Original Article

A note on a semi continuous and Darboux continuous function

Majid Mirmiran

Department of Mathematics, University of Isfahan Isfahan 81746-73441 Iran

*Corresponding Author
Majid Mirmiran
Department of Mathematics,
University of Isfahan Isfahan 81746-73441 Iran
E-mail: mirmir@sci.ui.ac.ir

Abstract
Present alternate proof for a function that is zero almost everywhere, not identically zero, of Baire class 1 (and indeed upper semi-continuous), and Darboux continuous.

1. Introduction

The purpose of this paper is to present alternate proof of well known result in real analysis [1]. An alternate proof of a theorem provides a new way of looking at the theorem and this fresh perspective is often enough to justify the new approach. However, a new proof of an old result that is conceptually easier has obvious benefits.

2. The main result

Before giving the main result, the necessary definitions are stated.

Definition 2.1.
A function is said to be Darboux continuous if it takes every intermediate value between any two values that it takes, for some intermediate argument.

Definition 2.2.
A function is said to be of Baire class 1 if it is the limit function of a set of continuous functions in the interval concerned.

We now give the following main result:

Theorem 2.1.
There exists in the closed interval I = [0, 1], a function that is zero almost everywhere, not identically zero, of Baire class 1 (and indeed upper semi-continuous), and Darboux continuous.

Proof.
Let \( f(x) = 0 \) if \( x \neq \frac{1}{2} \) and \( f(x) = 1 \) if \( x = \frac{1}{2} \).

Clearly \( f \) is a bounded upper semicontinuous function, thus by Theorem (A1) of [2], there exists a bounded Darboux upper semicontinuous function \( g \) such that \( \{x \in I : f(x) \neq g(x)\} \) is a first category null subset of \( I \), and \( f(x) \leq g(x) \) for all \( x \in I \). Since \( f(\frac{1}{2}) = 1 \) and \( f(\frac{1}{2}) \leq g(\frac{1}{2}) \), thus \( g \) is not identically zero. Also, \( \{x \in I : g(x) \neq 0\} \subset \{x \in I : f(x) \neq g(x)\} \cup \{\frac{1}{2}\} \). Thus, \( \lambda(\{x \in I : g(x) \neq 0\}) \leq \lambda(\{x \in I : f(x) \neq g(x)\}) + \lambda(\{\frac{1}{2}\}) = 0 + 0 + 0 \);

where \( \lambda \) is the Lebesgue measure. Therefore, \( g \) is zero almost everywhere. Also, since closed interval \( I = [0, 1] \) is a perfectly normal space, by Theorem 3 of [3] \( g \) is the limit of a monotonically decreasing sequence of continuous functions, thus \( g \) is of Baire class 1.

Theorem 2.2.
There exists in the closed interval I = [0, 1], a function that is zero almost everywhere, not identically zero, of Baire class 1 (and indeed lower semi-continuous), and Darboux continuous.

Proof.
Since \( f \) is zero almost everywhere, not identically zero, of Baire class 1 (and indeed upper semi-continuous), and Darboux continuous if and only if \( -f \) is zero almost everywhere, not identically zero, of Baire class 1 (and indeed lower semi-continuous), and Darboux continuous. Therefore, by Theorem 2.1. there exists in the closed interval \( I = [0, 1] \), a function that is zero almost everywhere, not identically zero, of Baire class 1 (and indeed lower semi-continuous), and Darboux continuous.

Acknowledgement
This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan).

References