Invariant monotone vector fields on Riemannian manifolds

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Abstract

Various concepts of invariant monotone vector fields on Riemannian manifolds are introduced. Some examples of invariant monotone vector fields are given. Several notions of invexities for functions on Riemannian manifolds are defined and their relations with invariant monotone vector fields are studied.

Keywords: Generalized invex functions; Monotone vector fields; Invariant monotone vector fields; Riemannian manifolds

1. Introduction

The theory of variational inequality has proven its applicability in many fields of mathematics and physics. On the other hand monotonicity has played a very important role in the study of the existence of solutions of variational inequality problems. It is also well known that generalized monotonicity is a powerful tool in the study of the existence and the sensitivity analysis of solutions for variational inclusion and complementarity problems.

The convexity of a real valued function is equivalent to the monotonicity of the corresponding gradient function; see [3,5,19]. The relation between generalized convexity of functions and generalized monotone operators has been investigated by many authors; for example see [4,7,16,18]. In the nonsmooth case the relationships between monotonicity and convexity of subdifferential operators as an application of the approximate Mean Value Theorem are studied in Section 3.2.2 of [14].

In [11] Hanson introduced the concept of invexity which was a generalization of convexity. Generalized invexity and its relation with generalized invex monotonicity has been investigated in [16,23]. The notion of invariant monotonicity as a generalization of monotonicity was studied in [24].

In general a manifold is not a linear space and extensions of concepts and techniques from linear spaces to Riemannian manifolds are natural. Rapšcak [20] and Udriste [22] considered a generalization of convexity called geodesic convexity and extended many results of convex analysis and optimization theory to Riemannian manifolds. They also studied the monotonicity of the gradient of a geodesic convex function. The interested reader is referred to the books [20,22] and references therein.

The concept of monotone vector field on Riemannian manifolds which was a generalization of monotone operator was introduced in [15] by Németh. Since then numerous articles have appeared in the literature reflecting further generalizations and applications in this category; for example see [6,10].

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In [17] the notion of invex function on Riemannian manifolds was introduced and in [2] invex functions and preinvex functions on invex subsets of Riemannian manifolds are studied.

In this paper various concepts of generalized invexities for functions on Riemannian manifolds are introduced. Then, invariant monotone vector fields and generalized invariant monotone vector fields on Riemannian manifolds are defined and their relations with generalized invexities are investigated.

The organization of the paper is as follows.

In Section 2 some concepts and facts from Riemannian geometry are collected.

In Section 3 we recall the notion of invexity and introduce the concept of strong invexity on Riemannian manifolds. Section 4 is devoted to the notions of monotone and invariant monotone vector fields. Then the relation between invexity and monotonicity is studied.

Finally in Section 5 we introduce pseudoinvexity and invariant pseudomonotonicity of functions on Riemannian manifolds.

2. Preliminaries

In this section, we recall some definitions and known results concerning Riemannian manifolds which will be used throughout the paper. We refer the reader to [12,13] for the standard material of differential geometry.

Throughout this paper $M$ is a $C^\infty$ smooth manifold modelled on a Hilbert space $H$, either finite dimensional or infinite dimensional, endowed with a Riemannian metric $g_p(.,.)$ on the tangent space $T_pM \cong H$. Therefore, we have a smooth assignment of an inner product to each tangent space. It is usual to write

$$g_p(v, w) = \langle v, w \rangle_p \quad \text{for all } v, w \in T_pM.$$  

Thus, $M$ is now a Riemannian manifold. The corresponding norm is denoted by $\| . \|_p$. Let us recall that the length of a piecewise $C^1$ curve $\gamma : [a, b] \rightarrow M$ is defined by $L(\gamma) := \int_a^b \| \gamma'(t) \|_{\gamma(t)} dt$. Minimizing this length functional on the set of all piecewise $C^1$ curves joining $p, q \in M$, we obtain a distance function $(p, q) \rightarrow d(p, q)$. Then $d$ is a distance which induces the original topology on $M$. Given a manifold $M$, denote by $\mathcal{X}(M)$ the space of all vector fields over $M$. The metric induces a map $f \mapsto \text{grad} f \in \mathcal{X}(M)$ which associates with each $f$ its gradient via the rule $\langle df, X \rangle = df(X)$ for each $X \in \mathcal{X}(M)$.

On every Riemannian manifold there exists exactly one covariant derivation called the Levi-Civita connection, denoted by $\nabla_X Y$ for any vector fields $X, Y$ on $M$. We also recall that a geodesic is a $C^\infty$ smooth path $\gamma$ whose tangent is parallel along the path $\gamma$, that is, $\gamma$ satisfies the equation $\nabla_{\gamma'(t)} \gamma'(t)/dt = 0$. Any path $\gamma$ joining $p$ and $q$ in $M$ such that $L(\gamma) = d(p, q)$ is a geodesic, and it is called a minimal geodesic. The existence theorem for ordinary differential equations implies that for every $v \in TM$ there exist an open interval $J(v)$ containing 0 and a unique geodesic $\gamma_v : J(v) \rightarrow M$ with $\gamma(0)/dt = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold $M$ of $TM$ such that for every $v \in \tilde{T}M$ the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $\exp : \tilde{T}M \rightarrow M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of $\exp$ to a fiber $T_pM$ in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$. Since $(T_pM, \| . \|_p)$ is a Hilbert space, there is a linear isometric identification between this space and its dual $(T_pM^*, \| . \|_p)$. If $\gamma : [a, b] \rightarrow M$ is a geodesic then for each $t_1, t_2 \in [a, b]$ the Levi-Civita connection $\nabla$ induces an isometry $P_{\gamma(t_1)}^{\gamma(t_2)} : T_{\gamma(t_1)}M \rightarrow T_{\gamma(t_2)}M$, the so called parallel translation along $\gamma$ from $\gamma(t_1)$ to $\gamma(t_2)$.

We also recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Cartan–Hadamard manifold.

If $f$ is a differentiable map from manifold $M$ to manifold $N$, we shall denote by $df_x$ the differential of $f$ at $x$.

3. Invexity on Riemannian manifolds

In this section, motivated by [17], the notion of strong invexity for functions is introduced.

Let $M$ be a Riemannian manifold and $\eta : M \times M \rightarrow TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_yM$.

**Definition 3.1.** Let $M$ be a Riemannian manifold and $f : M \rightarrow \mathbb{R}$ be a differentiable function; then, $f$ is said to be:

(i) **Invex with respect to $\eta$ on $M$** if

$$f(x) - f(y) \geq df_y(\eta(x, y)) \quad \text{for all } x, y \in M.$$
If the inequality (1) is strict for \( x \neq y \), then \( f \) is said to be strictly invex with respect to \( \eta \).

(ii) Strongly invex with respect to \( \eta \) on \( M \) if there exists a constant \( \alpha > 0 \) such that for every \( x, y \in M \) we have

\[
f(x) - f(y) \geq df_y(\eta(x, y)) + \alpha \|\eta(x, y)\|_y^2.
\]

**Example 3.1.** Let \( M \) be a Riemannian manifold and \( f : M \to \mathbb{R} \) be a differentiable function such that for every \( y \in M, df_y \neq 0 \). Suppose that the function \( \eta : M \times M \to TM \) is defined by

\[
\eta(x, y) := \frac{(f(x) - f(y))}{\|df_y\|_y^2} df_y.
\]

Then, for every \( x, y \in M \) we have

\[
\langle df_y, \eta(x, y) \rangle_y = \left( \frac{df_y, (f(x) - f(y))}{\|df_y\|_y^2} df_y \right)_y = \frac{(f(x) - f(y))}{\|df_y\|_y^2} \langle df_y, df_y \rangle_y = \frac{(f(x) - f(y))}{\|df_y\|_y^2} \|df_y\|_y^2 = f(x) - f(y).
\]

Therefore, \( f \) is an invex function with respect to \( \eta \).

The following definitions are introduced in [17].

**Definition 3.2.** Let \( M \) be a Riemannian manifold and \( \alpha_{x,y} : [0, 1] \to M \) be a curve on \( M \) such that \( \alpha_{x,y}(0) = y \) and \( \alpha_{x,y}(1) = x \). Then, \( \alpha_{x,y} \) is said to possess the property (P) with respect to \( y, x \in M \) if

\[
\alpha'_{x,y}(s)(t-s) = \eta(\alpha_{x,y}(t), \alpha_{x,y}(s)), \quad \text{for all } s, t \in [0, 1].
\]

**Definition 3.3.** Let \( M \) be a Riemannian manifold. Then, the function \( \eta : M \times M \to TM \) is said to be integrable if for every \( x, y \in M \) there exists at least one curve \( \alpha_{x,y} \) possessing the property (P) with respect to \( y, x \in M \).

For an example of an integrable map \( \eta \), see [17, p. 305].

**Remark 3.1.** Let \( M \) be a Riemannian manifold and function \( \eta : M \times M \to TM \) be integrable. Then, we have

\[
\eta(x, y) = \eta(\alpha_{x,y}(1), \alpha_{x,y}(0)) = \alpha'_{x,y}(0).
\]

In the case where \( \alpha_{x,y} \) is a geodesic, then

\[
\eta(y, \alpha_{x,y}(s)) = -s \alpha'_{x,y}(s) = -s P_{0,\alpha_{x,y}}^s[\alpha'_{x,y}(0)] = -s P_{0,\alpha_{x,y}}^s[\eta(x, y)]
\]

and

\[
\eta(\alpha_{x,y}(1), \alpha_{x,y}(s)) = (1-s) \alpha'_{x,y}(s) = (1-s) P_{0,\alpha_{x,y}}^s[\eta(x, y)].
\]

4. **Monotonicity on Riemannian manifolds**

Let us recall the definition of a monotone vector field on a Riemannian manifold; see [10,15] for the details.

**Definition 4.1.** Let \( M \) be a Riemannian manifold and \( X \) be a vector field on \( M \). Then, \( X \) is said to be monotone on \( M \) if for every \( x, y \in M \) we have

\[
\langle \alpha'_{x,y}(0), P_{1,\alpha_{x,y}}^0[X(x)] - X(y) \rangle_y \geq 0,
\]

where \( \alpha_{x,y} \) is a geodesic joining \( x \) and \( y \).
Motivated by Yang et al. [24] we generalize the notion of invariant monotonicity to Riemannian manifolds.

**Definition 4.2.** Let \( M \) be a Riemannian manifold and \( X \) be a vector field on \( M \); then:

(i) \( X \) is said to be invariant monotone on \( M \) with respect to \( \eta \) if for every \( x, y \in M \) one has

\[
(X(y), \eta(x, y))_y + (X(x), \eta(y, x))_x \leq 0.
\]

(ii) \( X \) is said to be strongly invariant monotone on \( M \) with respect to \( \eta \) if there exists an \( \alpha > 0 \) such that for every \( x, y \in M \) one has

\[
(X(y), \eta(x, y))_y + (X(x), \eta(y, x))_x \leq -\alpha(\|\eta(x, y)\|^2 + \|\eta(y, x)\|^2).
\]

It should be noted that on every Cartan–Hadamard manifold \( M \) each monotone vector field \( X \) is an invariant monotone vector field with respect to \( \eta(x, y) := \exp^{-1}_y(x) \). Indeed, for every \( x, y \in M \),

\[
\alpha_{x,y}(t) := \exp_y(t \exp^{-1}_y(x)), \quad \text{for all } t \in [0, 1],
\]

is the unique geodesic joining \( y \) to \( x \) and \( \alpha'_{x,y}(0) := \exp^{-1}_y(x) \). By (2) and the property \( P^1_{0,\alpha_{x,y}}[\exp^{-1}_y(x)] = -\exp^{-1}_y(y) \) of parallel translation we have

\[
(X(y), \eta(x, y))_y + (X(x), \eta(y, x))_x \leq 0.
\]

Therefore,

\[
(X(y), \eta(x, y))_y + (\exp^{-1}_y(x), \eta(x, y))_y \leq 0.
\]

Considering \( \eta(x, y) = \exp^{-1}_y(x) \) for every \( x, y \in M \), we conclude that \( X \) is an invariant monotone vector field on \( M \) with respect to \( \eta \).

Despite the above fact, in the following example we show that on every Cartan–Hadamard manifold \( M \) there exists a strictly invariant monotone vector field \( X \) with respect to an \( \eta : M \times M \to TM \) which is not monotone.

**Example 4.1.** Let \( M \) be a Cartan–Hadamard manifold and \( x \in M \). Then, the distance function \( z \mapsto d(x, z) \) is smooth on \( M \setminus \{x\} \). We denote the partial derivatives of the function \( d : M \times M \to \mathbb{R} \) by \( \frac{\partial d(x, y)}{\partial x} \) and \( \frac{\partial d(x, y)}{\partial y} \). Suppose that \( x \neq y \) and \( l := d(x, y) = \|\exp^{-1}_y(x)\|_x \).

If

\[
\gamma(t) := \exp_y(t \exp^{-1}_y(x)/\|\exp^{-1}_y(x)\|_x) \quad \text{for all } t \in [0, l]
\]

is the unique minimizing geodesic joining \( y \) to \( x \), then, by [1, p. 347], we have the following antisymmetry properties:

\[
P^1_{0,\gamma} \left[ \frac{\partial d(x, y)}{\partial y} \right] = -\frac{\partial d(x, y)}{\partial x}.
\]

On the other hand, by [21, p. 108] we have

\[
\frac{\partial d(x, y)}{\partial x} = \gamma'(l),
\]

and

\[
\left\| \frac{\partial d(x, y)}{\partial y} \right\|_y = \left\| \frac{\partial d(x, y)}{\partial x} \right\|_x = 1.
\]

Since \( \gamma'(0) = \frac{1}{t} \exp^{-1}_y(x) \), by Eq. (6) we can write

\[
\frac{\partial d(x, y)}{\partial x} = \gamma'(l) = P^1_{0,\gamma}[\gamma'(0)]
\]
\[
\begin{align*}
&= P_{0,y}^t \left[ \frac{1}{l} \exp_y^{-1}(x) \right] \\
&= \frac{1}{l} P_{0,y}^t[\exp_y^{-1}(x)] \\
&= -\frac{1}{l} \exp_x^{-1}(y).
\end{align*}
\]

(8)

Now, we define the function \( \eta \) and vector field \( X \);

\[ \eta(x, y) := -l \exp_y^{-1}(x) \]

and

\[ X(y) := -\frac{\partial d(x, y)}{\partial y}. \]

The vector field \( X \) is invariant monotone with respect to \( \eta \). Indeed, by utilizing (8) we get

\[
\langle X(x), \eta(y, x) \rangle_x = \left\langle -\frac{\partial d(x, y)}{\partial x}, -l \exp_x^{-1}(y) \right\rangle_x
\]

\[
= l \left\langle -\frac{1}{l} \exp_x^{-1}(y), \exp_x^{-1}(y) \right\rangle_x
\]

\[
= -\|\exp_x^{-1}(y)\|_x^2
\]

\[
= -l^2.
\]

Similarly,

\[
\langle X(y), \eta(x, y) \rangle_y = -l^2.
\]

Hence,

\[
\langle X(x), \eta(y, x) \rangle_x + \langle X(y), \eta(x, y) \rangle_y < 0.
\]

We show that the vector field \( X \) is not monotone. By (5)–(7) we get

\[
\langle \gamma'(0), P_{1,y}^0 [X(x)] \rangle_y - \langle X(y) \rangle_y = \langle \gamma'(0), P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] - \left( -\frac{\partial d(x, y)}{\partial y} \right) \rangle_y
\]

\[
= \langle \gamma'(0), -P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] - P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] \rangle_y
\]

\[
= -\langle \gamma'(0), P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] + P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] \rangle_y
\]

\[
= -2 \langle \gamma'(0), P_{1,y}^0 \left[ \frac{\partial d(x, y)}{\partial x} \right] \rangle_y
\]

\[
= -2 \left\langle \gamma'(0), P_{0,y}^1 \left[ \frac{\partial d(x, y)}{\partial x} \right] \right\rangle_x
\]

\[
= -2 \left\langle \frac{\partial d(x, y)}{\partial x}, \frac{\partial d(x, y)}{\partial x} \right\rangle_x
\]

\[
= -2 \left\| \frac{\partial d(x, y)}{\partial x} \right\|^2_x
\]

\[
= -2 < 0.
\]

The following example shows that there exists a strictly invariant monotone vector field on every Riemannian manifold.
Example 4.2. Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function such that for every $y \in M$, $df_y \neq 0$. Suppose that for every $x, y \in M$ the function $\eta : M \times M \to TM$ is defined by

$$\eta(x, y) := -\frac{1}{2\|df_y\|_y^2} df_y.$$ 

Then, $df$ is a strictly invariant monotone vector field with respect to $\eta$. Indeed, for every $x, y \in M$ with $x \neq y$ we have

$$\langle df_y, \eta(x, y) \rangle_y = \left\langle df_y, -\frac{1}{2\|df_y\|_y^2} df_y \right\rangle_y = -\frac{1}{2\|df_y\|_y^2} \|df_y\|_y^2 = -\frac{1}{2}. \tag{9}$$

Similarly, we have

$$\langle df_x, \eta(y, x) \rangle_x = -\frac{1}{2}. \tag{10}$$

Therefore,

$$\langle df_y, \eta(x, y) \rangle_y + \langle df_x, \eta(y, x) \rangle_x < 0.$$ 

Let $S$ be a convex subset of a finite dimensional Cartan–Hadamard manifold $M$ and $x \in M$. Then, there exists a unique point $p_S(x)$ such that for each $y \in S$, $d(x, p_S(x)) \leq d(x, y)$. The point $p_S(x)$ is called the projection of $x$ onto $S$; see [9, p. 262].

Now, we give an example of a strictly invariant monotone vector field which is not of gradient type.

Example 4.3. Let $M$ be a Cartan–Hadamard manifold of constant sectional curvature $K \neq 0$. Suppose that $x, y \in M$, $x \neq y$, $r > 0$. Moreover, assume that $B_x := \overline{B}(x, r)$ and $B_y := \overline{B}(y, r)$ are two closed balls such that $B_x \cap B_y = \emptyset$. Let $x' := P_{B_x}(x)$ and $y' := P_{B_y}(y)$ be projections of $x$ and $y$ on $B_x$ and $B_y$, respectively. Suppose that $l := \|\exp_y^{-1} x'\|_y$.

Let

$$\gamma(t) := \exp_y(t \exp_y^{-1} x' / \|\exp_y^{-1} x'\|_y), \quad \text{for all } t \in [0, l],$$

be the normalized geodesic joining $y$ to $x'$. Assume that $J$ is a Jacobi field along $\gamma$ normal to $\gamma'$ and $w(t)$ is a parallel field along $\gamma$ with $\|w(t)\|_{\gamma(t)} = 1$ and $\langle \gamma', w(t) \rangle_{\gamma(t)} = 0$ for every $t \in [0, l]$.

We define the function $\eta : M \times M \to TM$ by

$$\eta(x, y) := P_{l,y}[−w(l)].$$

We also define the vector field $X$ as follows:

$$X(y) := P_{l,y}[J(l)].$$

By [8, p. 112] we have

$$J(l) = \frac{\sinh(l\sqrt{-K})}{\sqrt{-K}}w(l). \tag{11}$$

By the definitions of $X$ and $\eta$ and utilizing (11) we get

$$\langle X(y), \eta(x, y) \rangle_y = \langle P_{l,y}[J(l)], P_{l,y}[−w(l)] \rangle_y = \langle J(l), −w(l) \rangle_y = \left\langle \frac{\sinh(l\sqrt{-K})}{\sqrt{-K}}w(l), −w(l) \right\rangle_{x'}.$$
Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function. Suppose that the function $\eta : M \times M \to TM$ is integrable. Then, $f$ is an invex function on $M$ if and only if $g_{x,y}(t) = f(\alpha_{x,y}(t))$ is convex on $[0, 1]$ for some $\alpha_{x,y}$.

**Proposition 4.2.** Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function. Suppose that the function $\eta : M \times M \to TM$ is integrable. Then, $f$ is an invex function on $M$ if and only if $g_{x,y}(t) = f(\alpha_{x,y}(t))$ is convex on $[0, 1]$ for some $\alpha_{x,y}$.

**Theorem 4.1.** Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function. Suppose that the function $\eta : M \times M \to TM$ is integrable. If $df$ is invariant monotone on $M$ with respect to $\eta$, then $f$ is an invex function on $M$.

**Proof.** We prove only the result when $f$ is strongly invex. The proofs for other cases are similar. Let $f$ be strongly invex on $M$ and $x, y \in M$. Since $f$ is strongly invex, we have

$$f(x) - f(y) \geq df_x(\eta(x, y)) + \alpha\|\eta(x, y)\|_x^2,$$

and

$$f(y) - f(x) \geq df_y(\eta(y, x)) + \alpha\|\eta(y, x)\|_y^2.$$

By adding these two inequalities we get

$$0 \geq df_x(\eta(x, y)) + df_y(\eta(y, x)) + \alpha(\|\eta(x, y)\|_x^2 + \|\eta(y, x)\|_y^2).$$

We need the following proposition from [17].

**Proposition 4.1.** Let $M$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function. If $f$ is (strongly, strictly) invex with respect to $\eta$, then $df$ is (strongly, strictly) invariant monotone with respect to $\eta$ on $M$.

**Proof.** We prove only the result when $f$ is strongly invex. The proofs for other cases are similar. Let $f$ be strongly invex on $M$ and $x, y \in M$. Since $f$ is strongly invex, we have

$$f(x) - f(y) \geq df_x(\eta(x, y)) + \alpha\|\eta(x, y)\|_x^2,$$

and

$$f(y) - f(x) \geq df_y(\eta(y, x)) + \alpha\|\eta(y, x)\|_y^2.$$

By adding these two inequalities we get

$$0 \geq df_x(\eta(x, y)) + df_y(\eta(y, x)) + \alpha(\|\eta(x, y)\|_x^2 + \|\eta(y, x)\|_y^2).$$

We need the following proposition from [17].
Let $\lambda f(\alpha_{x,y}(\lambda)) - f(x) > (1 - \lambda)[f(\alpha_{x,y}(\lambda)) - f(y)]$.

By the Mean Value Theorem there exist $\lambda_1, \lambda_2 \in (0, 1)$, $0 < \lambda_2 < \lambda < \lambda_1 < 1$ and

$$\lambda(\lambda - 1)df_{\alpha_{x,y}}(\alpha_{x,y}'(\lambda_1)) + (1 - \lambda)\lambda df_{\alpha_{x,y}}(\alpha_{x,y}'(\lambda_2)) > 0.$$ 

This is equivalent to

$$-d f_{\alpha_{x,y}}(\alpha_{x,y}'(\lambda_1)) + df_{\alpha_{x,y}}(\alpha_{x,y}'(\lambda_2)) > 0.$$ 

By the condition (P) we have

$$\frac{1}{\lambda_1 - \lambda_2} \eta(\alpha_{x,y}(\lambda_1), \alpha_{x,y}(\lambda_2)) = \alpha_{x,y}'(\lambda_2),$$

and

$$-\frac{1}{\lambda_1 - \lambda_2} \eta(\alpha_{x,y}(\lambda_2), \alpha_{x,y}(\lambda_1)) = \alpha_{x,y}'(\lambda_1).$$

Therefore,

$$d f_{\alpha_{x,y}}(\lambda_1)(\eta(\alpha_{x,y}(\lambda_2), \alpha_{x,y}(\lambda_1))) + d f_{\alpha_{x,y}}(\lambda_2)(\eta(\alpha_{x,y}(\lambda_1), \alpha_{x,y}(\lambda_2))) > 0.$$ 

This contradicts the invariant monotonicity of $f$. □

**Theorem 4.2.** Let $M$ be a Riemannian manifold such that there is a geodesic between every two points and let $f : \text{M} \to \mathbb{R}$ be a differentiable function. Suppose that the function $\eta : \text{M} \times \text{M} \to \text{TM}$ is integrable. If $df$ is strongly invariant monotone on $\text{M}$ with respect to $\eta$, then $f$ is a strongly invex function on $\text{M}$.

**Proof.** Let $x, y \in \text{M}$. Then, there exists a geodesic $\alpha_{x,y} : [0, 1] \to \text{M}$ such that $\alpha_{x,y}(0) = y, \alpha_{x,y}(1) = x$. Let $z := \alpha_{x,y}(\frac{1}{2})$; then, by the Mean Value Theorem there exist $t_1, t_2 \in (0, 1)$ such that $0 < t_2 < \frac{1}{2} < t_1 < 1$ and we have

$$f(x) - f(z) = \frac{1}{2} df_u(\alpha_{x,y}'(t_1)), \tag{16}$$

and

$$f(z) - f(y) = \frac{1}{2} df_v(\alpha_{x,y}'(t_2)), \tag{17}$$

where $u := \alpha_{x,y}(t_1)$, $v := \alpha_{x,y}(t_2)$. Since $df$ is strongly invariant monotone, then,

$$df_u(\eta(y, u)) + df_v(\eta(u, y)) \leq -\alpha(\|\eta(y, u)\|_u^2 + \|\eta(u, y)\|_v^2). \tag{18}$$

Since condition (P) holds, then, by Remark 3.1 and utilizing the parallel translation, we have

$$\eta(y, \alpha_{x,y}(t_1)) = -t_1 P_{0,\alpha_{x,y}}^l[\eta(x, y)], \tag{19}$$

and

$$\eta(\alpha_{x,y}(t_1), y) = t_1 \eta(x, y). \tag{20}$$

By combining (18)–(20) we have

$$df_u(-t_1 P_{0,\alpha_{x,y}}^l[\eta(x, y)]) + df_v(t_1[\eta(x, y)]) \leq -\alpha(\| - t_1 P_{0,\alpha_{x,y}}^l[\eta(x, y)]\|_u^2 + \|t_1[\eta(x, y)]\|_y^2).$$

This is equivalent to

$$-d f_u(P_{0,\alpha_{x,y}}^l[\eta(x, y)]) + df_v([\eta(x, y)]) \leq -2t_1 \alpha[\|\eta(x, y)\|_y^2]. \tag{21}$$

Since $P_{0,\alpha_{x,y}}^l[\eta(x, y)] = \alpha_{x,y}'(t_1)$ we have

$$\frac{1}{2} df_u(\alpha_{x,y}'(t_1)) \geq \frac{1}{2} df_v([\eta(x, y)]) + t_1 \alpha[\|\eta(x, y)\|_y^2]. \tag{22}$$
In a similar way we get
\[
\frac{1}{2} df_y(\alpha_x', (t_2)) \geq \frac{1}{2} df_y(\eta(x, y)) + t_2 \alpha \| \eta(x, y) \|_y^2.
\]
(23)

From (16), (17), (22) and (23) we conclude that
\[
f(x) - f(z) \geq \frac{1}{2} df_y(\eta(x, y)) + t_1 \alpha \| \eta(x, y) \|_y^2,
\]
and
\[
f(z) - f(y) \geq \frac{1}{2} df_y(\eta(x, y)) + t_2 \alpha \| \eta(x, y) \|_y^2.
\]

By adding these two inequalities we obtain
\[
f(x) - f(y) \geq df_y(\eta(x, y)) + \alpha(t_1 + t_2)\| \eta(x, y) \|_y^2
\leq \frac{1}{2} df_y(\eta(x, y)) + \frac{1}{2} \alpha \| \eta(x, y) \|_y^2.
\]

Therefore, \( f \) is strongly invex. \( \square \)

5. Pseudoinvexity and invariant pseudomonotonicity

Definition 5.1. Let \( M \) be a Riemannian manifold and \( f : M \rightarrow \mathbb{R} \) be a differentiable function; then:

(i) \( f \) is said to be pseudoinvex with respect to \( \eta \) on \( M \) if for every \( x, y \in M \), one has
\[
df_y(\eta(x, y)) \geq 0 \Rightarrow f(x) \geq f(y).
\]

(ii) \( f \) is said to be strictly pseudoinvex with respect to \( \eta \) on \( M \) if for every \( x, y \in M \) with \( x \neq y \) one has
\[
df_y(\eta(x, y)) > 0 \Rightarrow f(x) > f(y).
\]

(iii) \( f \) is said to be strongly pseudoinvex with respect to \( \eta \) on \( M \) if there exists a constant \( \alpha > 0 \) such that for every \( x, y \in M \), one has
\[
df_y(\eta(x, y)) \geq 0 \Rightarrow f(x) \geq f(y) + \alpha \| \eta(x, y) \|_y^2.
\]

Definition 5.2. Let \( M \) be a Riemannian manifold and \( X \) be a vector field on \( M \); then:

(i) \( X \) said to be invariant pseudomonotone on \( M \) with respect to \( \eta \) if for every \( x, y \in M \) one has
\[
\langle X(x), \eta(y, x) \rangle \geq 0 \Rightarrow \langle X(y), \eta(x, y) \rangle \leq 0.
\]

(ii) \( X \) is said to be strictly invariant pseudomonotone on \( M \) with respect to \( \eta \) if for every \( x, y \in M \) with \( x \neq y \) one has
\[
\langle X(x), \eta(y, x) \rangle \geq 0 \Rightarrow \langle X(y), \eta(x, y) \rangle > 0.
\]

(iii) \( X \) is said to be strongly invariant pseudomonotone on \( M \) with respect to \( \eta \) if there exists a constant \( \alpha > 0 \) such that for every \( x, y \in M \) one has
\[
\langle X(x), \eta(y, x) \rangle \geq 0 \Rightarrow \langle X(y), \eta(x, y) \rangle \leq -\alpha \| \eta(x, y) \|_y^2.
\]

The following theorem is a generalization of Theorem 4.1 in [24] to Riemannian manifolds.

Theorem 5.1. Let \( M \) be a Riemannian manifold and \( f : M \rightarrow \mathbb{R} \) be a differentiable function. Suppose that the function \( \eta : M \times M \rightarrow TM \) is integrable. Then, \( f \) is strictly pseudoinvex with respect to \( \eta \) on \( M \) if and only if \( df \) is strictly invariant pseudomonotone on \( M \) with respect to \( \eta \).
Suppose that \( df_y(\eta(x, y)) \geq 0 \). Then, by the strict pseudoinvexity of \( f \) we have

\[
f(x) \geq f(y).
\]  

(24)

We need to show that \( df_x(\eta(y, x)) < 0 \). Assume that \( df_x(\eta(y, x)) \geq 0 \). Again utilizing the strict pseudoinvexity of \( f \) with respect to \( \eta \) yields \( f(y) \geq f(x) \), which contradicts (24).

Conversely, suppose that \( df \) is strictly invariant pseudomonotone on \( M \). Let \( x, y \in M \), \( x \neq y \) be such that

\[
df_y(\eta(x, y)) \geq 0.
\]  

(25)

Since \( \eta \) is integrable, there exists at least one curve \( \alpha_{x,y} \) possessing the Property (P) such that \( \alpha_{x,y}(0) = y \) and \( \alpha_{x,y}(1) = x \). We need to show that \( f(x) > f(y) \). Assume

\[
f(x) \leq f(y),
\]  

(26)

By the Mean Value Theorem there exists \( \lambda \in (0, 1) \) such that

\[
f(x) - f(y) = df_{\alpha_{x,y}(\lambda)}(\alpha'_{x,y}(\lambda)).
\]  

(27)

Now, (26) and (27) implies that

\[
df_{\alpha_{x,y}(\lambda)}(\alpha'_{x,y}(\lambda)) \leq 0.
\]

Hence, by the Property (P) we have

\[
df_{\alpha_{x,y}(\lambda)}(\alpha'_{x,y}(\lambda)) = df_{\alpha_{x,y}(\lambda)} \left[ \frac{-1}{\lambda} \eta(y, \alpha_{x,y}(\lambda)) \right] = \frac{1}{\lambda} df_{\alpha_{x,y}(\lambda)}[\eta(y, \alpha_{x,y}(\lambda))] \leq 0.
\]

Therefore,

\[
df_{\alpha_{x,y}(\lambda)}(\eta(y, \alpha_{x,y}(\lambda))) \geq 0.
\]  

(28)

Since \( df \) is strictly invariant pseudomonotone, from (28) we have

\[
df_y(\eta(\alpha_{x,y}(\lambda), y)) < 0,
\]

and by the Property (P) we conclude that

\[
df_y(\lambda \alpha'_{x,y}(0)) = \lambda df_y(\eta(x, y)) < 0.
\]

Therefore, \( df_y(\eta(x, y)) < 0 \) which contradicts (25). Hence, \( f \) is strictly pseudoinvex with respect to \( \eta \) on \( M \).

\[ \square \]

**Theorem 5.2.** Let \( M \) be a Riemannian manifold such that there is a geodesic between every two points and let \( f : M \to \mathbb{R} \) be a differentiable function. Suppose that the function \( \eta : M \times M \to TM \) is integrable. If \( f \) is strongly invariant pseudomonotone on \( M \) with respect to \( \eta \), then \( f \) is strongly pseudoinvex with respect to \( \eta \) on \( M \).

**Proof.** Suppose that

\[
df_y\eta((x, y)) \geq 0.
\]  

(29)

Let \( x, y \in M \). Then, there exists a geodesic \( \alpha_{x,y} : [0, 1] \to M \) such that \( \alpha_{x,y}(0) = y \), \( \alpha_{x,y}(1) = x \). Let \( z := \alpha_{x,y}(\frac{1}{2}) \); then, by the Mean Value Theorem, there exist \( t_1, t_2 \in (0, 1) \) such that \( 0 < t_2 < \frac{1}{2} < t_1 < 1 \) and

\[
f(x) - f(z) = \frac{1}{2} df_u(\alpha'_{x,y}(t_1))
\]  

(30)

and

\[
f(z) - f(y) = \frac{1}{2} df_v(\alpha'_{x,y}(t_2)),
\]  

(31)
where \( u \equiv \alpha_{x,y}(t_1) \), \( v \equiv \alpha_{x,y}(t_2) \). By the property (P) and equalities (30) and (31) we obtain

\[
f(x) - f(z) = -\frac{1}{2t_1} df_u \eta((y, u)),
\]

and

\[
f(z) - f(y) = -\frac{1}{2t_2} df_v \eta((y, v)).
\]

The Property (P) and (29) imply that

\[
0 \leq df_z(\eta(x, y)) = \frac{1}{t_1} df_z(\eta(u, y)) = \frac{1}{t_2} df_z(\eta(v, y)).
\]

Since \( df \) is strongly invariant pseudomonotone on \( M \) with respect to \( \eta \), by utilizing (34) and Property (P) we have

\[
df_u(\eta(y, u)) \leq -\alpha \| \eta(y, u) \|^2_u
= -\alpha \| -t_1 P^u_{0,\alpha_{x,y}}[\eta(x, y)] \|^2_u
= -\alpha t_1^2 \| P^u_{0,\alpha_{x,y}}[\eta(x, y)] \|^2_u
= -\alpha t_1^2 \| \eta(x, y) \|^2_y,
\]

and

\[
df_v(\eta(y, v)) \leq -\alpha \| \eta(y, v) \|^2_v
= -\alpha \| -t_1 P^v_{0,\alpha_{x,y}}[\eta(x, y)] \|^2_v
= -\alpha t_1^2 \| P^v_{0,\alpha_{x,y}}[\eta(x, y)] \|^2_v
= -\alpha t_2^2 \| \eta(x, y) \|^2_y.
\]

Now, from (32) and (35) we conclude that

\[
f(x) - f(z) \geq \frac{\alpha}{2} t_1 \| \eta(x, y) \|^2_y.
\]

Similarly, by using (33) and (36) we have

\[
f(z) - f(y) \geq \frac{\alpha}{2} t_2 \| \eta(x, y) \|^2_y.
\]

By adding (37) and (38) we obtain

\[
f(x) - f(y) \geq \frac{\alpha}{2} (t_1 + t_2) \| \eta(x, y) \|^2_y
\geq \frac{\alpha}{4} \| \eta(x, y) \|^2_y.
\]

Therefore, \( f \) is strongly pseudoinvex with respect to \( \eta \) on \( M \). □

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References


\[ \text{References:} \]


