ON SKEW TRIANGULAR MATRIX RINGS

A. R. NASR-ISFAHANI

Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran.
E-mail: a_nasr_isfahani@yahoo.com

Abstract. For a ring \( R \), endomorphism \( \alpha \) of \( R \) and positive integer \( n \) we define a skew triangular matrix ring \( T_n(R, \alpha) \). By using an ideal theory of a skew triangular matrix ring \( T_n(R, \alpha) \) we can determine prime, primitive, maximal ideals and radicals of the ring \( R[x; \alpha]/(x^n) \), for each positive integer \( n \), where \( R[x; \alpha] \) is the skew polynomial ring, and \( (x^n) \) is the ideal generated by \( x^n \).

Keywords : Jacobson radical, Prime ideals, Skew polynomial rings.
AMS Subject Classification: 16S36; 16N20

1. Introduction

Throughout this paper \( R \) denotes an associative ring with unity and \( \alpha : R \to R \) is an endomorphism. We denote \( R[x; \alpha] \) the Ore extension whose elements are the polynomials \( \sum_{i=0}^n r_i x^i \), \( r_i \in R \), where the addition is defined as usual and the multiplication subject to the relation \( xa = \alpha(a)x \) for any \( a \in R \). For a ring \( R \), endomorphism \( \alpha \) of \( R \) and positive integer \( n \), we define skew triangular matrix ring

\[
T_n(R, \alpha) := \left\{ \begin{pmatrix}
  a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
  0 & a_0 & a_1 & \cdots & a_{n-2} \\
  & 0 & a_0 & \cdots & a_{n-3} \\
  & & \ddots & \ddots & \vdots \\
  & & & 0 & a_0
\end{pmatrix} \mid a_i \in R \right\},
\]

with addition pointwise and multiplication given by

\[
\begin{pmatrix}
  a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
  0 & a_0 & a_1 & \cdots & a_{n-2} \\
  & 0 & a_0 & \cdots & a_{n-3} \\
  & & \ddots & \ddots & \vdots \\
  & & & 0 & a_0
\end{pmatrix}
\begin{pmatrix}
  b_0 & b_1 & b_2 & \cdots & b_{n-1} \\
  0 & b_0 & b_1 & \cdots & b_{n-2} \\
  & 0 & b_0 & \cdots & b_{n-3} \\
  & & \ddots & \ddots & \vdots \\
  & & & 0 & b_0
\end{pmatrix} =
\begin{pmatrix}
  c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
  0 & c_0 & c_1 & \cdots & c_{n-2} \\
  & 0 & c_0 & \cdots & c_{n-3} \\
  & & \ddots & \ddots & \vdots \\
  & & & 0 & c_0
\end{pmatrix},
\]

where

\[ c_i = a_0 \alpha^0(b_i) + a_1 \alpha^1(b_{i-1}) + \cdots + a_i \alpha^i(b_0) \] for each \( 0 \leq i \leq n - 1 \). We denote elements of \( T_n(R, \alpha) \) by \( (a_0, a_1, \ldots, a_{n-1}) \). If \( \alpha \) be an identity, \( T_n(R, \alpha) \) is a subring of triangular matrix ring and we denote it by \( T_n(R) \).
There is a ring isomorphism \( \varphi : R[x; \alpha]/\langle x^n \rangle \to T_n(R, \alpha) \), given by, \( \varphi(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + \langle x^n \rangle) = (a_0, a_1, \cdots, a_{n-1}) \), with \( a_i \in R \), \( 0 \leq i \leq n - 1 \). So \( T_n(R, \alpha) \cong R[x; \alpha]/\langle x^n \rangle \), where \( R[x; \alpha] \) is the skew polynomial rings and \( \langle x^n \rangle \) is the ideal generated by \( x^n \).

Ore extensions, since their discovery in 1933, have been very important in constructing interesting mathematical objects. They have provided many important (counter)examples in ring theory. This topic has been extensively studied in various directions for a few decades. Recently the authors of \([2, 5, 6]\) studied some ring theory properties of \( R[x; \alpha]/\langle x^n \rangle \). In this note we continue study of \( R[x; \alpha]/\langle x^n \rangle \).

We shall try to describe the left, right and 2-sided ideals in \( T_n(R, \alpha) \) in terms of those in \( R \). We show that, there exists a one to one correspondence between the prime (left(right) primitive, semiprime, completely prime, completely semiprime, maximal left(right)) ideals of \( R \) and those of \( T_n(R, \alpha) \), respectively. Using this characterization of the ideals of the ring \( T_n(R, \alpha) \), we will show that there is a satisfactory classification of the radicals of the ring \( R[x; \alpha]/\langle x^n \rangle \).

In particular we determine the lower nil radical, the Levitzky radical, the upper nil radical and the Jacobson radical of the ring \( R[x; \alpha]/\langle x^n \rangle \). As an application, we prove that:

A ring \( R \) is Jacobson if and only if \( R[x; \alpha]/\langle x^n \rangle \) is Jacobson (i.e., a ring for which every prime ideal is semiprimitive).

A ring \( R \) is 2-primal if and only if \( R[x; \alpha]/\langle x^n \rangle \) is 2-primal (i.e., every minimal prime ideal \( P \subseteq R \) be completely prime (i.e. \( R/P \) be a domain)).

A ring \( R \) is left(right) quasi-duo if and only if \( R[x; \alpha]/\langle x^n \rangle \) is left(right) quasi-duo (i.e., every maximal left(right) ideal of \( R \) is two-sided).

We use \( \text{Nil}_*(R) \), \( L\text{-rad}R \), \( \text{Nil}^*(R) \), \( J(R) \) and \( \text{Nil}(R) \), to denote the lower nil radical (i.e., the prime radical), the Levitzky radical (i.e., sum of all locally nilpotent ideals), the upper nil radical (i.e., sum of all nil ideals), the Jacobson radical and the set of all nilpotent elements of \( R \), respectively. Wedderburn radical of a ring \( R \) is the largest nilpotent ideal in \( R \).

2. Prime and semiprime ideals of \( T_n(R, \alpha) \)

In this section we first show that, there exists a one to one correspondence between the prime (semiprime) ideals of \( R \) and those of \( T_n(R, \alpha) \). Then we shall determine the upper nil radical of the ring \( R[x; \alpha]/\langle x^n \rangle \), and deduce that, for every ring \( R \) and positive integer \( n \geq 2 \), the ring \( T_n(R, \alpha) \) and hence \( R[x; \alpha]/\langle x^n \rangle \) satisfies the Köthe’s Conjecture.

We start by proving the following result on the classification of (left, right) ideals in the ring \( T_n(R, \alpha) \).

**Lemma 2.1.** Let \( R \) be a ring, \( \alpha \) an endomorphism of \( R \) and \( n \) a positive integer. If for \( 0 \leq i \leq n - 1 \), \( \{I_i\} \) is a family of left ideals of \( R \), such that \( \alpha(I_j) \subseteq I_{j+1} \), for each \( 0 \leq j \leq n-2 \), then \( J = \{(a_0, a_1, \cdots, a_{n-1}) \mid a_i \in I_i \} \) is a left ideal of \( T_n(R, \alpha) \).

**Lemma 2.2.** Let \( R \) be a ring, \( \alpha \) an endomorphism of \( R \) and \( n \) a positive integer. If for \( 0 \leq i \leq n - 1 \), \( \{I_i\} \) is a family of right ideals of \( R \), such that \( I_j \subseteq I_{j+1} \), for each \( 0 \leq j \leq n-2 \), then \( J = \{(a_0, a_1, \cdots, a_{n-1}) \mid a_i \in I_i \} \) is a right ideal of
Let $R$ be a ring, $\alpha$ an epimorphism of $R$ and $n$ a positive integer. Then for each right ideal $I$ of $T_n(R, \alpha)$ and $0 \leq i \leq n-1$, $I_i = \{a \in R \mid (b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2}) \in I, \text{ for some } b_0, b_1, \cdots, b_{n-2} \in R\}$ is a right ideal of $R$ and $I_i \subseteq I_{i+1}$, for each $0 \leq i \leq n-2$.

**Proof.** Let $I$ be a right ideal of $T_n(R, \alpha)$. Let $a \in I$ and $s \in R$, so there are $b_0, \cdots, b_{n-2} \in R$, such that $(b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2}) \in I$. Let $r \in R$ such that $\alpha^i(r) = s$. Since $I$ is a right ideal of $T_n(R, \alpha)$, $(b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2})(r, 0, \cdots, 0) \in I$. Hence $aa^i(r) = as \in I_i$. Next we see that for each $0 \leq i \leq n-2$, $I_i \subseteq I_{i+1}$. Let $a \in I_i$. So there are $b_0, \cdots, b_{n-2} \in R$, such that $(b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2}) \in I$. Since $I$ is a right ideal of $T_n(R, \alpha)$, $(b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2})(0, 1, 0, \cdots, 0) = (c_0, \cdots, c_i, a, c_{i+1}, \cdots, c_{n-2}) \in I$. So $a \in I_{i+1}$ and the result follows.

Theorem 2.3. shows that right ideal structures in $T_n(R, \alpha)$ are closely tied to those of the ring $R$. Therefore we denote each right ideal $I$ of $T_n(R, \alpha)$ by $I = (I_0, I_1, \cdots, I_{n-1}) \cap I$, where for $0 \leq i \leq n-1$, $I_i$ is the right ideal of $R$ uniquely determined in Theorem 2.3.

The following example shows that the epimorphism condition in Theorem 2.3. is not superfluous.

**Example 2.4.** Let $F$ be a field and for each positive integer $i$, $R_i = F[t_i]$ be the polynomial ring with indeterminate $t_i$ and $R = \prod_{i=1}^{\infty} R_i$. Define $\alpha : R \to R$ given by $\alpha(f_1(t_1), f_2(t_2), \cdots) = (f_1(0), f_1(t_2), f_2(t_3), \cdots)$. It is easy to see that $\alpha$ is a non-surjective monomorphism of $R$. Let $I$ be a right ideal of $T_2(R, \alpha)$, generated by $(0, 1)$. We show that $I_1 = \{a \in R \mid (b, a) \in I, \text{ for some } b \in R\}$. Since $I_1 \subseteq I$ but, $(1, 1, 1, \cdots) \not\in I_1$. Thus $I_1$ is a right ideal of $R$ and $I$ is the right ideal of $R$ uniquely determined in Theorem 2.3.

**Theorem 2.5.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. Then for each left ideal $I$ of $T_n(R, \alpha)$ and $0 \leq i \leq n-1$, $I_i = \{a \in R \mid (b_0, \cdots, b_{i-1}, a, b_i, \cdots, b_{n-2}) \in I, \text{ for some } b_0, b_1, \cdots, b_{n-2} \in R\}$ is a left ideal of $R$ and $\alpha(I_j) \subseteq I_{j+1}$, for each $0 \leq j \leq n-2$.

**Proof.** The proof is similar to that of the Theorem 2.3.

We denote each left ideal $I$ of $T_n(R, \alpha)$ by $I = (I_0, I_1, \cdots, I_{n-1}) \cap I$, where for each $0 \leq i \leq n-1$, $I_i$ is the left ideal of $R$ uniquely determined in Theorem 2.5.

The following example shows that, for an ideal $I$ of the ring $T_n(R, \alpha)$ and the ideals $I_i$’s introduced in Theorems 2.3 and 2.5, it is no need to have $I = \cdots$
Let $Z$ be the ring of integers and $id$ be the identity endomorphism of $Z$. Let $R = T_2(Z, id)$ and $I$ be the ideal of $R$ generated by $(2, 1)$, and consider the ideals $I_0 = 2Z$ and $I_1 = Z$. We show that $I \neq (2Z, Z)$. We first observe that, each element $x \in I$ is of the form $x = (2a, 2b + a)$, for some $a, b \in Z$. We have $x = (a_1, b_1)(2, 1) + \cdots + (a_n, b_n)(2, 1) = (2(a_1 + \cdots + a_n), (a_1 + \cdots + a_n) + 2(b_1 + \cdots + b_n))$, for $a_i, b_i \in Z$, with $1 \leq i \leq n$. But $(0, 1) \in (2Z, Z)$, and $(0, 1) \notin I$, since otherwise, $(0, 1) \in I$, implies that $(0, 1) = (2a, 2b + a)$, for some $a, b \in Z$, which is a contradiction.

Notice that, for a left (right) Noetherian ring $R$ and automorphism $\alpha$ of $R$, $R[x; \alpha]$ is left (right) Noetherian, by skew Hilbert basis theorem. But the skew Hilbert basis theorem can fail when $\alpha$ is not an automorphism.

Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. Then $T_n(R, \alpha)$ is finitely generated left $R$-module and hence $R$ is a left Noetherian (resp., left Artinian) if and only if $T_n(R, \alpha)$ is a left Noetherian (resp., left Artinian). In case $\alpha$ is an epimorphism of $R$, $T_n(R, \alpha)$ is finitely generated right $R$-module and hence $R$ is a right Noetherian (resp., right Artinian) if and only if $T_n(R, \alpha)$ is a right Noetherian (resp., right Artinian). The argument above motivate the following question.

**Question 2.7.** Let $\alpha$ be an endomorphism of ring $R$ which is not epimorphism. Is it true that if a ring $R$ is right Noetherian (resp., right Artinian), then $T_n(R, \alpha)$ is a right Noetherian (resp., right Artinian)?

Now we show that, there exists a one to one correspondence between the prime(resp., semiprime) ideals of $R$ and those of $T_n(R, \alpha)$.

**Theorem 2.8.** An ideal $Q$ of the ring $T_n(R, \alpha)$ is prime (resp., semiprime) if and only if $Q = (P, R, \cdots, R)$, for some prime (resp., semiprime) ideal $P$ of $R$.

**Proof.** Let $P$ be a prime (resp., semiprime) ideal of $R$ then $Q = (P, R, \cdots, R)$ is an ideal of $T_n(R, \alpha)$. Suppose that $(x_1, \cdots, x_n)T_n(R, \alpha)(y_1, \cdots, y_n) \subseteq Q$, with $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in T_n(R, \alpha)$. Since $P$ is a prime ideal of $R$ and $x_1Ry_1 \subseteq P$, $x_1 \in P$ or $y_1 \in P$. Then $(x_1, \cdots, x_n) \in Q$ or $(y_1, \cdots, y_n) \in Q$ and hence $Q$ is a prime ideal of $T_n(R, \alpha)$. Conversely assume that $Q$ is a prime (resp., semiprime) ideal of $T_n(R, \alpha)$. Then $(0, 0, \cdots, 0, 1)T_n(R, \alpha)(0, 0, \cdots, 0, 1) = 0 \in Q$, so $(0, 0, \cdots, 0, 1) \in Q$. But $(0, 0, \cdots, 1, 0)T_n(R, \alpha)(0, 0, \cdots, 1, 0) = (0, 0, \cdots, 1, 0) \in Q$, so $(0, 0, \cdots, 1, 0) \in Q$. Continuing in this way, after $n - 1$ steps, we get $(0, 1, \cdots, 0)T_n(R, \alpha)(0, 1, \cdots, 0) = (0, 0, 1, 0, \cdots, 0) \in Q$, and hence $(0, 1, \cdots, 0) \in Q$. Let $Q_1 := \{a \in R \mid (a, b_2, \cdots, b_n) \in Q, \text{ for some } b_2, \cdots, b_n \in R\}$. $Q_1$ is an ideal of $R$. Since $(0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1) \in Q$, then $Q = (Q_1, R, \cdots, R)$. Now we show that $Q_1$ is a prime (resp., semiprime) ideal of $R$. Assume that $aRb \subseteq Q_1$ with $a, b \in R$. Then $(a, 0, 0, \cdots, 0)T_n(R, \alpha)(b, 0, 0, \cdots, 0) \subseteq Q$, and hence $(a, 0, 0, \cdots, 0) \in Q$ or $(b, 0, 0, \cdots, 0) \in Q$. So $a \in Q_1$ or $b \in Q_1$, and hence $Q_1$ is a prime (resp., semiprime) ideal of $R$, and the result follows.
Corollary 2.9. An ideal $Q$ of the ring $R[x;\alpha]/\langle x^n \rangle$, is prime (resp., semiprime) if and only if $Q = P + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$, for some prime (resp., semiprime) ideal $P$ of $R$.

By the above results we deduce that $\text{Nil}_4(T_n(R,\alpha)) = (\text{Nil}_4(R),R,\cdots,R)$ and hence $\text{Nil}_4(R[x;\alpha]/\langle x^n \rangle) = \text{Nil}_4(R) + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$.

The following result shows that, there exists a one to one correspondence between the completely prime (resp., semiprime) ideals of $R$ and those of $T_n(R,\alpha)$.

Theorem 2.10. An ideal $Q$ of the ring $T_n(R,\alpha)$, is completely prime (resp., semiprime) if and only if $Q = (P,R,\cdots,R)$, for some completely prime (resp., semiprime) ideal $P$ of $R$.

Proof. The proof is similar to that of Theorem 2.8.

Corollary 2.11. An ideal $Q$ of the ring $R[x;\alpha]/\langle x^n \rangle$, is completely prime (resp., semiprime) if and only if $Q = P + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$, for some completely prime (resp., semiprime) ideal $P$ of $R$.

In a commutative ring, the set of nilpotent elements form an ideal that coincides with the intersection of all prime ideals. This property is also possessed by certain noncommutative rings, which are known as 2-primal rings. A necessary and sufficient condition for a ring $R$ to be 2-primal is that every minimal prime ideal $P$ of $R$ be completely prime (i.e. $R/P$ be a domain); this result is Proposition 1.11 in [10]. In [8] G. Marks examined several conditions on a noncommutative ring which imply that it is 2-primal. G.F. Birkenmeier et.al, proved [1, Proposition 2.6] that the 2-primal condition is inherited by ordinary polynomial extensions.

As a corollary, we may extend the class of 2-primal rings from the skew triangular matrix rings over a 2-primal ring:

Theorem 2.12. A ring $R$ is 2-primal if and only if $T_n(R,\alpha)$ is 2-primal.

Proof. Assume that $R$ is 2-primal and $(a_1,\cdots,a_n) \in \text{Nil}_4(T_n(R,\alpha))$. Then for some positive integer $m$, $(a_1,\cdots,a_n)^m = 0$, which gives $a_1^m = 0$ and hence $a_1 \in \text{Nil}_4(R)$, since $R$ is 2-primal. So $(a_1,\cdots,a_n) \in (\text{Nil}_4(R),R,\cdots,R) = \text{Nil}_4(T_n(R,\alpha))$, and hence $T_n(R,\alpha)$ is 2-primal. Conversely, assume that $T_n(R,\alpha)$ is 2-primal. Since $(R,0,\cdots,0) \subseteq T_n(R,\alpha)$, we deduce from [1, Proposition 2.2] that $R \cong (R,0,\cdots,0)$ is also 2-primal.

Corollary 2.13. A ring $R$ is 2-primal if and only if $R[x;\alpha]/\langle x^n \rangle$ is 2-primal.

There is a famous unsolved problem: If $R$ has no nonzero nil ideals, does it follows that $R$ has no nonzero nil one-sided ideals. The truth of this was Conjectured many years ago by G. Köthe. In spite of the many great advances made in ring theory in recent times, Köthe’s Conjecture has remained unsolved in general. For several special classes of rings, the Conjecture has been shown to be true. Next we shall determine the upper nil radical of the ring $R[x;\alpha]/\langle x^n \rangle$, and deduce that, for
every ring $R$ and positive integer $n \geq 2$, the ring $T_n(R, \alpha)$ and hence $R[x; \alpha]/(x^n)$ satisfies the Köthe Conjecture.

**Theorem 2.14.** Let $R$ be a ring, the upper nil radical of the ring $T_n(R, \alpha)$, \( \text{Nil}^*(T_n(R, \alpha)) = (\text{Nil}^*(R), R, \cdots, R) \).

**Proof.** Since $\text{Nil}_*(T_n(R, \alpha)) \subseteq \text{Nil}^*(T_n(R, \alpha))$ and $\text{Nil}_*(T_n(R, \alpha)) = (\text{Nil}_*(R), R, \cdots, R)$, we have $(0, 0, \cdots, 0, 1), (0, 0, \cdots, 0, 1, 0), \cdots, (0, 1, \cdots, 0) \in \text{Nil}^*(T_n(R, \alpha))$. On the other hand $A = \{a \in R \mid (a, a_2, \cdots, a_n) \in \text{Nil}^*(R, a)\}$ is an ideal of $R$, so we have $\text{Nil}^*(T_n(R, \alpha)) = (A, R, \cdots, R)$. Now we show that $A = \text{Nil}^*(R)$. To see this, let $a \in A$, so we have $(ras, 1, 1, \cdots, 1) \in \text{Nil}^*(T_n(R, \alpha))$, for each $r, s \in R$. So $(ras, 1, 1, \cdots, 1)^m = 0$ and hence $(ras)^m = 0$, for some positive integer $m$, so $a \in \text{Nil}^*(R)$. Conversely, assume that $a \in \text{Nil}^*(R)$. So the principal ideal $(a)$ is nilpotent in $R$. Then for each $r, s \in R$, $ras$ is nilpotent and hence $(ras, 0, 0, \cdots, 0)$ is nilpotent. Thus $(a, 0, \cdots, 0) \in \text{Nil}^*(T_n(R, \alpha))$ and that $a \in A$. Hence $A = \text{Nil}^*(R)$. Therefore we have $\text{Nil}^*(T_n(R, \alpha)) = (\text{Nil}^*(R), R, \cdots, R)$.

**Corollary 2.15.** Let $R$ be a ring, the upper nil radical of the ring $R[x; \alpha]/(x^n)$, \( \text{Nil}^*(R[x; \alpha]/(x^n)) = \text{Nil}^*(R) + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle \).

Recall that Köthe’s conjecture is equivalent to the condition that, if $\text{Nil}^*(R) = 0$, then $R$ has no non zero nil one-sided ideals.

**Theorem 2.16.** For every ring $R$ and positive integer $n \geq 2$, the ring $T_n(R, \alpha)$ and hence $R[x; \alpha]/(x^n)$ satisfies the Köthe Conjecture.

**Proof.** If $\text{Nil}^*(R[x; \alpha]/(x^n)) = 0$, then $R = 0$, by Corollary 2.15. Thus $R$ has no non zero nil one-sided ideal. Therefore for any ring $R$ and any positive integer $n \geq 2$, $R[x; \alpha]/(x^n)$ satisfies the Köthe’s conjecture.

**Corollary 2.17.** For every ring $R$ and positive integer $n \geq 2$, \( J(R[x; \alpha]/(x^n))[y] = (\text{Nil}^*R + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle)[y] \).

**Proof.** Since the ring $R[x; \alpha]/(x^n)$ satisfies the Köthe’s conjecture, by Theorem 2.16., the result follows by [4, Exercise 10.25].

**Corollary 2.18.** For every ring $R$ and positive integer $n \geq 2$, \( \text{Nil}^*(M_n(T_n(R, \alpha))) = M_n((\text{Nil}^*(R), R, \cdots, R)) \).

**Proof.** Since the ring $T_n(R, \alpha)$ satisfies the Köthe’s conjecture, by Theorem 2.16., the result follows by [4, Exercise 10.25].

**Theorem 2.19.** Let $R$ be a ring, the Levitzky radical of the ring $T_n(R, \alpha)$, \( \text{L-rad}(T_n(R, \alpha)) = (\text{L-rad}(R), R, \cdots, R) \).

**Proof.** Since $\text{Nil}_*(T_n(R, \alpha)) = (\text{Nil}_*(R), R, \cdots, R) \subseteq \text{L-rad}(T_n(R, \alpha))$, so $(0, 1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1, 0), (0, 0, \cdots, 0, 1) \in \text{L-rad}(T_n(R, \alpha))$. Thus we have, $\text{L-rad}(T_n(R, \alpha)) = (I, R, \cdots, R)$, for some ideal $I$ of $R$. By the same method


Corollary 2.20. Let \( R \) be a ring, the Levitzky radical of the ring \( R[x;\alpha]/(x^n) \), \( L-rad(R[x;\alpha]/(x^n)) = L-rad(R) + Rx + \cdots + Rx^{n-1} + (x^n) \).

3. Primitive and maximal ideals of \( T_n(R,\alpha) \)

In this section we determine left(right) primitive and maximal left(right) ideals of the ring \( T_n(R,\alpha) \) and hence those of \( R[x;\alpha]/(x^n) \), and as an application, we shall determine the Jacobson radical and the Brown-McCoy radical of the ring \( R[x;\alpha]/(x^n) \).

Theorem 3.1. Let \( R \) be a ring, \( \alpha \) an endomorphism of \( R \) and \( n \) a positive integer. A left ideal \( M \) of the ring \( T_n(R,\alpha) \) is a maximal left ideal if and only if \( M = (M_1, R, \cdots, R) \), for some maximal left ideal \( M_1 \) of \( R \).

Proof. Let \( M_1 \) be a maximal left ideal of \( R \). We show that \( M = (M_1, R, \cdots, R) \) is a maximal left ideal of \( T_n(R,\alpha) \). It is easy to show that \( M \) is a left ideal of \( T_n(R,\alpha) \). Let \( M' \) be a left ideal of \( T_n(R,\alpha) \) with \( M \subseteq M' \). \( M' = (M'_1, M'_2, \cdots, M'_n) \subseteq M' \), where \( M'_i \) is a left ideal of \( R \), for each \( 1 \leq i \leq n \). So \( M \subseteq (M'_1, M'_2, \cdots, M'_n) \) and \( M \subseteq (M'_1, M'_2, \cdots, M'_n) \). Thus \( M_1 \subseteq M'_1 \) and \( M'_i = R \) for \( 2 \leq i \leq n \), so \( M = M' \) or \( M' = T_n(R,\alpha) \). Conversely, assume that \( M \) is a maximal left ideal of \( T_n(R,\alpha) \). By Theorem 2.5., \( M = (M_1, M_2, \cdots, M_n) \cap M \) for some left ideals \( M_i \) of \( R \), with \( 1 \leq i \leq n \). Consider \( M' = (M_1, R, \cdots, R) \). Since \( M \subseteq M' \) and \( M \) is a maximal left ideal, we have \( M = M' \), so \( M = (M_1, R, \cdots, R) \). Thus it is enough to show that \( M_1 \) is a maximal left ideal of \( R \). Assume that \( M_1 \subseteq N \) for some left ideal \( N \) of \( R \). It is easy to see that \( (N, R, \cdots, R) \) is a left ideal of \( T_n(R,\alpha) \) and we have \( M = (M_1, R, \cdots, R) \subseteq (N, R, \cdots, R) \). But \( M \) is a maximal left ideal of \( T_n(R,\alpha) \), so \( M = (N, R, \cdots, R) \) or \( (N, R, \cdots, R) = T_n(R,\alpha) \). It implies that \( M_1 \) is a maximal left ideal of \( R \).

As an application of Theorem 3.1., we shall determine the Jacobson radical of the ring \( T_n(R,\alpha) \).

Corollary 3.2. Let \( R \) be a ring, \( \alpha \) an endomorphism of \( R \) and \( n \) a positive integer. Then \( J(T_n(R,\alpha)) = (J(R), R, \cdots, R) \).

Proof. Assume that \( J(R) = \bigcap M_i \), where \( M_i \) is a maximal left ideal of \( R \). So \( J(T_n(R,\alpha)) = \bigcap (M_i, R, \cdots, R) = (\bigcap M_i, R, \cdots, R) = (J(R), R, \cdots, R) \).

Corollary 3.3. Let \( R \) be a ring, \( \alpha \) an endomorphism of \( R \) and \( n \) a positive integer. A right ideal \( M \) of the ring \( T_n(R,\alpha) \) is a maximal right ideal if and only if \( M = (M_1, R, \cdots, R) \), for some maximal right ideal \( M_1 \) of \( R \).

Proof. Let \( M_1 \) be a maximal right ideal of \( R \), then it is easy to see that \( (M_1, R, \cdots, R) \) is a maximal right ideal of \( T_n(R,\alpha) \). Now let \( M \) be a maximal right ideal of \( T_n(R,\alpha) \). So \( J(T_n(R,\alpha)) = (J(R), R, \cdots, R) \subseteq M \) and hence \( M = (M_1, R, \cdots, R) \) for some right ideal \( M_1 \) of \( R \). By the using the same argument in the proof of
Theorem 3.1. we can see that $M_1$ is a maximal right ideal of $R$.

**Corollary 3.4.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. Then $J(R[x;\alpha]/(x^n)) = \{ a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mid a_0 \in J(R), a_i \in R, 1 \leq i \leq n-1 \} = J(R) + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$.

Notice by Corollary 3.4. that, the ring $T_n(R,\alpha)$ is semiprimitive if and only if $R = 0$.

A Jacobson ring is a ring for which every prime ideal is semiprimitive. Watters [11] proved that $R[x]$ is Jacobson for any Jacobson ring $R$ and Goldie and Michler [3] showed that if $R$ is a Noetherian Jacobson and $\alpha$ an automorphism of $R$ then $R[x;\alpha]$ is a Jacobson. On the other hand, an example has been constructed of a non-Noetherian commutative Jacobson ring $R$ with a skew polynomial extension $R[x;\alpha]$ that is not Jacobson; see Pearson and Stephenson [9]. Now we show that for any positive integer $n$, $R$ is a Jacobson ring if and only if $R[x;\alpha]/\langle x^n \rangle$ is a Jacobson ring.

**Theorem 3.5.** A ring $R$ is a Jacobson ring if and only if $R[x;\alpha]/\langle x^n \rangle$ is a Jacobson ring.

**Proof.** By Theorem 2.8., every prime ideal $Q$ of $R[x;\alpha]/\langle x^n \rangle$ is of the form $Q = P + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$, where $P$ is a prime ideal of $R$. We have also $R[x;\alpha]/\langle x^n \rangle = R + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$. So for each prime ideal $Q$ of $R[x;\alpha]/\langle x^n \rangle$, $(R[x;\alpha]/\langle x^n \rangle)/Q \cong R/P$, for some prime ideal $P$ of $R$ and the result follows.

A ring $R$ is called right (left) quasi-duo if every maximal right (left) ideal of $R$ is two-sided or, equivalently, every right (left) primitive homomorphic image of $R$ is a division ring. Leroy et. al. in [7] proved that for a ring $R$ with an endomorphism $\alpha$, $R[x;\alpha]$ is a right (left) quasi-duo if and only if $R$ is right (left) quasi-duo, $J(R[x;\alpha]) = (J(R) \cap N(R)) + N(R)[x;\alpha]$, $N(R)$ is a $\alpha$-stable ideal of $R$, the factor ring $R/N(R)$ is commutative and the endomorphism $\alpha$ induces identity on $R/N(R)$, where $N(R) = \{ a \in R \mid 3n \geq 1, \alpha a(a) \cdots \alpha^n(a) = 0 \}$. By Theorem 3.1. and Corollary 3.3. we have the following Corollary.

**Corollary 3.6.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. Then $R[x;\alpha]/\langle x^n \rangle$ is a left (right) quasi-duo if and only if $R$ is a left (right) quasi-duo.

**Theorem 3.7.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. A left(resp., right) ideal $M$ of the ring $T_n(R,\alpha)$ is a left(resp., right) primitive if and only if $M = (M_1, R, \cdots, R)$, for some left(resp., right) primitive ideal $M_1$ of $R$.

**Proof.** Let $M_1$ be a left(resp., right) primitive ideal of $R$, then $T_n(R,\alpha)/(M_1, R, \cdots, R) \cong R/M_1$ and hence $(M_1, R, \cdots, R)$ is a left(resp., right) primitive. Now let $M$ be a left(resp., right) primitive ideal of $T_n(R,\alpha)$. So $J(T_n(R,\alpha)) = (J(R), R, \cdots, R) \subseteq M$ and hence $M = (M_1, R, \cdots, R)$ for some
left (resp., right) ideal $M_1$ of $R$. $T_n(R,\alpha)/(M_1, R, \cdots, R)$ is a left (resp., right) primitive ring and hence $M_1$ is a left (resp., right) primitive ideal of $R$.

**Corollary 3.8.** A left (resp., right) ideal $M$ of the ring $R(x;\alpha)/(x^n)$ is a left (resp., right) primitive if and only if $M = M_1 + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$, for some left (resp., right) primitive ideal $M_1$ of $R$.

**Theorem 3.9.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. An ideal $M$ of the ring $T_n(R,\alpha)$ is a maximal ideal if and only if $M = (M_1, R, \cdots, R)$ for some maximal ideal $M_1$ of $R$.

**Proof.** The proof is similar to that of Theorem 3.1.

**Corollary 3.10.** Let $R$ be a ring, $\alpha$ an endomorphism of $R$ and $n$ a positive integer. An ideal $M$ of the ring $R[x;\alpha]/\langle x^n \rangle$ is a maximal ideal if and only if $M = M_1 + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$ for some maximal ideal $M_1$ of $R$.

The Brown-McCoy radical $\text{rad} R$ of $R$, is the intersection of all maximal ideals of $R$ and $J(R) \subseteq \text{rad} R$.

Now we shall try to relate the Brown-McCoy radical of a ring $R$ and to those of the ring $T_n(R,\alpha)$.

**Corollary 3.11.** Let $R$ be a ring, the Brown-McCoy radical of the ring $T_n(R,\alpha)$, $\text{rad}(T_n(R,\alpha)) = (\text{rad}(R), R, \cdots, R)$.

**Corollary 3.12.** Let $R$ be a ring, the Brown-McCoy radical of the ring $R[x;\alpha]/\langle x^n \rangle$, $\text{rad}(R[x;\alpha]/\langle x^n \rangle) = \text{rad}(R) + Rx + \cdots + Rx^{n-1} + \langle x^n \rangle$.

**Acknowledgment.** This research was partially supported by the Center of Excellence for Mathematics, University of Isfahan.

**REFERENCES**

