that \( f(x)r = 0 \), for some \( 0 \neq r \in R \).

Suppose that the classical right quotient ring \( Q(R) \) of \( R \) exists. Then for an automorphism \( \sigma \) of \( R \) and any \( pq^{-1} \in Q(R) \) where \( p, q \in R \) with \( q \) regular, the induced map \( \bar{\sigma} : Q(R) \rightarrow Q(R) \) defined by \( \bar{\sigma}(pq^{-1}) = \sigma(p)\sigma(q)^{-1} \) is also an automorphism. Başer et al. posed a question whether the classical (right) quotient ring \( Q(R) \) of a \( \sigma \)-skew McCoy ring \( R \) has to be \( \sigma \)-skew McCoy. In this note we first give a positive answer to this question. Namely, we will prove that if \( R \) is a \( \sigma \)-skew McCoy ring then the classical right (left) quotient ring \( Q(R) \) of \( R \) is a \( \sigma \)-skew McCoy ring. An endomorphism \( \sigma \) of a ring \( R \) is called right (resp. left) reversible if \( b\sigma(a) = 0 \) (resp. \( \sigma(b)a = 0 \)) whenever \( ab = 0 \) for \( a, b \in R \) (for more details see [1]). We show that for a semiprime right Goldie (von Neumann regular) ring \( R \) with an automorphism \( \sigma \), \( R \) is \( \sigma \)-skew McCoy, \( \sigma \)-reversible \( \iff \) \( Q \) is \( \sigma \)-skew McCoy, \( \sigma \)-reversible \( \iff \) \( R \) is \( \sigma \)-skew Armendariz (i.e. for polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n \) and \( g(x) = b_0 + b_1x + \cdots + b_mx^m \) in \( R[x; \sigma] \), \( f(x)g(x) = 0 \) implies \( a_i\sigma^i(b_j) = 0 \) for each \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \)) \( \iff \) \( Q \) is \( \sigma \)-skew Armendariz \( \iff \) \( R \) is \( \sigma \)-rigid (i.e. \( a = 0 \) whenever \( a\sigma(a) = 0 \) for \( a \in R \)) \( \iff \) \( Q \) is \( \sigma \)-rigid, where \( Q \) is a classical right quotient ring of \( R \). We also show that for a semiprime right Goldie ring \( R \), \( R \) is right linearly McCoy (i.e. for each non-zero linear polynomials
such that \( Q \) is a \( \sigma \)-skew McCoy ring.

Thus we have

\[ \text{Corollary 2.3. Let } R \text{ be a right (resp. left) Ore ring and } \sigma \text{ an automorphism of } R. \text{ Then for each element } g \in T = Q(R)[x; \sigma] \text{ there exists a regular element } e \in R \text{ such that } ge \in S = R[x; \sigma] \text{ (resp. } cg \in S = R[x; \sigma]). \]

\[ \begin{align*}
\text{Proof. If } g & \in Q(R), \text{ then for some regular element } c \in R \text{ and } a \in R \text{ we have } g = ac^{-1} \text{ and } ge \in S. \text{ Now assume inductively that for all elements } g \in T \text{ of degree less than } n \text{ the assertion is hold and let } g = q_0 + \cdots + q_n x^n \in T. \text{ Let } q_0 = ac^{-1} \text{ with } a \in R \text{ and regular element } c \in R. \text{ Let } \sigma^n(b) = c \text{ for some regular element } b \in R.
\end{align*} \]

Then \( gb = (q_0 + \cdots + q_{n-1} x^{n-1})b + q_n \sigma^n(b) x^n. \) Now we have \( gb = h' + ac^{-1}cx^n \) with \( h' \in T \) and \( deg(h') < n. \) By induction hypothesis there exists a regular element \( d \in R \) with \( h'd \in S. \) Thus we have \( gbd = h'd + ax^nd \in S \) and the result follows.

\[ \begin{align*}
\text{Theorem 2.2. Let } R \text{ be a right (resp. left) Ore ring and } \sigma \text{ an automorphism of } R. \text{ If } R \text{ is a } \sigma \text{-skew McCoy ring then the classical right (resp. left) quotient ring } Q(R) \text{ of } R \text{ is a } \sigma \text{-skew McCoy ring.} \]

\[ \begin{align*}
\text{Proof. Let } R \text{ be a right Ore, } \sigma \text{-skew McCoy ring and } f(x), g(x) \in Q(R)[x; \sigma] \text{ such that } f(x)g(x) = 0. \text{ By Lemma 2.1., there exist regular element } e \in R \text{ and } f_1 \in R[x; \sigma] \text{, such that } f(x) = f_1 c^{-1}. \text{ So we have } f_1 c^{-1}g(x) = 0. \text{ There exist regular element } e \in R \text{ and } f_2 \in R[x; \sigma] \text{, such that } e^{-1}g(x) = f_2 e^{-1}, \text{ by Lemma 2.1. Thus we have } f_1 f_2 e^{-1} = 0 \text{ and hence } f_1 f_2 = 0. \text{ So there exists } 0 \neq r \in R \text{ such that } f_1 r = 0 \text{ and hence } f(x)cr = f_1 c^{-1}cr = 0. \text{ Thus the classical right quotient ring } Q(R) \text{ of } R \text{ is a } \sigma \text{-skew McCoy. Now assume that } R \text{ is a left Ore, } \sigma \text{-skew McCoy ring and } f(x), g(x) \in Q(R)[x; \sigma] \text{ such that } f(x)g(x) = 0. \text{ By Lemma 2.1., there exist regular element } c \in R \text{ and } f_1 \in R[x; \sigma], \text{ such that } cg(x) = f_1. \text{ Since } f(x)c^{-1} \in Q(R)[x; \sigma] \text{ so there exist regular element } e \in R \text{ and } f_2 \in R[x; \sigma], \text{ such that } ef(x)c^{-1} = f_2, \text{ by Lemma 2.1. Thus we have } f_2 f_1 = ef(x)c^{-1}cg(x) = 0 \text{ and hence there exists } 0 \neq r \in R \text{ such that } f_2 r = 0. \text{ So } ef(x)c^{-1}r = 0 \text{ and hence } f(x)c^{-1}r = 0. \text{ Thus the classical left quotient ring } Q(R) \text{ of } R \text{ is } \sigma \text{-skew McCoy.} \end{align*} \]

\[ \begin{align*}
\text{Corollary 2.3. Let } R \text{ be a right (resp. left) Ore ring. If } R \text{ is a right (resp. left) McCoy ring then the classical right (resp. left) quotient ring } Q(R) \text{ of } R \text{ is a right (resp. left) McCoy ring.} \end{align*} \]
**Lemma 2.4.** Let σ be an automorphism of a ring R and R be a σ-reversible ring. Then for each \( e = e^2 \in R \), \( σ(e) = e \) and R is an abelian ring.

**Proof.** Let \( e = e^2 \in R \). We have \( e(1 - e) = (1 - e)e = 0 \) and hence \( σ(e)(1 - e) = (1 - e)σ(e) = σ(1 - e)e = eσ(1 - e) = 0 \). Thus \( σ(e) = e \). Now let \( r ∈ R \). There exists \( s ∈ R \) such that \( σ(s) = r \). For each \( e = e^2 ∈ R \), \( (1 - e)es = e(1 - e)s = 0 \) and hence \( σ(es)(1 - e) = σ((1 - e)s)e = 0 \). Thus \( er(1 - e) = (1 - e)re = 0 \) and hence \( er = re \).

**Theorem 2.5.** Let R be a right (resp. left) Ore ring and σ an automorphism of R. If R is a right (resp. left) σ-reversible ring then the classical right (resp. left) quotient ring \( Q(R) \) of R is a right (resp. left) \( σ \)-reversible ring.

**Proof.** Let \( R \) be a right Ore, right \( σ \)-reversible ring and \( ab^{-1}, cd^{-1} ∈ Q(R) \) such that \( ab^{-1}cd^{-1} = 0 \). \( b^{-1}c = c_1b_1^{-1} \) for some \( c_1 ∈ R \) and regular element \( b_1 ∈ R \). So \( ac_1 = 0 \) and hence \( c_1σ(a) = 0 \). Since \( c_1σ(a)σ(b) = 0 \), then \( σ(ab)σ(c_1) = 0 \) and hence \( acb_1 = abc_1 = 0 \). Thus \( ac = 0 \) and hence \( cσ(a) = 0 \). There exist regular element \( d_1 ∈ R \) and \( a_1 ∈ R \) such that \( σ(a)d_1 = da_1 \). So \( cσ(a)d_1 = 0 \) and hence \( da_1σ(c) = σ(a)d_1σ(c) = 0 \). Thus \( a_1σ(c) = 0 \) and so \( σ(c)a_1 = 0 \). \( cd^{-1}σ(ab^{-1}) = cd^{-1}σ(a)σ(b)^{-1} = cd^{-1}(da_1d_1^{-1})σ(b)^{-1} = ca_1d_1^{-1}σ(b)^{-1} = 0 \) and hence \( Q(R) \) is a right \( σ \)-reversible ring. Now assume that \( R \) is a left \( σ \)-reversible ring and \( ab^{-1}, cd^{-1} ∈ Q(R) \) such that \( ab^{-1}cd^{-1} = 0 \). \( b^{-1}c = c_1b_1^{-1} \) for some \( c_1 ∈ R \) and regular element \( b_1 ∈ R \). So \( ac_1 = 0 \) and hence \( σ(c_1)a = 0 \). Since \( σ(c_1)ab = 0 \), then \( σ(ab)σ(c_1) = 0 \) and hence \( acb_1 = abc_1 = 0 \). Thus \( ac = 0 \) and hence \( σ(c)a = 0 \). There exist regular element \( d_1 ∈ R \) and \( a_1 ∈ R \) such that \( ad_1 = σ(a)d_1 \). So \( σ(c)ad_1 = 0 \) and hence \( σ(c)σ(d)a_1 = σ(c)ad_1 = 0 \) and then \( σ(a_1)σ(c)d = 0 \). Thus \( a_1c = 0 \) and so \( σ(c)a_1 = 0 \). \( σ(c)d^{-1}ab^{-1} = σ(c)σ(d)^{-1}σ(d)ad_1^{-1}b^{-1} = 0 \) and hence \( Q(R) \) is a left \( σ \)-reversible ring.

**Corollary 2.6.** Let R be a right (resp. left) Ore ring. If R is a reversible ring then the classical right (resp. left) quotient ring \( Q(R) \) of R is a reversible ring.

**Theorem 2.7.** Let R be a semiprime right Goldie ring and σ an automorphism of R. Then the following are equivalent:

1. R is σ-skew McCoy, σ-reversible;
2. Q is σ-skew McCoy, σ-reversible;
3. R is σ-skew Armendariz;
4. Q is σ-skew Armendariz;
5. R is σ-rigid;
6. Q is σ-rigid;

where Q is the classical right quotient ring of R.

**Proof.** The statements 3, 4, 5 and 6 are equivalent by [3, Theorem 4.9.]. For the implication 1 → 5, Q is σ-skew McCoy by Theorem 2.2. Also Q is a semisimple artinian by Goldie’s Theorem and hence \( Q ∼ M_{n_1}(D_1) × ⋯ × M_{n_k}(D_k) \), where \( D_i \) is a division ring and \( n_i \) is a positive integer for each \( 1 ≤ i ≤ k \). By Theorem 2.5. and Lemma 2.4., Q is an abelian ring. So for each \( 1 ≤ i ≤ k \), \( n_i = 1 \) and hence Q
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is a reduced ring. Thus \( R \) is a reduced \( \sigma \)-reversible and hence \( \sigma \)-rigid ring. If \( R \) is a \( \sigma \)-rigid it is easy to see that \( R \) is a \( \sigma \)-skew McCoy, \( \sigma \)-reversible ring. By the same argument we have the equivalence (2) \( \Leftrightarrow \) (6).

**Theorem 2.8.** Let \( R \) be a von Neumann regular ring and \( \alpha \) an automorphism of \( R \). Then the following statements are equivalent:

1. \( R \) is \( \sigma \)-skew McCoy, \( \sigma \)-reversible;
2. \( Q \) is \( \bar{\sigma} \)-skew McCoy, \( \bar{\sigma} \)-reversible;
3. \( R \) is \( \sigma \)-skew Armendariz;
4. \( Q \) is \( \bar{\sigma} \)-skew Armendariz;
5. \( R \) is \( \sigma \)-rigid;
6. \( Q \) is \( \bar{\sigma} \)-rigid;

where \( Q \) is the classical right quotient ring of \( R \).

**Proof.** Since any von Neumann regular ring is an Ore ring and for each automorphism \( \sigma \), \( \sigma(c) \) is regular for each regular element \( c \in R \), then the statements 3, 4, 5 and 6 are equivalent by [3, Theorem 6.9.]. The implication 1 \( \Rightarrow \) 5, follows from Theorems 2.2., 2.5., Lemma 2.4. and the fact that abelian von Neumann regular rings are reduced. If \( R \) is a \( \sigma \)-rigid it is easy to see that \( R \) is a \( \sigma \)-skew McCoy, \( \sigma \)-reversible ring. By the same argument we have the equivalence (2) \( \Leftrightarrow \) (6).

**Corollary 2.9.** Let \( R \) be a von Neumann regular ring. Then the following statements are equivalent:

1. \( R \) is right (left) McCoy, reversible;
2. \( Q \) is right (left) McCoy, reversible;
3. \( R \) is Armendariz;
4. \( Q \) is Armendariz;
5. \( R \) is reduced;
6. \( Q \) is reduced;

where \( Q \) is the classical right quotient ring of \( R \).

**Proposition 2.10.** The matrix ring \( M_n(R) \) is not right linearly McCoy, for any ring \( R \) and \( n > 1 \).

**Proof.** Let \( f(x) = \begin{pmatrix} A & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \) and \( g(x) = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \), where \( A, B, C, D \in M_2(R) \) such that, \( A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \). Thus \( f(x), g(x) \in M_n(R)[x] \) and \( f(x)g(x) = 0 \). But if \( E \in M_n(R) \) and \( f(x)E = 0 \) then \( E = 0 \). So \( M_n(R) \) is not right linearly McCoy ring.

Let \( R \) be a ring and \( \sigma = 0 \) be a zero homomorphism of \( M_n(R) \). It is easy to see that \( M_n(R) \) is \( \sigma \)-skew McCoy ring. Proposition 2.10. motivate the following question:
Question 2.11. Is it true that for any ring $R$, any positive integer $n > 1$ and any nonzero homomorphism $\sigma$ of $M_n(R)$, $M_n(R)$ is not $\sigma$-skew McCoy?

Theorem 2.12. Let $R$ be a semiprime right Goldie ring. Then the following are equivalent:

1. $R$ is a right McCoy ring;
2. $Q$ is a right McCoy ring;
3. $R$ is a right linearly McCoy ring;
4. $Q$ is a right linearly McCoy ring;
5. $R$ is a McCoy ring;
6. $Q$ is a McCoy ring;
7. $R$ is an Armendariz ring;
8. $Q$ is an Armendariz ring;
9. $R$ is a weak Armendariz ring;
10. $Q$ is a weak Armendariz ring;
11. $R$ is a reversible ring;
12. $Q$ is a reversible ring;
13. $R$ is a semicommutative ring;
14. $Q$ is a semicommutative ring;
15. $R$ is a symmetric ring;
16. $Q$ is a symmetric ring;
17. $R$ is a 2-primal ring;
18. $Q$ is a 2-primal ring;
19. $R$ is a reduced ring;
20. $Q$ is a reduced ring;
21. $Q$ is a finite direct product of division rings;

where $Q$ is the classical right quotient ring of $R$.

Proof. The statements 7, 8, 9, 10, 13, 14, 19, 20 and 21 are equivalent by [3, Corollary 4.11]. If $R$ is a right McCoy then $Q$ is a right McCoy by Corollary 2.3. Also $Q$ is a semisimple artinian by Goldie’s Theorem and hence $Q \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where $D_i$ is a division ring and $n_i$ is a positive integer for each $1 \leq i \leq k$. So by [2, Theorem 4.4], for each $1 \leq i \leq k$, $M_{n_i}(D_i)$ is a right McCoy and hence $i = 1$, by Proposition 2.10. Thus $R$ is a reduced ring and hence, $R$ is 2-primal, symmetric, semicommutative, reversible and Armendariz. If $R$ is a symmetric ring, then $R$ is reversible and hence $Q$ is abelian semisimple. Thus $Q$ is a reduced ring and hence $R$ is reduced. Thus $R$ is a McCoy ring. If $R$ is 2-primal then $R$ is reduced, since $R$ is semiprime. So we have $1 \leftrightarrow 19$, $1 \leftrightarrow 17$, $1 \leftrightarrow 15$ and $1 \leftrightarrow 11$. By the same argument we have $2 \leftrightarrow 20$, $2 \leftrightarrow 18$, $2 \leftrightarrow 16$ and $2 \leftrightarrow 12$. If $R$ is right McCoy, then $Q$ is right McCoy and by the argument above $R$ and $Q$ are reduced. So we have $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6$ and the result follows.

We know that, there exists a McCoy ring with 1 which is not abelian [2, Theorem 7.1]. Theorem 2.12. shows that each semiprime right Goldie McCoy ring is abelian. Corollary 2.9. and Theorem 2.12. motivate the following question:
Question 2.13. Is it true that any von Neumann regular McCoy ring is abelian?

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