WEIGHTED SEMIGROUP MEASURE ALGEBRA AS A WAP-ALGEBRA

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A Banach algebra \( \mathfrak{A} \) for which the natural embedding \( x \mapsto \hat{x} \) of \( \mathfrak{A} \) into \( WAP(\mathfrak{A})^* \) is bounded below; that is, for some \( m \in \mathbb{R} \) with \( m > 0 \) we have \( ||\hat{x}|| \geq m||x|| \), is called a WAP-algebra. Through we mainly concern with weighted measure algebra \( M_b(S, \omega) \), where \( \omega \) is a weight on a semi-topological semigroup \( S \). We study those conditions under which \( M_b(S, \omega) \) is a WAP-algebra (respectively dual Banach algebra). In particular, \( M_b(S) \) is a WAP-algebra (respectively dual Banach algebra) if and only if \( wap(S) \) separates the points of \( S \) (respectively \( S \) is compactly cancellative semigroup).

We apply our results for improving some older results in the case where \( S \) is discrete.

Keywords: WAP-algebra, dual Banach algebra, Arens regularity, weak almost periodicity.


1. Introduction and Preliminaries

Throughout this paper, we study those conditions under which \( M_b(S, \omega) \) is either a WAP-algebra or a dual Banach algebra. Our main result in section 2 is that for a locally compact topological semigroup and a continuous weight \( \omega \) on \( S \), the measure algebra \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \) if and only if for all compact subsets \( F \) and \( K \) of \( S \), the maps \( \frac{\chi_F}{\omega} \) and \( \frac{\chi_K}{\omega} \) vanishes at infinity. This improved the result of Abolghasemi, Rejali, and Ebrahimi Vishki [1] to include the case where \( S \) is not necessarily discrete. As a consequence in non-weighted case, we conclude for a locally compact topological semigroup \( S \), the measure algebra \( M_b(S) \) is a dual Banach algebra with respect to \( C_0(S) \) if and only if \( S \) is a compactly cancellative semigroup. The

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later result improved the well known result of Dales, Lau and Strauss [7, Theorem 4.6], \( \ell_1(S) \) is dual Banach algebra with respect to \( c_0(S) \) if and only if \( S \) is weakly cancellative semigroup.

Section 3 is devoted to study WAP-algebras on a semigroup \( S \). For every weighted locally compact semi-topological semigroup \( (S, \omega) \), \( M_0(S, \omega) \) is a WAP-algebra if and only if the evaluation map \( \epsilon : S \to \hat{X} \) is one to one, where \( \hat{X} = MM(wap(S, 1/\omega)) \). Our main result of this section is that \( M_0(S, \omega) \) is a WAP-algebra if and only if \( \omega \) separate the points of \( S \). If \( C_0(S, 1/\omega) \subseteq wap(S, 1/\omega) \) then \( wap(S, 1/\omega) \) separate the points of \( S \). Thus \( M_0(S, \omega) \) is a WAP-algebra. We may ask whether, if \( M_0(S, \omega) \) is a WAP-algebra then \( C_0(S, 1/\omega) \subseteq wap(S, 1/\omega) \). We answer to this question negatively by a counter example. Then we exhibit some necessary and sufficient condition for \( \omega \) separate the points of \( S \) if \( C_0(S, 1/\omega) \subseteq wap(S, 1/\omega) \). We may ask whether, if \( M_0(S, \omega) \) is a WAP-algebra then \( \omega \) separate the points of \( S \). We end the paper by some examples which show that our results cannot be improved.

The dual \( \mathfrak{A}^* \) of a Banach algebra \( \mathfrak{A} \) can be turned into a Banach \( \mathfrak{A} \)-module in a natural way, by setting

\[
\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle \quad (a, b \in \mathfrak{A}, f \in \mathfrak{A}^*).
\]

A dual Banach algebra is a Banach algebra \( \mathfrak{A} \) such that \( \mathfrak{A} = (\mathfrak{A}_s)^* \), as a Banach space, for some Banach space \( \mathfrak{A}_s \), and such that \( \mathfrak{A}_s \) is a closed \( \mathfrak{A} \)-submodule of \( \mathfrak{A}^* \); or equivalently, the multiplication on \( \mathfrak{A} \) is separately weak*-continuous. We call \( \mathfrak{A}_s \) the predual of \( \mathfrak{A} \). It should be remarked that the predual of a dual Banach algebra need not be unique, in general (see [5, 10]); so we usually point to the involved predual of a dual Banach algebra.

A functional \( f \in \mathfrak{A}^* \) is said to be weakly almost periodic if \( \{ f \cdot a : \|a\| \leq 1 \} \) is relatively weakly compact in \( \mathfrak{A}^* \). We denote by \( WAP(\mathfrak{A}) \) the set of all weakly almost periodic elements of \( \mathfrak{A}^* \). It is easy to verify that, \( WAP(\mathfrak{A}) \) is a (norm) closed subspace of \( \mathfrak{A}^* \).

It is known that the multiplication of a Banach algebra \( \mathfrak{A} \) has two natural but, in general, different extensions (called Arens products) to the second dual \( \mathfrak{A}^{**} \) each turning \( \mathfrak{A}^{**} \) into a Banach algebra. When these extensions are equal, \( \mathfrak{A} \) is said to be (Arens) regular. It can be verified that \( \mathfrak{A} \) is Arens regular if and only if \( WAP(\mathfrak{A}) = \mathfrak{A}^* \). Further information for the Arens regularity of Banach algebras can be found in [5, 6].

WAP-algebras, as a generalization of the Arens regular algebras, has been introduced and intensively studied in [9]. A Banach algebra \( \mathfrak{A} \) for which the natural embedding \( x \mapsto \hat{x} \) of \( \mathfrak{A} \) into \( WAP(\mathfrak{A})^* \) where \( \hat{x}(\gamma) = \gamma(x) \) for \( \gamma \in WAP(\mathfrak{A}) \), is bounded below; that is, for
some \( m \in \mathbb{R} \) with \( m > 0 \) we have \( \|ix\| \geq m\|x\| \), is called a WAP-algebra. When \( \mathfrak{A} \) is Arens regular or dual Banach algebra, the natural embedding of \( \mathfrak{A} \) into \( WAP(\mathfrak{A})^* \) is isometric [16, Corollary 4.6]. Also Theorem 3.1 shows that \( M_b(S,\omega) \) is a WAP-algebra if and only if this embedding is isometric and of course bounded below, however in general \( M_b(S,\omega) \) is neither Arens regular nor dual Banach algebra. It has also known that \( \mathfrak{A} \) is a WAP-algebra if and only if it admits an isometric representation on a reflexive Banach space.

Moreover, group algebras are also always WAP-algebras, however; they are neither dual Banach algebras, nor Arens regular in the case where the underlying group is not discrete, see [17]. Ample information about WAP-algebras with further details can be found in the impressive paper [9].

A character on an abelian algebra \( \mathfrak{A} \) is a non-zero homomorphism \( \tau : \mathfrak{A} \to \mathbb{C} \). The set of all characters on \( \mathfrak{A} \) endowed with relative weak*-topology is called character space of \( \mathfrak{A} \).

Following [3], a semi-topological semigroup is a semigroup \( S \) equipped with a Hausdorff topology under which the multiplication of \( S \) is separately continuous. If the multiplication of \( S \) is jointly continuous, then \( S \) is said to be a topological semigroup. We write \( \ell^\infty(S) \) for the commutative \( C^* \)-algebra of all bounded complex-valued functions on \( S \). In the case where \( S \) is locally compact we also write \( C(S) \) and \( C_0(S) \) for the \( C^* \)-subalgebras of \( \ell^\infty(S) \) consist of continuous elements and continuous elements which vanish at infinity, respectively. We also denote the space of all weakly almost periodic functions on \( S \) by \( wap(S) \) which is defined by

\[
wap(S) = \{ f \in C(S) : \{ R_s f : s \in S \} \text{ is relatively weakly compact} \},
\]

where \( R_s f(t) = f(ts), \ (s,t \in S) \). Then \( wap(S) \) is a \( C^* \)-subalgebra of \( C(S) \) and its character space \( S^{wap} \), endowed with the Gelfand topology, enjoys a (Arens type) multiplication that turns it into a compact semi-topological semigroup. The evaluation mapping \( \epsilon : S \to S^{wap} \) is a homomorphism with dense image and it induces an isometric \( * \)-isomorphism from \( C(S^{wap}) \) onto \( wap(S) \). Many other properties of \( wap(S) \) and its inclusion relations among other function algebras are completely explored in [3].

Let \( M_b(S) \) be the Banach space of all complex regular Borel measures on \( S \), which is known as a Banach algebra with the total variation norm and under the convolution product \(*\) defined by the equation

\[
\langle \mu * \nu, g \rangle = \int_S \int_S g(xy) d\mu(x) d\nu(y) \quad (g \in C_0(S))
\]
and as dual of $C_0(S)$. Throughout, a weight on $S$ is a Borel measurable function $\omega : S \to (0, \infty)$ such that
\[
\omega(st) \leq \omega(s)\omega(t), \quad (s, t \in S).
\]
For $\mu \in M_b(S)$ we define $(\mu \omega)(E) = \int_E \omega d\mu$, ($E \subseteq S$ is Borel set). If $\omega \geq 1$, then
\[
M_b(S, \omega) = \{\mu \in M_b(S) : \mu \omega \in M_b(S)\}
\]
is known as a Banach algebra which is called the weighted semigroup measure algebra (see [6, 12, 13, 14] for further details about such algebras and arbitrary weight functions). Let $S$ be a locally compact semigroup, and let $B(S)$ denote the space of all Borel measurable and bounded functions on $S$. Set $B(S, 1/\omega) = \{f : S \to \mathbb{C} : f/\omega \in B(S)\}$. A standard predual for $M_b(S, \omega)$ is
\[
C_0(S, 1/\omega) = \{f \in B(S, 1/\omega) : f/\omega \in C_0(S)\}.
\]
Let $f \in C(S, 1/\omega)$ then $f$ is called $\omega$-weakly almost periodic if the set $\{R_s f/\omega(s): s \in S\}$ is relatively weakly compact in $C(S)$, where $R_s$ is defined as above. The set of all $\omega$-weakly almost periodic functions on $S$ is denoted by $wap(S, 1/\omega)$.

In the case where $S$ is discrete we write $\ell_1(S, \omega)$ instead of $M_b(S, \omega)$ and $c_0(S, 1/\omega)$ instead of $C_0(S, 1/\omega)$. Then the space
\[
\ell_1(S, \omega) = \{f : f = \sum_{s \in S} f(s)\delta_s, \quad ||f||_{1, \omega} = \sum_{s \in S} |f(s)|\omega(s) < \infty\}
\]
(where, $\delta_s \in \ell_1(S, \omega)$ be the point mass at $s$ which can be thought as the vector basis element of $\ell_1(S, \omega)$ ) equipped with the multiplication
\[
f * g = \sum_{r \in S} \sum_{st = r} f(s)g(t)\delta_r
\]
(and also define $f * g = 0$ if for each $r \in S$ the equation $st = r$ has no solution;) is a Banach algebra which will be called weighted semigroup algebra. We also suppress 1 from the notation whenever $w = 1$.

2. Semigroup Measure Algebras as Dual Banach Algebras

It is known that the semigroup algebra $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if $S$ is weakly cancellative semigroup, see [7, Theorem4.6]. Throughout this section $\omega$ is a continuous weight on $S$. This result has been extended for the weighted
semigroup algebras \( \ell_1(S, \omega) \); [1, 8]. In this section we extend this results to the non-discrete case. We provide some necessary and sufficient conditions that the measure algebra \( M_b(S, \omega) \) becomes a dual Banach algebra with respect to the predual \( C_0(S, 1/\omega) \).

Let \( F \) and \( K \) be nonempty subsets of a semigroup \( S \) and \( s \in S \). We put

\[
s^{-1}F = \{ t \in S : st \in F \}, \quad Fs^{-1} = \{ t \in S : ts \in F \}
\]

and we also write \( s^{-1}t \) for the set \( s^{-1}\{t\} \), \( FK^{-1} \) for \( \cup\{Fs^{-1} : s \in K\} \) and \( K^{-1}F \) for \( \cup\{s^{-1}F : s \in K\} \).

A semigroup \( S \) is called left (respectively, right) zero semigroup if \( xy = x \) (respectively, \( xy = y \)), for all \( x, y \in S \). A semigroup \( S \) is called zero semigroup if there exist \( z \in S \) such that \( xy = z \) for all \( x, y \in S \). A semigroup \( S \) is said to be left (respectively, right) weakly cancellative semigroup if \( s^{-1}F \) (respectively, \( Fs^{-1} \)) is finite for each \( s \in S \) and each finite subset \( F \) of \( S \). A semigroup \( S \) is said to be weakly cancellative semigroup if it is both left and right weakly cancellative semigroup.

A semi-topological semigroup \( S \) is said to be compactly cancellative semigroup if for every compact subsets \( F \) and \( K \) of \( S \) the sets \( F^{-1}K \) and \( KF^{-1} \) are compact set.

**Lemma 2.1.** Let \( S \) be a topological semigroup. For every compact subsets \( F \) and \( K \) of \( S \) the sets \( F^{-1}K \) and \( KF^{-1} \) are closed.

**Proof.** If \( F^{-1}K \) is empty, then it is closed. Let \( x \) be in the closure of \( F^{-1}K \). Then there is a net \( (x_\alpha) \) in \( F^{-1}K \) such that \( x_\alpha \to x \). Since \( x_\alpha \in F^{-1}K \) there is a net \( (f_\alpha) \) in \( F \) such that \( f_\alpha x_\alpha \in K \). Using the compactness of \( F \) and \( K \), by passing to a subnet, if necessary, we may suppose that \( f_\alpha x_\alpha \to k \) and \( f_\alpha \to f \), for some \( f \in F \) and \( k \in K \). So \( fx = k \in K \), that is \( x \in F^{-1}K \). Therefore \( F^{-1}K \) is closed. A similar argument shows that \( KF^{-1} \) is also closed. \( \square \)

In the next result we study \( M_b(S, \omega) \) from the dual Banach algebra point of view.

**Theorem 2.1.** Let \( S \) be a locally compact topological semigroup and \( \omega \) be a continuous weight on \( S \). Then the measure algebra \( M_b(S, \omega) \) is a dual Banach algebra with respect to the predual \( C_0(S, 1/\omega) \) if and only if for all compact subsets \( F \) and \( K \) of \( S \), the maps \( \overline{\lambda_{F^{-1}K}} \omega \) and \( \overline{\lambda_{KF^{-1}}} \omega \) vanishes at infinity.

**Proof.** Suppose that \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \) and let \( \epsilon > 0 \). Let \( K, F \) be nonempty compact subsets of \( S \) with a net \( (x_\alpha) \) in \( \{ t \in F^{-1}K : 1/\omega(t) \geq \epsilon \} \). Let \( C^+_{00}(S) \) denote the non-negative continuous functions with compact support on \( S \) and set \( C^+_{00}(S, 1/\omega) = \{ f \in C_0(S, 1/\omega) : f/\omega \in C^+_{00}(S) \} \). Since \( \omega \) is
continuous we may choose \( f \in C^+_{00}(S, 1/\omega) \) with \( f(K) = 1 \). There is a net \((t_\alpha) \subset F\) such that \( t_\alpha x_\alpha \in K \) and the compactness of \( F \) guarantees the existence of a subnet \((t_\gamma)\) of \((t_\alpha)\) such that \( t_\gamma \to t_0 \) for some \( t_0 \) in \( S \). Indeed, for \( s \in S \),
\[
\lim_{\gamma} \left( \frac{\delta_{t_\gamma}}{\omega}(s) \right) = \lim_{\gamma} \frac{f(t_\gamma s)}{\omega(s)} = \frac{f(t_0 s)}{\omega(s)} = \delta_{t_0} \frac{f}{\omega}(s)
\]
there is a \( \gamma_0 \) such that
\[
\{ t \in \bigcup_{\gamma \geq \gamma_0} t_\gamma^{-1} K : 1/\omega(t) \geq \varepsilon \} \subseteq \bigcup_{\gamma \geq \gamma_0} \{ r \in S : \left( \frac{\delta_{t_\gamma}}{\omega} \right)(r) \geq \varepsilon \}
\]
Let \( H = \{ t_\gamma : \gamma \geq \gamma_0 \} \cup \{ t_0 \} \). Then
\[
\{ t \in H^{-1} K : 1/\omega(t) \geq \varepsilon \} = \{ t \in \bigcup_{\gamma \geq \gamma_0} t_\gamma^{-1} K \cup t_0^{-1} K : 1/\omega(t) \geq \varepsilon \}
\]
and so \( \{ t \in H^{-1} K : 1/\omega(t) \geq \varepsilon \} \) is compact. Furthermore, \( t_\gamma x_\gamma \in K \), that is \((x_\gamma)\) is a net in compact set \( \{ t \in H^{-1} K : 1/\omega(t) \geq \varepsilon \} \). This means that \((x_\alpha)\) has a convergent subnet. Thus \( \{ t \in F^{-1} K : 1/\omega(t) \geq \varepsilon \} \) is compact set and \( \frac{x_f}{\omega} \) vanishes at infinity. Similarly \( \frac{x_f}{\omega} \) vanishes at infinity. This is the proof of necessity.

The sufficiency can be adopted from [1, Proposition 3.1] with some modifications. Let \( f \in C_0(S, 1/\omega) \), \( \mu \in M_b(S, \omega) \) and \( \varepsilon > 0 \) be arbitrary. There exist compact subsets \( F \) and \( K \) of \( S \) such that \( |\frac{f}{\omega}(s)| < \varepsilon \) for all \( s \in K \) and \( |(\mu \omega)|(S \setminus F) < \varepsilon \).

Let \( s \not\in \{ t \in F^{-1} K : \omega(t) \leq \frac{1}{\varepsilon} \} \), which is compact by hypothesis. Then
\[
|\frac{\mu f}{\omega}(s)| = \left| \int_S \frac{f(t s)}{\omega(s)} d\mu(t) \right|
\leq \left| \int_F \frac{f(t s)}{\omega(s)} d\mu(t) \right| + \left| \int_{S \setminus F} \frac{f(t s)}{\omega(s)} d\mu(t) \right|
\leq \int_F \left| \frac{f(t s)}{\omega(t s)} \right| \omega(t) d|\mu|(t) + \int_{S \setminus F} \left| \frac{f(t s)}{\omega(t s)} \right| \omega(t) d|\mu|(t)
\leq \varepsilon \int_S \omega(t) d|\mu|(t) + \| f \|_{\omega, \infty} \int_{S \setminus F} \omega(t) d|\mu|(t)
\leq \varepsilon \| \mu \|_\omega + \| f \|_{\omega, \infty}
\]
That is, \( \mu f \in C_0(S, 1/\omega) \). Therefore \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \).

The next Corollaries are immediate consequences of Theorem 2.1.
Corollary 2.1. Let $S$ be a locally compact topological semigroup. Then the measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if $S$ is a compactly cancellative semigroup.

Corollary 2.2. [1, Theorem 2.2] For a semigroup $S$ the semigroup algebra $\ell_1(S, \omega)$ is a dual Banach algebra with respect to the predual $c_0(S, 1/\omega)$ if and only if for all $s, t \in S$, the maps $\frac{X_{t^{-1}s}}{\omega}$ and $\frac{X_{s^{-1}t}}{\omega}$ are in $c_0(S)$.

Corollary 2.3. For a locally compact topological semigroup $S$, if $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ then $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$.

Corollary 2.4. Let $S$ be either a left zero (right zero) or a zero locally compact semigroup. There is a weight $\omega$ such that $M_b(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$ if and only if $S$ is $\sigma$-compact.

Proof. Let $K$ and $F$ be compact subsets of $S$. It can be readily verified that in either cases (being left zero, right zero or zero) the sets $F^{-1}K$ and $KF^{-1}$ are equal to either empty or $S$. Put

$$S_m = \{ t \in F^{-1}K : \omega(t) \leq m \} = \{ t \in S : \omega(t) \leq m \} \quad (m \in \mathbb{N}).$$

Then $S = \cup_{m \in \mathbb{N}} S_m$ and so $S$ is $\sigma$-compact. For the converse let $S = \cup_{n \in \mathbb{N}} S_n$ as a disjoint union of compact sets and let $z$ be a (left or right) zero for $S$. Define $\omega(z) = 1$ and $\omega(x) = 1 + n$ for $x \in S_n$ then $\omega$ is a weight on $S$ and $M_b(S, \omega)$ is a dual Banach algebra. \qed

Examples 2.1. (1) The set $S = \mathbb{R}^+ \times \mathbb{R}$ equipped with the multiplication

$$(x, y)(x', y') = (x + x', y') \quad ((x, y), (x', y') \in S)$$

and the weight $\omega(x, y) = e^{-x}(1 + |y|)$ is a weighted semigroup. In this example $[a, b]$ denotes a closed interval. As for $F = [a, b] \times [c, d]$ and $K = [e, f] \times [g, h]$, with
\[ F^{-1}K = \bigcup_{(x,y) \in F} (x,y)^{-1}K \]
\[ = \bigcup_{(x,y) \in F} \{(s,t) \in S : (x,y)(s,t) \in K\} \]
\[ = \bigcup_{(x,y) \in F} \{(s,t) \in S : (x+s,t) \in K\} \]
\[ = \bigcup_{(x,y) \in F} [e - x, f - x] \times [g, h] = [e - b, f - a] \times [g, h] \]

and
\[ KF^{-1} = \bigcup_{(x,y) \in F} K(x,y)^{-1} \]
\[ = \bigcup_{(x,y) \in F} \{(s,t) \in S : (s,t)(x,y) \in K\} \]
\[ = \bigcup_{(x,y) \in F} \{(s,t) \in S : (x+s,y) \in K\} \]
\[ = \bigcup_{(x,y) \in F} [e - x, f - x] \times \mathbb{R} = [e - b, f - a] \times \mathbb{R} \]

Thus
\[ F^{-1}K = [e - b, f - a] \times [g, h] \quad \text{and} \quad KF^{-1} = \begin{cases} [e - b, f - a] \times \mathbb{R} & \text{if} \ [c, d] \cap [g, h] \neq \emptyset \\ \emptyset & \text{if} \ [c, d] \cap [g, h] = \emptyset \end{cases} \]

\( M_0(S) \) is not a dual Banach algebra by Corollary 2.1. However, for all compact subsets \( F \) and \( K \) of \( S \), the maps \( \mathcal{X}_{F^{-1}K} \) and \( \mathcal{X}_{KF^{-1}} \) vanishes at infinity. So \( M_0(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \). This shows that the converse of Corollary 2.3 may not be valid.

(2) For the semigroup \( S = [0, \infty) \) endowed with the zero multiplication, neither \( M_0(S) \) nor \( \ell_1(S) \) is a dual Banach algebra. In fact, \( S \) is neither compactly nor weakly cancellative semigroup.

3. Semigroup Measure Algebras as WAP-Algebras

In this section, for a weighted locally compact semi-topological semigroup \((S, \omega)\), we investigate some necessary and sufficient condition for \( M_0(S, \omega) \) being WAP-algebra. First, we provide some preliminaries.
**Definition 3.1.** Let $\tilde{\mathcal{F}}$ be a linear subspace of $B(S, 1/\omega)$, and let $\tilde{\mathcal{F}}_r$ denote the set of all real-valued members of $\tilde{\mathcal{F}}$. A mean on $\tilde{\mathcal{F}}$ is a linear functional $\tilde{\mu}$ on $\tilde{\mathcal{F}}$ with the property that
\[
\inf_{s \in S} \frac{f}{\omega}(s) \leq \tilde{\mu}(f) \leq \sup_{s \in S} \frac{f}{\omega}(s) \quad (f \in \tilde{\mathcal{F}}_r).
\]
The set of all means on $\tilde{\mathcal{F}}$ is denoted by $M(\tilde{\mathcal{F}})$. If $\tilde{\mathcal{F}}$ is also an algebra with the multiplication given by $f \odot g := (f,g)/\omega$ $(f,g \in \tilde{\mathcal{F}})$ and if $\tilde{\mu} \in M(\tilde{\mathcal{F}})$ satisfies
\[
\tilde{\mu}(f \odot g) = \tilde{\mu}(f)\tilde{\mu}(g) \quad (f,g \in \tilde{\mathcal{F}}),
\]
then $\tilde{\mu}$ is said to be multiplicative. The set of all multiplicative means on $\tilde{\mathcal{F}}$ will be denoted by $MM(\tilde{\mathcal{F}})$.

Let $\tilde{\mathcal{F}}$ be a conjugate closed, linear subspace of $B(S, 1/\omega)$ such that $\omega \in \tilde{\mathcal{F}}$.

(i) For each $s \in S$ define $\epsilon(s) \in M(\tilde{\mathcal{F}})$ by $\epsilon(s)(f) = (f/\omega)(s)$ $(f \in \tilde{\mathcal{F}})$. The mapping $\epsilon : S \rightarrow M(\tilde{\mathcal{F}})$ is called the evaluation mapping. If $\tilde{\mathcal{F}}$ is also an algebra, then $\epsilon(S) \subseteq MM(\tilde{\mathcal{F}})$.

(ii) Let $\tilde{X} = M(\tilde{\mathcal{F}})$ (resp. $\tilde{X} = MM(\tilde{\mathcal{F}})$, if $\tilde{\mathcal{F}}$ is a subalgebra) be endowed with the relative weak* topology. For each $f \in \tilde{\mathcal{F}}$ the function $\hat{\mu} \in C(\tilde{X})$ is defined by
\[
\hat{\mu}(f) := \tilde{\mu}(f) \quad (\tilde{\mu} \in \tilde{X}).
\]
Furthermore, we define $\hat{\mathcal{F}} := \{\hat{\mu} : f \in \tilde{\mathcal{F}}\}$.

**Remark 3.1.** 

(i) The mapping $f \mapsto \hat{f} : \hat{\mathcal{F}} \rightarrow C(\tilde{X})$ is clearly linear and multiplicative if $\mathcal{F}$ is an algebra and $\tilde{X} = MM(\tilde{\mathcal{F}})$. Also it preserves complex conjugation, and is an isometry, since for any $f \in \mathcal{F}$
\[
||\hat{f}|| = \sup\{||\hat{\mu}(f)|| : \tilde{\mu} \in \tilde{X}\} = \sup\{||\mu(f/\omega)|| : \mu \in X, ||\mu|| \leq 1\}
\]
\[
= \sup\{||\mu(f/\omega)|| : \mu \in X\} \leq \sup\{||\mu(f/\omega)|| : \mu \in C(X)^*, ||\mu|| \leq 1\}
\]
\[
= ||f/\omega|| = ||f|| = \sup\{||\mu(f/\omega)|| : s \in S\} = \sup\{||\mu(f)|| : \mu \in C(X)^*, ||\mu|| \leq 1\}
\]
\[
= \sup\{||\mu(r)|| : s \in S\} \leq ||\hat{f}||,
\]
where $X = M(\mathcal{F})$ and $\mathcal{F} = \{f/\omega : f \in \tilde{\mathcal{F}}\}$. Note that $\hat{\epsilon}(s) = \epsilon(s)(f) = (\hat{\mu}(s)(f \in \tilde{\mathcal{F}}, s \in S)$. This identity may be written in terms of dual map $\epsilon^* : C(\tilde{X}) \rightarrow C(S, 1/\omega)$ as $\epsilon^*(\hat{f}) = f$ for $f \in \tilde{\mathcal{F}}$.

(ii) Let $\tilde{\mathcal{F}}$ be a conjugate closed linear subspace of $B(S, 1/\omega)$, containing $\omega$. Then $M(\tilde{\mathcal{F}})$ is convex and weak* compact, $co(\epsilon(S))$ is weak* dense in $M(\tilde{\mathcal{F}})$, $\hat{\mathcal{F}}^*$ is the weak*
closed linear span of $\epsilon(S)$, $\epsilon : S \rightarrow M(\tilde{F})$ is weak* continuous, and if $\tilde{F}$ is also an algebra, then $MM(\tilde{F})$ is weak* compact and $\epsilon(S)$ is weak* dense in $MM(\tilde{F})$.

(iii) Let $\tilde{F}$ be a $C^*$-subalgebra of $B(S, 1/\omega)$, containing $\omega$. If $\tilde{X}$ denotes the space $MM(\tilde{F})$ with the relative weak* topology, and if $\epsilon : S \rightarrow \tilde{X}$ denotes the evaluation mapping, then the mapping $f \rightarrow \tilde{\epsilon} : \tilde{F} \rightarrow C(\tilde{X})$ is an isometric isomorphism with the inverse $\epsilon^* : C(\tilde{X}) \rightarrow \tilde{F}$.

Let $\tilde{F} = \text{wap}(S, 1/\omega)$. Then $\tilde{F}$ is a $C^*$-algebra and a subspace of $WAP(M_b(S, \omega))$, see [11, Theorem1.6, Theorem3.3]. Set $\tilde{X} = MM(\tilde{F})$. By the above remark $\text{wap}(S, 1/\omega) \cong C(\tilde{X})$ and so

$$M_b(\tilde{X}) \cong C(\tilde{X})^* \cong \text{wap}(S, 1/\omega)^* \subset WAP(M_b(S, \omega))^*.$$ 

Let $\epsilon : S \rightarrow \tilde{X}$ be the evaluation mapping. We also define $\bar{\epsilon} : M_b(S, \omega) \rightarrow M_b(\tilde{X})$, by $\langle \bar{\epsilon}(\mu), f \rangle = \int S f \omega d\mu$ for $f \in \text{wap}(S, 1/\omega) \equiv C(\tilde{X})$. Then for every Borel set $B$ in $\tilde{X}$,

$$\bar{\epsilon}(\mu)(B) = (\mu \omega)(\epsilon^{-1}(B)).$$

In particular, $\bar{\epsilon}(\frac{\delta_x}{\omega}) = \delta_{\epsilon(x)}$.

The next theorem is the main result of this section.

**Theorem 3.1.** For every weighted locally compact semi-topological semigroup $(S, \omega)$ the following statements are equivalent:

1. The evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(\text{wap}(S, 1/\omega))$;
2. $\bar{\epsilon} : M_b(S, \omega) \rightarrow M_b(\tilde{X})$ is an isometric isomorphism;
3. $M_b(S, \omega)$ is a WAP-algebra.

**Proof.** (1) $\Rightarrow$ (2). Take $\mu \in M_b(S, \omega)$, say $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_j \in M_b(S, \omega)^+$. Set $\nu_j = \bar{\epsilon}(\mu_j) \in M_b(\tilde{X})^+$ for $j = 1, 2, 3, 4$, and set

$$\nu = \bar{\epsilon}(\mu) = \nu_1 - \nu_2 + i(\nu_3 - \nu_4).$$

Take $\delta > 0$. For each $j$, there exists Borel set $B_j$ in $\tilde{X}$ such that $\nu_j(B) \geq 0$ for each Borel subset $B$ of $B_j$ and $\sum_{j=1}^4 \nu_j(B_j) > ||\nu|| - \delta$. In fact, by Hahn decomposition theorem for signed measures $\lambda_1 = \nu_1 - \nu_2$ and $\lambda_2 = \nu_3 - \nu_4$ there exist four Borel sets $P_1, P_2, N_1$ and $N_2$ in $\tilde{X}$ such that

$$P_1 \cup N_1 = \tilde{X}, \quad P_1 \cap N_1 = \emptyset, \quad P_2 \cup N_2 = \tilde{X}, \quad P_2 \cap N_2 = \emptyset$$

and for every Borel set $E$ of $\tilde{X}$ we have,

$$\nu_1(E) = \lambda_1(P_1 \cap E), \quad \nu_2(E) = -\lambda_1(N_1 \cap E), \quad \nu_3(E) = \lambda_2(P_2 \cap E), \quad \nu_4(E) = -\lambda_2(N_2 \cap E).$$
that is \(\nu_1, \nu_2, \nu_3, \nu_4\) are concentrated respectively on \(P_1, N_1, P_2, N_2\).

Set \(D_1 := P_1 \cap N_2, D_2 := N_1 \cap P_2, D_3 := P_2 \cap P_1, D_4 := N_2 \cap N_1\). Then the family \(\{D_1, D_2, D_3, D_4\}\) is a partition of \(\tilde{X}\). Also for \(\delta > 0\) there is a compact set \(K\) for which

\[
||\nu|| - \delta \leq \sum_{j=1}^{4} ||\nu_{j|D_j}| - \delta \leq \sum_{j=1}^{4} \nu_{j|D_j}(K) = \sum_{j=1}^{4} \nu_j(D_j \cap K).
\]

Set \(B_j = D_j \cap K\). Then the sets \(B_1, B_2, B_3, B_4\) are pairwise disjoint.

Set \(C_j = (\epsilon)^{-1}(B_j)\), a Borel set in \(S\). Then \((\mu|_\omega)(C_j) = \nu_j(B_j)\). Since \(\epsilon\) is injection, the sets \(C_1, C_2, C_3, C_4\) are pairwise disjoint, and so

\[
||\mu||_\omega \geq \sum_{j=1}^{4} (\mu|_\omega(C_j)) \geq \sum_{j=1}^{4} (\mu_j|_\omega)(C_j) = \sum_{j=1}^{4} \nu_j(B_j) > ||\nu|| - \delta.
\]

This holds for each \(\delta > 0\), so \(||\mu||_\omega \geq ||\nu||\). A similar argument shows that \(||\mu||_\omega \leq ||\nu||\).

Thus \(\|\nu\| = ||\nu||\).

(2) \(\Rightarrow\) (1). Let \(P(S, \omega)\) denote the subspace of all probability measures of \(M_b(S, \omega)\) and \(ext(P(S, \omega))\) the extreme points of unit ball of \(P(S, \omega)\). Then \(ext(P(S, \omega)) = \{ \frac{\delta_x}{\omega(x)} : x \in S\} \cong S\) and \(ext(P(\tilde{X}) \cong \tilde{X}\), see [4, p.151]. By injectivity of \(\epsilon\), it maps the extreme points of the unit ball onto the extreme points of the unit ball, thus \(\epsilon : S \rightarrow \tilde{X}\) is a one to one map.

(2) \(\Rightarrow\) (3). Since \(\tilde{X}\) is compact, \(M_b(\tilde{X})\) is a dual Banach algebra with respect to \(C(\tilde{X})\), so it has an isometric representation \(\psi\) on a reflexive Banach space \(E\), see [9]. In the following commutative diagram,

\[
\begin{array}{ccc}
M_b(S, \omega) & \xrightarrow{\epsilon} & M_b(\tilde{X}) \\
\phi \downarrow & & \psi \downarrow \\
B(E) & & 
\end{array}
\]

If \(\epsilon\) is isometric, then so is \(\phi\).

Thus \(M_b(S, \omega)\) has an isometric representation on a reflexive Banach space \(E\) if \(\epsilon\) is an isometric isomorphism. So \(M_b(S, \omega)\) is a WAP-algebra if \(\epsilon\) is an isometric isomorphism.

(3) \(\Rightarrow\) (1). Let \(M_b(S, \omega)\) be a WAP-algebra. Since \(\ell_1(S, \omega)\) is a norm closed subalgebra of \(M_b(S, \omega)\), then \(\ell_1(S, \omega)\) is a WAP-algebra. Using the double limit criterion, it is a simple matter to check that \(wap(S, 1/\omega) = WAP(\ell_1(S, \omega))\) (see also [11, Theorem3.7]) where we treat \(\ell^\infty(S, 1/\omega)\) as an \(\ell_1(S, \omega)\)-bimodule. Then \(\tilde{\epsilon} : \ell_1(S, \omega) \rightarrow wap(S, 1/\omega)^\star\) is an isometric isomorphism. Since \(wap(S, 1/\omega)\) is a \(C^*\)-algebra, as (2) \(\Rightarrow\) (1), \(\epsilon : S \rightarrow \tilde{X}\) is one to one.

\(\square\)
**Corollary 3.1.** The following statements are equivalent.

1. $\ell_1(S,\omega)$ is a WAP-algebra;
2. $M_b(S,\omega)$ is a WAP-algebra.

For $\omega = 1$, it is clear that $\check{X} = S^{wap}$, and the map $\epsilon : S \rightarrow S^{wap}$ is one to one if and only if $wap(S)$ separates the points of $S$, see [3].

**Corollary 3.2.** For a locally compact semi-topological semigroup $S$, the following statements are equivalent:

1. $M_b(S)$ is a WAP-algebra;
2. $\ell_1(S)$ is a WAP-algebra;
3. The evaluation map $\epsilon : S \rightarrow S^{wap}$ is one to one;
4. $wap(S)$ separates the points of $S$.

**Definition 3.2.** Let $X, Y$ be sets and $f$ be a complex-valued function on $X \times Y$.

1. We say that $f$ is a cluster on $X \times Y$ if for each pair of sequences $(x_n), (y_m)$ of distinct elements of $X, Y$, respectively
   \[\lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m)\] (1) whenever both sides of (1) exist.
2. If $f$ is cluster and both sides of 1 are zero (respectively positive) in all cases, we say that $f$ is 0-cluster (respectively positive cluster).

In general $\{f \omega : f \in wap(S)\} \neq wap(S, 1/\omega)$. By using [2, Lemma1.4] the following is immediate.

**Lemma 3.1.** Let $\Omega(x, y) = \frac{\omega(xy)}{\omega(x)\omega(y)}$, for $x, y \in S$. Then

1. If $\Omega$ is cluster, then $\{f \omega : f \in wap(S)\} \subseteq wap(S, 1/\omega)$;
2. If $\Omega$ is positive cluster, then $wap(S, 1/\omega) = \{f \omega : f \in wap(S)\}$.

It should be noted that if $M_b(S)$ is Arens regular (resp. dual Banach algebra) then $M_b(S,\omega)$ is so. We don’t know that if $M_b(S)$ is WAP-algebra, then $M_b(S,\omega)$ is so. The following Lemma give a partial answer to this question.

**Corollary 3.3.** Let $S$ be a locally compact topological semigroup with a Borel measurable weight function $\omega$ such that $\Omega$ is cluster on $S \times S$.

1. If $M_b(S)$ is a WAP-algebra, then so is $M_b(S,\omega)$;
2. If $\ell_1(S)$ is a WAP-algebra, then so is $\ell_1(S,\omega)$. 
Proof. (1) Suppose that $M_b(S)$ is a WAP-algebra so $wap(S)$ separates the points of $S$. By lemma 3.1 for every $f \in wap(S)$, $f\omega \in wap(S,1/\omega)$. Thus the evaluation map $\epsilon : S \rightarrow \hat{X}$ is one to one.

(2) follows from (1). □

Corollary 3.4. For a locally compact semi-topological semigroup $S$,

(1) If $C_0(S) \subseteq wap(S)$, then the measure algebra $M_b(S)$ is a WAP-algebra.

(2) If $S$ is discrete and $c_0(S) \subseteq wap(S)$, then $\ell_1(S)$ is a WAP-algebra.

Proof. (1) By [3, Corollary 4.2.13] the map $\epsilon : S \rightarrow S \subseteq wap$ is one to one, thus $M_b(S)$ is a WAP-algebra.

(2) follows from (1). □

Dales, Lau and Strauss [7, Theorem 4.6, Proposition 8.3] showed that for a semi-group $S$, $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if $S$ is weakly cancellative. If $S$ is left or right weakly cancellative semigroup, then $\ell_1(S)$ is a WAP-algebra. The next example shows that the converse is not true, in general.

Example 3.1. Let $S = (\mathbb{N}, \min)$ then $wap(S) = c_0(S) \oplus \mathbb{C}$. So $\ell_1(S)$ is a WAP-algebra but $S$ is neither left nor right weakly cancellative. In fact, for $f \in wap(S)$ and all sequences $\{a_n\}, \{b_m\}$ with distinct element in $S$, we have $\lim_n f(b_m) = \lim_m \lim_n f(a_n b_m) = \lambda = \lim_n \lim_m f(a_n b_m) = \lim_n f(a_n)$, for some $\lambda \in \mathbb{C}$. This means $f - \lambda \in c_0(S)$ and $wap(S) \subseteq c_0(S) \oplus \mathbb{C}$. The other inclusion is clear.

If $\{x_n\}$ and $\{y_m\}$ are sequences in $S$ we obtain an infinite matrix $\{x_n y_m\}$ which has $x_n y_m$ as its entry in the $n$th row and $m$th column. As in [2], a matrix is said to be of row type $C$ (resp. column type $C$) if the rows (resp. columns) of the matrix are all constant and distinct. A matrix is of type $C$ if it is constant or of row or column type $C$.

J.W.Baker and A. Rejali in [2, Theorem 2.7(v)] showed that $\ell_1(S)$ is Arens regular if and only if for each pair of sequences $\{x_n\}, \{y_m\}$ with distinct elements in $S$ there is a submatrix of $\{x_n y_m\}$ of type $C$.

A matrix $\{x_n y_m\}$ is said to be upper triangular constant if $x_n y_m = s$ if and only if $m \geq n$ and it is lower triangular constant if $x_n y_m = s$ if and only if $m \leq n$. A matrix $\{x_n y_m\}$ is said to be $W$-type if every submatrix of $\{x_n y_m\}$ is neither upper triangular constant nor lower triangular constant.

Theorem 3.2. Let $S$ be a semigroup. The following statements are equivalent:

(1) $c_0(S) \subseteq wap(S)$. 


(2) For each pair \( \{x_n\}, \{y_m\} \) of sequences in \( S \),
\[
\{ \chi_s(x_n y_m) : n < m \} \cap \{ \chi_s(x_n y_m) : n > m \} \neq \emptyset;
\]

(3) For each pair \( \{x_n\}, \{y_m\} \) of sequences in \( S \) with distinct elements, \( \{x_n y_m\} \) is a \( W \)-type matrix;

(4) For every \( s \in S \), every infinite set \( B \subset S \) contains a finite subset \( F \) such that
\[
\cap \{ sb^{-1} : b \in F \} \backslash \left( \cap \{ sb^{-1} : b \in B \backslash F \} \right) \text{ and } \cap \{ b^{-1} s : b \in F \} \backslash \left( \cap \{ b^{-1} s : b \in B \backslash F \} \right)
\]
are finite.

\textbf{Proof.} (1)\( \iff \) (2). For all \( s \in S \), \( \chi_s \in \text{wap}(S) \) if and only if
\[
\{ \chi_s(x_n y_m) : n < m \} \cap \{ \chi_s(x_n y_m) : n > m \} \neq \emptyset.
\]

(3)\( \Rightarrow \) (1). Let \( c_0(S) \not\subseteq \text{wap}(S) \) then there are sequences \( \{x_n\}, \{y_m\} \) in \( S \) with distinct elements such that for some \( s \in S \),
\[
1 = \lim_{m \to \infty} \lim_{n \to \infty} \chi_s(x_n y_m) \neq \lim_{n \to \infty} \lim_{m \to \infty} \chi_s(x_n y_m) = 0.
\]

Since \( \lim_{n \to \infty} \lim_{m \to \infty} \chi_s(x_n y_m) = 0 \), for \( 1 > \epsilon > 0 \) there is a \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \lim_{m \to \infty} \chi_s(x_n y_m) < \epsilon \). This implies for all \( n \geq N \), \( \lim_{m \to \infty} \chi_s(x_n y_m) = 0 \). Then for \( n \geq N \), \( 1 > \epsilon > 0 \) there is a \( M_n \in \mathbb{N} \) such that for all \( m \geq M_n \) we have \( \chi_s(x_m y_n) < \epsilon \). So if we omit finitely many terms, for all \( n \in \mathbb{N} \) there is \( M_n \in \mathbb{N} \) such that for all \( m \geq M_n \) we have \( x_m y_n \neq s \). As a similar argument, for all \( m \in \mathbb{N} \) there is \( N_m \in \mathbb{N} \) such that for all \( n \geq N_m \), \( x_m y_n = s \).

Let \( a_1 = x_1, b_1 \) be the first \( y_n \) such that \( a_1 y_n = s \). Suppose \( a_m, b_n \) have been chosen for \( 1 \leq m, n < r \), so that \( a_n b_m = s \) if and only if \( n \geq m \). Pick \( a_r \) to be the first \( x_m \) not belonging to the finite set \( \bigcup_{1 \leq n < r} \{ x_m : x_m y_n = s \} \). Then \( a_r b_n \neq s \) for \( n < r \). Pick \( b_r \) to be the first \( y_n \) belonging to the cofinite set \( \cap_{1 \leq n \leq r} \{ y_n : x_m y_n = s \} \). Then \( a_n b_m = s \) if and only if \( n \geq m \). The sequences \( (a_m), (b_n) \) so constructed satisfy \( a_m b_n = s \) if and only if \( n \geq m \). That is, \( \{a_n b_m\} \) is not of \( W \)-type and this is a contradiction.

(1)\( \Rightarrow \) (3). Let there are sequences \( \{x_n\}, \{y_m\} \) in \( S \) such that \( \{x_n y_m\} \) is not a \( W \)-type matrix, (say) \( x_n y_m = s \) if and only if \( m \leq n \). Then
\[
1 = \lim_{m \to \infty} \lim_{n \to \infty} \chi_s(x_n y_m) \neq \lim_{n \to \infty} \lim_{m \to \infty} \chi_s(x_n y_m) = 0.
\]
So \( \chi_s \not\in \text{wap}(S) \). Thus \( c_0(S) \not\subseteq \text{wap}(S) \).

(4)\( \Rightarrow \) (1) This is Ruppert criterion for \( \chi_s \in \text{wap}(S) \), see [15, Theorem 4].

\[ \square \]
We conclude with some examples which show that some of the above results cannot be improved.

**Examples 3.1.**

(i) Let $S = \mathbb{N}$. Then for $S$ equipped with min multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra but is not neither Arens regular nor a dual Banach algebra. While, if we replace the min multiplication with max then $\ell_1(S)$ is a dual Banach algebra (so a WAP-algebra) which is not Arens regular. If we change the multiplication of $S$ to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

(ii) Let $S$ be the set of all sequences with $0, 1$ values. We equip $S$ with coordinate wise multiplication. We denote by $e_n$ the sequence with all zero unless a $1$ in the $n$-th place. Let $s = \{x_n\} \in S$, and let $F_w(S)$ be the set of all elements of $S$ such that $x_i = 0$ for only finitely index $i$. It is easy to see that $F_w(S)$ is countable. Let $F_w(S) = \{s_1, s_2, \cdots\}$. Recall that, every element $g \in \ell_\infty(S)$ can be denoted by $g = \sum_{s \in S} g(s) \chi_s$, see [6, p.65]. Suppose

$$g = \sum_{s \in S \setminus F_w(S)} g(s) \chi_s$$

be in $wap(S)$, we show that $g = 0$. Let $s = \{x_n\} \in S$, and $\{k \in \mathbb{N} : x_k = 0\} = \{k_1, k_2, \cdots\}$ be an infinite set. Put $a_n = s + \sum_{j=1}^{n} e_{k_j}$ and $b_m = s + \sum_{i=m}^{\infty} e_{k_i}$. Then

$$a_nb_m = \begin{cases} \sum_{j=m}^{n} e_{k_j} + s & \text{if } m \leq n \\ s & \text{if } m > n \end{cases}$$

Thus $g(s) = \lim_n \lim_m g(a_nb_m) = \lim_m \lim_n g(a_nb_m) = \lim_m g(s + \sum_{i=m}^{\infty} e_{k_i}) = 0$.

In fact,

$$wap(S) = \{f \in \ell_\infty(S) : f = \sum_{i=1}^{\infty} f(s_i) \chi_{s_i}, \; s_i \in F_w(S)\} \oplus \mathbb{C}$$

It is clear that $F_w(S)$ is the subsemigroup of $S$ and $wap(F_w(S)) = \ell_\infty(F_w(S))$. So $\ell_1(F_w(S))$ is Arens regular. Let $T$ consists of those sequences $s = \{x_n\} \in S$ such that $x_i = 0$ for infinitely index $i$, then $T$ is a subsemigroup of $S$ and $wap(T) = \mathbb{C}$. Since $e_{\ell_1} : T \to S^{wap}$ isn’t one to one, $\ell_1(S)$ is not a WAP-algebra. This shows that in general $\ell_1(S)$ need not be a WAP-algebra.
(iii) If we equip \( S = \mathbb{R}^2 \) with the multiplication \((x, y).(x', y') = (xx', x'y + y')\), then \( M_b(S) \) is not a WAP-algebra. Indeed, every non-constant function \( f \) over \(-x-axis is not in \( \text{wap}(S) \). Let \( f(0, z_1) \neq f(0, z_2) \) and \( \{x_m\}, \{y_m\}, \{\beta_n\} \) be sequences with distinct elements satisfying the recursive equation
\[
\beta_n x_m + y_m = \frac{mz_1 + nz_2}{m + n}
\]

Then
\[
\lim_{n} \lim_{m} f((0, \beta_n).(x_m, y_m)) = \lim_{n} \lim_{m} f(0, \beta_n x_m + y_m) = \lim_{n} \lim_{m} f(0, \frac{mz_1 + nz_2}{m + n}) = f(0, z_1)
\]

and similarly
\[
\lim_{m} \lim_{n} f((0, \beta_n).(x_m, y_m)) = f(0, z_2).
\]

Thus the map \( \epsilon : S \rightarrow S^{\text{wap}} \) isn't one to one, so \( M_b(S) \) is not a WAP-algebra. This shows that in general \( M_b(S) \) need not be a WAP-algebra.

(iv) Let \( S \) be the interval \( [\frac{1}{2}, 1] \) with multiplication \( x.y = \max\{\frac{1}{2}, xy\} \), where \( xy \) is the ordinary multiplication on \( \mathbb{R} \). Then for all \( s \in S \setminus \{\frac{1}{2}\} \), \( x \in S \), \( x^{-1}s \) is finite. But \( x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}] \). Let \( B = [\frac{1}{2}, \frac{3}{4}] \). Then for all finite subset \( F \) of \( B \),
\[
\bigcap_{x \in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x \in B \setminus F} x^{-1}\frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}]
\]

where \( x_F = \max F \). By [15, Theorem 4] \( \chi_{\frac{1}{2}} \notin \text{wap}(S) \). So \( c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \subsetneq \text{wap}(S) \).

It can be readily verified that \( \epsilon : S \rightarrow S^{\text{wap}} \) is one to one, so \( \ell_1(S) \) is a WAP-algebra but \( c_0(S) \not\subseteq \text{wap}(S) \). This is a counter example for the converse of Corollary 3.4.

(v) Take \( T = (\mathbb{N} \cup \{0\}, .) \) with 0 as zero of \( T \) and the multiplication defined by
\[
n.m = \begin{cases} 
n & \text{if } n = m \\
0 & \text{otherwise.} \end{cases}
\]

Then \( S = T \times T \) is a semigroup with coordinate wise multiplication. Now let \( X = \{(k, 0) : k \in T\}, Y = \{(0, k) : k \in T\} \) and \( Z = X \cup Y \). We use the Ruppert criterion [15] to show that \( \chi_z \notin \text{wap}(S) \), for each \( z \in Z \). Let \( B = \{(k, n) : k, n \in T\}, \) then \( (k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\} \). Thus for all finite subsets \( F \)
of $B$,
\[
\bigcap \{(k,n)^{-1}(k,0) : (k,n) \in F\} \setminus \bigcap \{(k,0)(k,n)^{-1} : (k,n) \in F\} \\
= \bigcap \{(k,n)^{-1}(k,0) : (k,n) \in B \setminus F\} \\
= (B \setminus F) \setminus F = B \setminus F
\]
and the last set is infinite. This means $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$.

Let $f = \sum_{n=0}^{\infty} f(0,n) \chi_{(0,n)} + \sum_{m=1}^{\infty} f(m,0) \chi_{(m,0)}$ be in $\text{wap}(S)$. For arbitrary fixed $n$ and sequence $\{(n,k)\}$ in $S$, we have $\lim_k f(n,k) = \lim_k \lim_l f(n,l,k) = \lim_l \lim_k f(n,l,k) = f(n,0)$ implies $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Since $\text{wap}(S)$ can not separate the points of $S$ so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n,m) = 2^n3^m$ for $(n,m) \in S$. Then $\omega$ is a weight on $S$ such that $\omega \in \text{wap}(S,1/\omega)$, so the evaluation map $\epsilon : S \to \hat{X}$ is one to one. This means $\ell_1(S,\omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counter example for the converse of Corollary 3.3.

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