On the characters of nilpotent association schemes

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Abstract
In this paper we give some properties of the character values of nilpotent
commutative association schemes. As a main result, a class of commutative
schurian association schemes is given.

Key words : nilpotent association scheme, character, dual.
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1 Introduction
Nilpotent association schemes have been defined by Hanaki in [8] as a generalization
of nilpotent finite groups. Some properties of the character values of nilpotent asso-
ciation schemes have also been given in [8]. In this paper, we first determine the char-
acter values of nilpotent commutative association schemes of class 2. Then we show
that if \((X, G)\) is a nilpotent commutative association scheme with the upper central
series \( \{1_X\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{n-1} \subseteq G_n = G \) such that \( \pi(G_i/G_{i-1}), 1 \leq i \leq n, \)
are distinct sets, then for every faithful irreducible character \( \chi \in \text{Irr}(G/G_{i-1}) \) and
\( g \in G - G_i, \) where \( 1 \leq i \leq n - 1, \) we have \( \chi(\sigma_g) = 0. \) Moreover, we obtain a
condition which a commutative association scheme is equal to the wreath product
of some cyclic groups. More precisely, we show that if \((X, G)\) is a nilpotent com-
mutative association scheme of class \( n \) such that for every \( 1 \leq i \leq n, G_i/G_{i-1} \) is a
group of prime order \( p_i \) where \( p_i, 1 \leq i \leq n, \) are distinct prime numbers, then we
have \( G = G_1 \wr (G_2/G_1) \wr (G_3/G_2) \wr \cdots \wr (G_n/G_{n-1}).\)

2 Preliminaries
Let us first state some necessary definitions and notation. For details, we refer the
reader to [1] and [12] for the background of association schemes. Throughout this
paper, \( \mathbb{C} \) denotes the complex numbers and \( \mathbb{R} \) denotes the real numbers.
Definition 2.1. Let $X$ be a finite set and $G$ be a partition of $X \times X$. Then $(X, G)$ is called an association scheme if the following properties hold:

(i) $1_X \in G$, where $1_X := \{(x, x) | x \in X\}$.

(ii) For every $g \in G$, $g^*$ is also in $G$, where $g^* := \{(x, y) | (y, x) \in g\}$.

(iii) For every $g, h, k \in G$, there exists a nonnegative integer $\lambda_{ghk}$ such that for every $(x, y) \in k$, there exist exactly $\lambda_{ghk}$ elements $z \in X$ with $(x, z) \in g$ and $(z, y) \in h$.

For each $g \in G$, we call $n_g = \lambda_{gg^*1_X}$ the valency of $g$. For any nonempty subset $H$ of $G$, put $n_H = \sum_{h \in H} n_h$. We call $n_G$ the order of $(X, G)$.

Let $H$ and $K$ be nonempty subsets of $G$. We define $HK$ to be the set of all elements $t \in G$ such that there exist elements $h \in H$ and $k \in K$ with $\lambda_{hkt} \neq 0$. The set $HK$ is called the complex product of $H$ and $K$. If one of factors in a complex product consists of a single element $g$, then one usually writes $g$ for $\{g\}$.

An association scheme $(X, G)$ is called commutative if for all $g, h, d \in G$, $\lambda_{ghd} = \lambda_{hdg}$.

A nonempty subset $H$ of $G$ is called a closed subset if $HH \subseteq H$. A closed subset $H$ of $G$ is called strongly normal if $gHg^* = H$ for any $g \in G$.

Let $H$ be a closed subset of $G$. For every $h \in H$ we define $xH = \{y \in X | (x, y) \in h\}$. Put $X/H = \{xH | x \in X\}$ and $G/H = \{gH | g \in G\}$, where $xH = \bigcup_{h \in H} xH$ and $gH = \{(xH, yH) | y \in xHgH\}$. Then $(X/H, G/H)$ is an association scheme called the quotient scheme of $(X, G)$ over $H$. Note that a closed subset $H$ is strongly normal iff the quotient scheme $(X/H, G/H)$ is a group with respect to the relational product iff $gg^* \subseteq H$, for every $g \in G$.

For each closed subset $H$ of $G$, we define $O_\phi(H) = \{h \in H | n_h = 1\}$, called the thin radical of $H$. Note that $O_\phi(H)$ is a closed subset of $G$. In fact $O_\phi(H)$ is a group with respect to the relational product. The closed subset $H$ is called thin if $O_\phi(H) = H$. A commutative association scheme $(X, G)$ is called the nilpotent association scheme of class $n$, if it has the following sequence of closed subsets

$$\{1_X\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{n-1} \subseteq G_n = G$$

such that for all $1 \leq i \leq n$, $G_i/G_{i-1} = O_\phi(G/G_{i-1})$. We call this sequence the upper central series of $(X, G)$.

Let $(X, G)$ be an association scheme. For every $g \in G$, let $\sigma_g$ be the adjacency matrix of $g$. That is, $\sigma_g$ is the matrix whose rows and columns are indexed by the elements of $X$ and its $(x, y)$-entry is 1 if $(x, y) \in g$ and 0 otherwise. Note that for every $h, k \in G$, $\sigma_h \sigma_k = \sum_{t \in G} \lambda_{hkt} \sigma_t$. For any nonempty subset $H$ of $G$, we put $\sigma_H := \{\sigma_h | h \in H\}$.

It is known that $CG = \bigoplus_{g \in G} \mathbb{C}\sigma_g$, the adjacency algebra of $(X, G)$, is a semisimple algebra. The set of irreducible characters of $G$ is denoted by $\text{Irr}(G)$. An irreducible character $\chi \in \text{Irr}(G)$ is called faithful if $\text{Ker}(\chi) = \{1_X\}$, where $\text{Ker}(\chi) = \{g \in G | \chi(\sigma_g) = n_g \chi(\sigma_{1_X})\}$. 


Let \((X, G)\) be an association scheme and \(\chi, \psi \in \text{Irr}(G)\). In [6], the character product of \(\chi\) and \(\psi\) is defined by \(\chi \psi(\sigma_g) = \frac{1}{n_g} \chi(\sigma_g) \psi(\sigma_g)\). It is known that this character product need not be a character in general.

In the following, we deal with the dual of the complex adjacency algebra of a commutative association scheme in the sense of [1]. We refer the reader to [2] for the background of table algebras.

Suppose that \((C_G, \sigma_G)\) is the complex adjacency algebra of a commutative association scheme \((X, G)\), where \(\sigma_G = \{\sigma_g | g \in G\}\). Let \(\{\varepsilon_\chi | \chi \in \text{Irr}(G)\}\) be the set of the primitive idempotents of \(C_G\). Then from [1, Section 2.5] there are two matrices \(P = (p_g(\chi))\) and \(Q = (q_\chi(g))\) in \(\text{Mat}_{|X|}(C)\), where \(g \in G\) and \(\chi \in \text{Irr}(G)\), such that
\[
PQ = QP = |X|I,
\]
where \(I\) is the identity matrix in \(\text{Mat}_{|X|}(C)\), and
\[
\sigma_g = \sum_{\chi \in \text{Irr}(G)} p_g(\chi) \varepsilon_\chi \quad \text{and} \quad \varepsilon_\chi = \frac{1}{|X|} \sum_{g \in G} q_\chi(g) \sigma_g.
\]
The dual of \((C_G, \sigma_G)\) in the sense of [1] is as follows: with each linear representation \(\Delta_\chi : \sigma_g \mapsto p_g(\chi)\), we associate the linear mapping \(\Delta_\chi^* : \sigma_g \mapsto q_\chi(g) = \frac{m_\chi(\sigma_g^*)}{n_g}\). Since the matrix \(Q = (q_\chi(g))\) is non-singular, the set \(B = \{\Delta_\chi^* : \chi \in \text{Irr}(G)\}\) is linearly independent and so forms a base of the set of all linear mappings \(A\) of \(C_G\) into \(C\). From [1, Theorem 5.9 and Theorem 3.8] the pair \((A, B)\) is a table algebra (see [2]) with the identity \(1_A = \Delta_\rho^*\), where \(\rho \in \text{Hom}_C(C_G, C)\) such that \(\rho(\sigma_g) = n_g\), and involutory automorphism \(*\) which maps \(\Delta_\chi^*\) to \(\Delta_{\chi^*}^*\), where \(\chi^*\) is the complex conjugate to \(\chi\). The table algebra \((A, B)\) is called the dual of the association scheme \((X, G)\). Moreover, for every \(\chi, \psi \in \text{Irr}(G)\) we have
\[
\Delta_\chi^* \Delta_\psi^* = \sum_{\Delta_\varphi \in B} q_{\chi \psi}^\varphi \Delta_\varphi^*
\]
where the structure constants \(q_{\chi \psi}^\varphi, \varphi \in \text{Irr}(G)\) are nonnegative real numbers. For every \(\chi, \psi \in \text{Irr}(G)\), put
\[
\text{Supp}(\Delta_\chi^* \Delta_\psi^*) = \{\Delta_\varphi \in B | q_{\chi \psi}^\varphi \neq 0\}.
\]
A nonempty subset \(N\) of \(B\) is called a closed subset if for every \(\Delta_\chi^*, \Delta_\psi^* \in N\) we have \(\text{Supp}(\Delta_\chi^* \Delta_\psi^*) \subseteq N\).

Let \((A, B)\) be the dual of \((C_G, \sigma_G)\) and let \(H\) be a closed subset of \(G\). Put
\[
\text{Ker}(H) = \{\Delta_\chi^* \in B | \chi(\sigma_g) = n_g, \text{ for every } g \in H\}.
\]
Then from [2, Theorem 1 and 2], \(\text{Ker}(H)\) is a closed subset of \(B\) and
\[
\hat{H} \simeq B/\text{Ker}(H)
\]
where \(\hat{H}\) is the dual of \(C_H\).
3 Main Results

In this section we assume that $(X, G)$ is a commutative association scheme. We first give some properties of the character values of nilpotent association schemes of class 2.

**Lemma 3.1.** Let $(X, G)$ be a nilpotent association scheme of class 2. Then for every $\chi \in \text{Irr}(G)$ and every $g \in G - G_1$, $\chi(\sigma_g) = 0$ or $\chi(\sigma_g) = n_g \varepsilon_g$, where $\varepsilon_g$ is a $(n_G/n_{G_1})$-th root of unity. In particular, for every faithful irreducible character $\chi \in \text{Irr}(G)$ and every $g \in G - G_1$, $\chi(\sigma_g) = 0$.

**Proof.** Let $g \in G - G_1$. Since $G_1$ is a strongly normal closed subset of $G$, we have

$$\sigma_g \sigma_g^* = n_g \sigma_{1_X} + \sum_{t \in G_1} \lambda_{gg^*t} \sigma_t. \quad (1)$$

If $\lambda_{gg^*t} \neq 0$, then from the equality $\lambda_{gg^*t} n_t = \lambda_{gtg} n_g$ we get $\lambda_{gtg} \neq 0$ and it follows that $tg = g$. So if for some $\chi \in \text{Irr}(G)$, $\chi(\sigma_g) \neq 0$, then from the equality $\chi(\sigma_t) \chi(\sigma_g) = \chi(\sigma_g)$ we conclude that $\chi(\sigma_t) = 1$. Thus from (1) we have

$$\chi(\sigma_g) \chi(\sigma_g^*) = n_g + \sum_{t \in G_1} \lambda_{gg^*t} \chi(\sigma_t) = n_g + \sum_{t \in G_1} \lambda_{gg^*t} = n_g n_{g^*},$$

and thus we obtain $|\chi(\sigma_g)| = n_g$. So from [8, Proposition 3.2] we get $\chi(\sigma_g) = n_g \varepsilon_g$, for some root of unity $\varepsilon_g$. Let $m = n_G/n_{G_1}$. Then since $G/G_1$ is a finite group of order $m$, we have $\sigma_g^m \in G_1$. Suppose that

$$\sigma_g^m = \sum_{h \in G_1} \lambda_h \sigma_h,$$

where $\lambda_h, h \in G_1$, are nonnegative integers. Then we have

$$\chi(\sigma_g)^m = \sum_{h \in G_1} \lambda_h \chi(\sigma_h)$$

and so

$$n_g^m \varepsilon_g^m = \sum_{h \in G_1} \lambda_h \chi(\sigma_h). \quad (2)$$

But since

$$n_g^m = |n_g^m \varepsilon_g^m| = \left| \sum_{h \in G_1} \lambda_h \chi(\sigma_h) \right| \leq \sum_{h \in G_1} \lambda_h |\chi(\sigma_h)| \leq \sum_{h \in G_1} \lambda_h n_h = n_g^m,$$

we conclude that $|\chi(\sigma_h)| = \chi(\sigma_h)$, for every $h \in G_1$ such that $\lambda_h \neq 0$. So from (2) we have $\varepsilon_g^m = 1$. 

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Now to complete the proof, we must show that for every faithful irreducible character $\chi \in \text{Irr}(G)$ and every $g \in G - G_1$, $\chi(\sigma_g) = 0$. Since from [3, Lemma 2.8] we have

$$1 = n_G G_1 = \frac{n_g}{a}$$

where $a = |\{t \in G_1 | tg = g\}|$, we conclude that $a \neq 1$. This implies that there exists $t \in G_1$ such that $tg = g$. Then if for some $\chi \in \text{Irr}(G)$, $\chi(\sigma_t) \neq 1$, the equality $\chi(\sigma_g)\chi(\sigma_t) = \chi(\sigma_g)$ implies that $\chi(\sigma_g) = 0$. So if $\chi \in \text{Irr}(G)$ is a faithful character, then for every $g \in G - G_1$ we have $\chi(\sigma_g) = 0$, as desired. 

In the following, by $\pi(m)$ we mean the set of all prime divisors of the integer $m$.

**Theorem 3.2.** Let $(X,G)$ be a nilpotent association scheme of class $n$. If for some faithful irreducible character $\chi \in \text{Irr}(G)$ and $g \in G - G_{n-1}$, $\chi(\sigma_g) \neq 0$ then $G_1$ and $G/G_{n-1}$ are $p$-groups, for some prime $p$ and moreover, $\chi(\sigma_g)$ is a linear combination of the $p$-power roots of unity with real coefficients.

**Proof.** Proceed by induction on $n$. If $n = 2$, then the results follow from Lemma 3.1. So we can assume that $n \geq 3$. Let $n_{G_1} = m$ and $n_{G/G_{n-1}} = s$. Assume that $\chi \in \text{Irr}(G)$ is a faithful irreducible character and $g \in G - G_{n-1}$ such that $\chi(\sigma_g) \neq 0$. First we claim that $\pi(m) = \pi(s)$. To do so, let $p \in \pi(m)$. Then there exists an element $t \in G_1$ of order $p$. Since $G/G_{n-1}$ is a finite group of order $s$, it follows that $(\sigma_g \sigma_t)^s \in G_{n-1}$. Suppose that

$$ (\sigma_g \sigma_t)^s = \sum_{h \in G_{n-1}} \lambda_h \sigma_h, $$

where $\lambda_h, h \in G_{n-1}$, are nonnegative integers. Then we have

$$ \chi(\sigma_g \sigma_t)^s = \sum_{h \in G_{n-1}} \lambda_h \chi(\sigma_h). $$

If $\chi(\sigma_h) \neq 0$ for some $h \in G_i$, $2 \leq i \leq n - 1$, the inductive hypothesis shows that $n_{G_i}$ is a power of prime $p$ and $\chi(\sigma_h)$ is a linear combination of $p$-power roots of unity with real coefficients. Then the equality

$$ \chi(\sigma_g)^s = \sum_{h \in G_{n-1}} \lambda_h \chi(\sigma_g \sigma_t)^s $$

implies that $\chi(\sigma_g)^s$ is a linear combination of the $p$-power roots of unity with real coefficients.

Otherwise, we have

$$ \chi(\sigma_g \sigma_t)^s = \sum_{h \in G_1} \lambda_h \chi(\sigma_h) $$

and so

$$ \chi(\sigma_g)^s = \sum_{h \in G_1} \lambda_h \chi(\sigma_g \sigma_t)^s. $$
Then we conclude that $\chi(\sigma_g)^s$ is a linear combination of the $m$-th roots of unity with nonnegative integer coefficients.

On the other hand, let $(A, B)$ be the dual of $(X, G)$. Since $B/\text{Ker}(G_1) \simeq \widehat{G}_1$, where $\widehat{G}_1$ is the dual of $CG_1$, and $\widehat{G}_1 \simeq G_1$, we conclude that $\text{Supp}(\Delta^{sm}) \in \text{Ker}(G_1)$. Then $\text{Supp}(\chi^m) \in \text{Irr}(G/G_1)$. Thus we have

$$\chi^m(\sigma_g) = \sum_{\psi \in \text{Irr}(G/G_1)} \lambda_\psi \psi(\sigma_g),$$

where $\lambda_\psi \in \mathbb{R}$ for every $\psi \in \text{Irr}(G/G_1)$. Since $\chi^m(\sigma_g) \neq 0$, it follows that for some $\psi \in \text{Irr}(G/G_1)$, $\psi(\sigma_g) \neq 0$. If $\psi \in \text{Irr}(G_{n-1}/G_1)$, then we can consider $\psi$ as a faithful irreducible character of $G/\text{Ker}(\psi)$ and the inductive hypothesis yields

$$(G/\text{Ker}(\psi))/(G_{n-1}/\text{Ker}(\psi)) \simeq G/G_{n-1}$$

is a $q$-group, for some prime $q$. So $\psi(\sigma_g)$ is a linear combination of $q$-power roots of unity with real coefficients. Then $\chi^m(\sigma_g)$ is also a linear combination of the $q$-power roots of unity with real coefficients. Thus in this case from the equality

$$(n_q^{m-1} \chi^m(\sigma_g))^s = (\chi(\sigma_g)^m)^s = (\chi(\sigma_g)^s)^m$$

we conclude that $\pi(m) = q = \pi(s)$.

Otherwise, we have

$$\chi^m(\sigma_g) = \sum_{\psi \in \text{Irr}(G/G_{n-1})} \lambda_\psi \psi(\sigma_g)$$

and so $\chi^m(\sigma_g)$ is a linear combination of the $s$-th roots of unity with real coefficients. Then the equality

$$(n_q^{m-1} \chi^m(\sigma_g))^s = (\chi(\sigma_g)^m)^s = (\chi(\sigma_g)^s)^m$$

implies that $\pi(m) = \pi(s)$.

Now suppose that $\pi(m) = \{p_1, \ldots, p_f\}$. Then $G_1 = P_1 \times \ldots \times P_f$ where $P_i \in \text{Syl}_{p_i}(G_1)$, the set of Sylow $p_i$-subgroups of $G_1$. Assume that $|P_1| = p_1^{\alpha_1}$. Then $\text{Supp}(\chi_{P_1}^{p_1^{\alpha_1}}) \in \text{Irr}(G/P_1)$ and since $\chi_{P_1}^{p_1^{\alpha_1}}(\sigma_g) \neq 0$, it follows that there exists $\varphi \in \text{Irr}(G/P_1)$ such that $\varphi(\sigma_g) \neq 0$. Then we can consider $\varphi$ as a faithful irreducible character of $G/\text{Ker}(\varphi)$. Since $G/\text{Ker}(\varphi)$ is a nilpotent association scheme and we have

$$(G/\text{Ker}(\varphi))/(G_{n-1}/\text{Ker}(\varphi)) \simeq G/G_{n-1},$$

the first part of the proof yields

$$\pi(G_1/\text{Ker}(\varphi)) = \pi((G/\text{Ker}(\varphi))/(G_{n-1}/\text{Ker}(\varphi))) = \pi(G/G_{n-1}).$$

This implies that $\pi(G_1/\text{Ker}(\varphi)) = \pi(G_1)$. Therefore, we must have $\pi(m) = p_1 = \pi(s)$. Thus $G_1$ and $G/G_{n-1}$ are $p$-groups for some prime $p$ and moreover, $\chi(\sigma_g)$ is a linear combination of the $p$-power roots of unity with real coefficients, and the proof is complete.
Example 3.3. (This example is [9, as8, No. 12].) Let \((X, G)\) be the association scheme of order 8 with the following basic matrix

\[
\sum_{i=0}^{4} i\sigma_{g_i} = \begin{pmatrix}
0 & 1 & 2 & 2 & 3 & 4 & 4 & 4 \\
1 & 0 & 2 & 2 & 4 & 3 & 3 & 3 \\
2 & 2 & 0 & 1 & 3 & 4 & 3 & 4 \\
2 & 2 & 1 & 0 & 4 & 3 & 4 & 3 \\
4 & 3 & 4 & 3 & 0 & 2 & 2 & 1 \\
4 & 3 & 3 & 4 & 2 & 0 & 1 & 2 \\
3 & 4 & 4 & 3 & 2 & 1 & 0 & 2 \\
3 & 4 & 3 & 4 & 1 & 2 & 2 & 0
\end{pmatrix}
\]

where \(G = \{g_0, g_1, g_2, g_3, g_4\}\). Then from [9] the character table of the complex adjacency algebra of \(G\) is as follows

<table>
<thead>
<tr>
<th>(\sigma_{g_0})</th>
<th>(\sigma_{g_1})</th>
<th>(\sigma_{g_2})</th>
<th>(\sigma_{g_3})</th>
<th>(\sigma_{g_4})</th>
<th>(m_\chi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(\chi_2)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>(\chi_3)</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\chi_4)</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>(E(8) + E(8)^3)</td>
<td>(-E(8) - E(8)^3)</td>
</tr>
<tr>
<td>(\chi_5)</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>(-E(8) - E(8)^3)</td>
<td>(E(8) + E(8)^3)</td>
</tr>
</tbody>
</table>

where \(E(8)\) is a primitive 8-th root of 1. One can see that the sequence

\[\{g_0\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq G_3 = G\]

is the upper central series of \((X, G)\), where \(G_1 = \{g_0, g_1\}\) and \(G_2 = \{g_0, g_1, g_2\}\). Then the association scheme \((X, G)\) is a nilpotent association scheme of class 3. Since \(\chi_4(\sigma_{g_3}) \neq 0\), Theorem 3.2 implies that \(G_1\) and \(G/G_1\) are 2-groups and \(\chi_4(\sigma_{g_3})\) is a linear combination of 2-power roots of unity with real coefficients.

Corollary 3.4. Let \((X, G)\) be a nilpotent association scheme of class \(n\) such that \(\pi(G_i/G_{i-1}), 1 \leq i \leq n,\) are distinct sets. Then for every faithful irreducible character \(\chi \in \text{Irr}(G/G_{i-1})\) and \(g \in G - G_i\), \(\chi(\sigma_g) = 0\), where \(1 \leq i \leq n - 1\).

Proof. We first consider the case that \(i = 1\). Then since \(\pi(G_1)\) is different from \(\pi(G_i/G_{i-1}), 3 \leq i \leq n,\) Theorem 3.2 follows that for every faithful irreducible character \(\chi \in \text{Irr}(G)\) and every \(g \in G - G_1\), \(\chi(\sigma_g) = 0\). Now suppose that \(i \geq 2\) and consider the nilpotent association scheme \(G/G_{i-1}\) with the following upper central series

\[\{1_{G_{i-1}}\} \subseteq G_i/G_{i-1} \subseteq \cdots \subseteq G_{n-1}/G_{i-1} \subseteq G_n/G_{i-1} = G/G_{i-1}\]

Then an argument similar to that given for \(i = 1\) shows that \(\chi(\sigma_{g_{G_{i-1}}}) = 0\), for every faithful irreducible character \(\chi \in \text{Irr}(G/G_{i-1})\) and every \(g^{G_{i-1}} \in G/G_{i-1} - G_i/G_{i-1}\). This implies that for every \(\chi \in \text{Irr}(G/G_{i-1})\) and every \(g \in G - G_i\), \(\chi(\sigma_g) = \frac{n_\chi}{n_{\sigma_{g_{G_{i-1}}}}} \chi(\sigma_{g^{G_{i-1}}}) = 0\) (see [7, Theorem 3.5]). This completes the proof. \(\blacksquare\)
Remark 3.5. Let \((X, G)\) be an association scheme such that it satisfies the hypotheses of Corollary 3.4. Then from \([5, \text{Theorem 3.4}]\) it follows that for every faithful irreducible character \(\chi \in \text{Irr}(G/G_{i-1})\), there exists an irreducible character \(\psi \in \text{Irr}(G_i/G_{i-1})\) such that \(\chi = \psi^G\), where \(\psi^G\) is the induced character of \(\psi\) to \(G\).

In the following, we give a class of schurian association schemes. We recall that the association scheme \((X, G)\) is schurian if and only if \((X, G)\) is a quotient scheme of a thin association scheme, see \([12]\). We refer the reader to \([11]\) for the definition of wreath products of association schemes.

Corollary 3.6. Let \((X, G)\) be a nilpotent association scheme of class \(n\) such that for every \(1 \leq i \leq n\), \(G_i/G_{i-1}\) is a group of prime order \(p_i\). If \(p_i, 1 \leq i \leq n\), are distinct prime numbers, then

\[
G = G_1 \wr (G_2/G_1) \wr (G_3/G_2) \wr \cdots \wr (G_n/G_{n-1}).
\]

Proof. Proceed by induction on \(n\). Let \(n = 2\) and suppose that \(n_{G_1} = p_1\) and \(n_{G_2}/G_1 = p_2\), where \(p_1\) and \(p_2\) are distinct prime numbers. We claim that \(|G_2/G_1| = |G_2| - |G_1| + 1\). To do so, let \(g_1, g_2 \in G_2 - G_1\) such that \(g_1^{G_1} = g_2^{G_2}\). Then \(g_1g_2^* \in G_1\) and so \(g_2 = g_1t\), for some \(t \in G_1\).

On the other hand, one can see that the set

\[
T = \{t \in G_1 | g_1t = g_1\}
\]

is a subgroup of \(G_1\) and so \(T = \{1_X\} \) or \(T = G_1\). Suppose that \(T = \{1_X\}\). Then since \(1 = n_{G_1} = \frac{n_{G_2}}{p_1} = n_{G_1}\), we obtain a contradiction. So we have \(T = G_1\). Thus \(g_2 = g_1t = g_1\) and we conclude that \(G_2/G_1 = \{g^{G_1} | g \in G_2 - G_1\} \cup \{1_X\}\). Hence \(|G_2/G_1| = |G_2| - |G_1| + 1\) and the claim is proved. Then from \([4, \text{Corollary 4.5}]\) we conclude that \(G_2 = G_1 \wr (G_2/G_1)\).

Now suppose that \(n \geq 3\). Since \(G_1\) is a cyclic group of prime order, it follows that for every \(i \geq 2\), \(\chi \in \text{Irr}(G_i) - \text{Irr}(G_i/G_1)\) is a faithful irreducible character. This implies that for every \(g \in G_i - G_{i-1}\), \(\chi(\sigma_g) = 0\). In particular, for every \(\chi \in \text{Irr}(G) - \text{Irr}(G/G_1)\) and every \(g \in G - G_1\) we have \(\chi(\sigma_g) = 0\). Similarly if we consider the upper central series

\[
\{1_X/G_1\} \subseteq G_{i+1}/G_i \subseteq \cdots \subseteq G_{n-1}/G_i \subseteq G_n/G_i = G/G_i
\]

for the nilpotent association scheme \(G/G_i\), we conclude that for every \(\chi \in \text{Irr}(G/G_i) - \text{Irr}(G/G_{i+1})\) and every \(g \in G - G_i\), \(\chi(\sigma_g) = 0\). In particular, for every \(\chi \in \text{Irr}(G) - \text{Irr}(G/G_{n-1})\) and every \(g \in G - G_{n-1}\), \(\chi(\sigma_g) = 0\).

Now let \(g \in G - G_{n-1}\). Then since

\[
\sigma_g \sigma_g^* = \sum_{h \in G_{n-1}} \lambda_{gg^*h} \sigma_h,
\]


we conclude that
\[ \chi(\sigma_g \sigma_g^*) = \chi(\sigma_g)(\sigma_g^*) = \sum_{h \in G_{n-1}} \lambda_{gg_h} \chi(h) = n_g^2, \tag{3} \]
for every \( \chi \in \text{Irr}(G/G_{n-1}) \). So from (3) we have
\[ |\chi(\sigma_g)| = n_g. \tag{4} \]

One the other hand, from [10, Lemma 3.1] it follows that
\[ n_g|X| = \sum_{\chi \in \text{Irr}(G)} m_{\chi} \chi(\sigma_g) \chi(\sigma_g^*). \tag{5} \]
Since for every \( \chi \in \text{Irr}(G) - \text{Irr}(G/G_{n-1}) \) we have \( \chi(\sigma_g) = 0 \), equality (5) implies that
\[ n_g|X| = \sum_{\chi \in \text{Irr}(G/G_{n-1})} m_{\chi} |\chi(\sigma_g)|^2. \tag{6} \]
Furthermore, since from [7, Theorem 4.1] we have \( m_{\chi} = 1 \) for every \( \chi \in \text{Irr}(G/G_{n-1}) \), equality (6) shows that
\[ n_g|X| = \sum_{\chi \in \text{Irr}(G/G_{n-1})} |\chi(\sigma_g)|^2. \tag{7} \]
Then from equality (4) along with equality (7) we conclude that \( |\text{Irr}(G/G_{n-1})| n_g^2 = n_g|X| \). Thus \( \frac{n_G}{n_{G_{n-1}}} n_{G_{n-1}} = n_g|X| \) and so \( n_{G_{n-1}} = n_g \). Then from the equality
\[ n_g n_{G_{n-1}} = \frac{n_G}{n_{G_{n-1}}} n_{G_{n-1}} \]
it follows that \( gG_{n-1} = \{g\} \). This implies that if \( g_1 G_{n-1} = g_2 G_{n-1} \), for some \( g_1, g_2 \in G - G_{n-1} \), then \( \{g_1\} = g_1 G_{n-1} = g_2 G_{n-1} = \{g_2\} \). Thus we conclude that \( |G/G_{n-1}| = |G| - |G_{n-1}| + 1 \). So \( |G| = |G_{n-1}| + |G/G_{n-1}| - 1 \) and from [4, Corollary 4.5] we conclude that \( G = G_{n-1} \wr (G/G_{n-1}) \). Therefore, from the inductive hypothesis we obtain
\[ G = G_1 \wr (G_2/G_1) \wr (G_3/G_2) \wr \cdots \wr (G_n/G_{n-1}) \]
and we are done.

\[ \blacksquare \]

**Example 3.7.** (This example is [9, as30, No. 120].)
Let \((X, G)\) be the association scheme of order 30 No. 120, where \( G = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7\} \). Then from [9] the character table of the complex adjacency algebra of \( G \) is as follows
where $E(5)$ is a primitive 5-th root of 1. One can see that $(X, G)$ is a nilpotent association scheme with the upper central series
\[
\{g_0\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq G_3 = G,
\]
where $G_1 = \{g_0, g_1, g_2, g_3, g_4\}$, $G_2 = \{g_0, g_1, g_2, g_3, g_4, g_5\}$ and $G_3 = G$. Moreover, $G_1, G_2/G_1$ and $G/G_2$ are cyclic groups of order 5, 2 and 3 respectively. Then from Corollary 3.6 we conclude that $G = G_1 \wr (G_2/G_1) \wr (G/G_2)$.

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