Insertion of an $\alpha$-Continuous Function

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Received February 02, 2009

Abstract—A necessary and sufficient condition in terms of lower cut sets are given for the insertion of an $\alpha$-continuous function between two comparable real-valued functions.

2000 Mathematics Subject Classification: 54C08, 54C10, 54C50, 26A15, 54C30

DOI: 10.1134/S1995080209030068

Key words and phrases: Insertion, Strong binary relation, Preopen set, Semi-open set, $\alpha$-open set, Lower cut set.

1. INTRODUCTION

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [3]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [11], while the concept of a, locally dense, set was introduced by H.H. Corson and E. Michael [3].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [10]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semi-closed if its complement is semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open if $A$ is the difference of an open and a nowhere dense subset of $X$. A set $A$ is called $\alpha$-closed if its complement is $\alpha$-open or equivalently if $A$ is union of a closed and a nowhere dense set.

We have a set is $\alpha$-open if and only if it is semi-open and preopen.

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [13] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subset of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [4, 5].

Hence, a real-valued function $f$ defined on a topological space $X$ is called precontinuous (resp. semi-continuous or $\alpha$-continuous) if the preimage of every open subset of $\mathbb{R}$ is preopen (resp. semi-open or $\alpha$-open) subset of $X$.

Precontinuity was called by V.Ptk nearly continuity [14]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [6]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [1].

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a sufficient condition for the insertion of an $\alpha$-continuous function between two comparable real-valued functions.

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11This work was supported by University of Isfahan and Centre of Excellence for Mathematics (University of Isfahan).
If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all $x$ in $X$.

The following definitions are modifications of conditions considered in [9].

A property $P$ defined relative to a real-valued function on a topological space is an $\alpha$-property provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and any $\alpha$-continuous function also has property $P$. If $P_1$ and $P_2$ are $\alpha$-property, the following terminology is used: (i) A space $X$ has the weak $\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists an $\alpha$-continuous function $h$ such that $g \leq h \leq f$. (ii) A space $X$ has the $\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists an $\alpha$-continuous function $h$ such that $g < h < f$. (iii) A space $X$ has the weakly $\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$, $f$ has property $P_2$ and $f - g$ has property $P_2$, then there exists an $\alpha$-continuous function $h$ such that $g < h < f$.

In this paper, is given a sufficient condition for the weak $\alpha$-insertion property. Also for a space with the weak $\alpha$-insertion property, we give a necessary and sufficient condition for the space to have the $\alpha$-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. THE MAIN RESULT

Before giving a sufficient condition for insertability of an $\alpha$-continuous function, the necessary definitions and terminology are stated.

Let $(X, \tau)$ be a topological space, the family of all $\alpha$-open, $\alpha$-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $s O(X, \tau)$, $s C(X, \tau)$, $p O(X, \tau)$ and $p C(X, \tau)$, respectively.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. Respectively, we define the $\alpha$-closure, $\alpha$-interior, $s$-closure, $s$-interior, $p$-closure and $p$-interior of a set $A$, denoted by $\alpha Cl(A)$, $\alpha Int(A)$, $s Cl(A)$, $s Int(A)$, $p Cl(A)$ and $p Int(A)$ as follows:

- $\alpha Cl(A) = \cap \{F : F \supseteq A, F \in \alpha C(X, \tau)\}$,
- $\alpha Int(A) = \cup \{O : O \subseteq A, O \in \alpha O(X, \tau)\}$,
- $s Cl(A) = \cap \{F : F \supseteq A, F \in s C(X, \tau)\}$,
- $s Int(A) = \cup \{O : O \subseteq A, O \in s O(X, \tau)\}$,
- $p Cl(A) = \cap \{F : F \supseteq A, F \in p C(X, \tau)\}$ and
- $p Int(A) = \cup \{O : O \subseteq A, O \in p O(X, \tau)\}$.

Respectively, we have $\alpha Cl(A), s Cl(A), p Cl(A)$ are $\alpha$-closed, semi-closed, preclosed and $\alpha Int(A), s Int(A), p Int(A)$ are $\alpha$-open, semi-open, preopen.

The following first two definitions are modifications of conditions considered in [7, 8].

**Definition 2.2.** If $\rho$ is a binary relation in a set $S$ then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any $u$ and $v$ in $S$.

**Definition 2.3.** A binary relation $\rho$ in the power set $P(X)$ of a topological space $X$ is called a strong binary relation in $P(X)$ in case $\rho$ satisfies each of the following conditions:

1. If $A_1 \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2. If $A \subseteq B$, then $A \rho B$.
3. If $A \rho B$, then $\alpha Cl(A) \subseteq B$ and $A \subseteq \alpha Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a lower indefinite cut set in the domain of $f$ at the level $\ell$.

We now give the following main result:

**Theorem 2.1.** Let $g$ and $f$ be real-valued functions on a topological space $X$ with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$
and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists an $\alpha$-continuous function $h$ defined on $X$ such that $g \leq h \leq f$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$ such that $g \leq f$. By hypothesis there exists a strong binary relation $\rho$ on the power set of $X$ and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions $F$ and $G$ mapping the rational numbers $\mathbb{Q}$ into the power set of $X$ by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \rho F(t_2)$, $G(t_1) \rho G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [8] it follows that there exists a function $H$ mapping $\mathbb{Q}$ into the power set of $X$ such that if $t_1$ and $t_2$ are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$, and $H(t_1) \rho G(t_2)$.

For any $x$ in $X$, let $h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \}$.

We first verify that $g \leq h \leq f$: If $x$ is in $H(t)$ then $x$ is in $G(t')$ for any $t' > t$; since $x$ is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If $x$ is not in $H(t)$, then $x$ is not in $F(t')$ for any $t' < t$; since $x$ is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers $t_1$ and $t_2$ with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \alpha \text{Int}(H(t_2)) \setminus \alpha \text{Cl}(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is an $\alpha$-open subset of $X$, i.e., $h$ is an $\alpha$-continuous function on $X$.

The above proof used the technique of proof of Theorem 1 of [7].

**Theorem 2.2.** Let $P_1$ and $P_2$ be $\alpha$-property and $X$ a space that satisfies the weak $\alpha$-insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the $\alpha$-insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $(D_n)$ of subsets of $X$ with empty intersection and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by $\alpha$-continuous functions.

**Proof.** Theorem 2.1 of [12].

3. APPLICATIONS

The abbreviations $pc$ and $sc$ are used for precontinuous and semicontinuous, respectively.

**Corollary 3.1.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$ of $X$, there exist $\alpha$-open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the weak $\alpha$-insertion property for $(pc, pc)$ (resp. $(sc, sc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g < f$. If a binary relation $\rho$ is defined on $A(p \cup B)$ in case $p \text{Cl}(A) \subseteq p \text{Int}(B)$ (resp. $s \text{Cl}(A) \subseteq s \text{Int}(B)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);$$

since $\{ x \in X : f(x) \leq t_1 \}$ is a preclosed (resp. semi-closed) set and since $\{ x \in X : g(x) < t_2 \}$ is a preopen (resp. semi-open) set, it follows that $p \text{Cl}(A(f, t_1)) \subseteq p \text{Int}(A(g, t_2))$ (resp. $s \text{Cl}(A(f, t_1)) \subseteq s \text{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1.

**Corollary 3.2.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$, there exist $\alpha$-open sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every precontinuous (resp. semi-continuous) function is $\alpha$-continuous.

**Proof.** Let $f$ be a real-valued precontinuous (resp. semi-continuous) function defined on the $X$. Set $g = f$, then by Corollary 3.1, there exists an $\alpha$-continuous function $h$ such that $g = h = f$.

**Corollary 3.3.** If for each pair of disjoint preclosed (resp. semi-closed) sets $F_1, F_2$ of $X$, there exist $\alpha$-open sets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then $X$ has the $\alpha$-insertion property for $(pc, pc)$ (resp. $(sc, sc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are $pc$ (resp. $sc$), and $g < f$. Set $h = (f + g) / 2$, thus $g < h < f$, and by Corollary 3.2, since $g$ and $f$ are $\alpha$-continuous functions hence $h$ is $\alpha$-continuous function.
Corollary 3.4. If for each pair of disjoint subsets $F_1, F_2$ of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist $\alpha$-open subsets $G_1$ and $G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then $X$ has the weak $\alpha$-insertion property for $(pc, sc)$ and $(sc, pc)$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$, such that $g$ is $pc$ (resp. $sc$) and $f$ is $sc$ (resp. $pc$), with $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $sCl(A) \subseteq pInt(B)$ (resp. $pCl(A) \subseteq sInt(B)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $sCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (resp. $pCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. \qed

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

**Lemma 3.1.** The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $F_1, F_2$ of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed, there exist $\alpha$-open subsets $G_1, G_2$ of $X$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.

(ii) If $F$ is a semi-closed (resp. preclosed) subset of $X$ which is contained in a preopen (resp. semi-open) subset $G$ of $X$, then there exists an $\alpha$-open subset $H$ of $X$ such that $F \subseteq H \subseteq \alpha Cl(H) \subseteq G$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where $F$ and $G$ are semi-closed (resp. preclosed) and preopen (resp. semi-open) subsets of $X$, respectively. Hence, $G^c$ is a preclosed (resp. semi-closed) and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint $\alpha$-open subsets $G_1, G_2$ of $X$ s.t., $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since $G_2^c$ is an $\alpha$-closed set containing $G_1$ we conclude that $\alpha Cl(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq \alpha Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $F_1, F_2$ are two disjoint subsets of $X$, such that $F_1$ is preclosed and $F_2$ is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and $F_1^c$ is a preopen subset of $X$. Hence by (ii) there exists an $\alpha$-open set $H$ s.t., $F_2 \subseteq H \subseteq \alpha Cl(H) \subseteq F_1^c$.

But

$$H \subseteq \alpha Cl(H) \Rightarrow H \cap (\alpha Cl(H))^c = \emptyset$$

and

$$\alpha Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\alpha Cl(H))^c.$$ 

Furthermore, $(\alpha Cl(H))^c$ is an $\alpha$-open set of $X$. Hence $F_2 \subseteq H, F_1 \subseteq (\alpha Cl(H))^c$ and $H \cap (\alpha Cl(H))^c = \emptyset$. This means that condition (i) holds. \qed

**Lemma 3.2.** Suppose that $X$ is a topological space. If each pair of disjoint subsets $F_1, F_2$ of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed, can separate by $\alpha$-open subsets of $X$ then there exists an $\alpha$-continuous function $h : X \rightarrow [0, 1]$ s.t., $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.

**Proof.** Suppose $F_1$ and $F_2$ are two disjoint subsets of $X$, where $F_1$ is preclosed and $F_2$ is semi-closed. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. In particular, since $F_1^c$ is a preopen subset of $X$ containing semi-closed subset $F_2$ of $X$, by Lemma 3.1, there exists an $\alpha$-open subset $H_{1/2}$ of $X$ s.t.,

$$F_2 \subseteq H_{1/2} \subseteq \alpha Cl(H_{1/2}) \subseteq F_1^c.$$
Note that $H_{1/2}$ is also a preopen subset of $X$ and contains $F_2$, and $F_1^c$ is a preopen subset of $X$ and contains a semi-closed subset $\alpha Cl(H_{1/2})$ of $X$. Hence, by Lemma 3.1, there exists $\alpha$-open subsets $H_{1/4}$ and $H_{3/4}$ s.t.,

$$F_2 \subseteq H_{1/4} \subseteq \alpha Cl(H_{1/4}) \subseteq H_{1/2} \subseteq \alpha Cl(H_{1/2}) \subseteq H_{3/4} \subseteq \alpha Cl(H_{3/4}) \subseteq F_1^c.$$  

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $\alpha$-open subsets $H_t$ of $X$ with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function $h$ on $X$ by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., $h$ maps $X$ into $[0, 1]$. Also, we note that for any $t \in D$, $F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that $h$ is an $\alpha$-continuous function on $X$. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \emptyset$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are $\alpha$-open subsets of $X$. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(\alpha Cl(H_t))^c : t > \beta\}$, hence, every of them is an $\alpha$-open subset of $X$. Consequently $h$ is an $\alpha$-continuous function. \hfill $\Box$

**Lemma 3.3.** Suppose that $X$ is a topological space such that every two disjoint semi-closed and preclosed subsets of $X$ can be separated by $\alpha$-open subsets of $X$. The following conditions are equivalent:

(i) Every countable covering of semi-open (resp. preopen) subsets of $X$ has a refinement consisting of preopen (resp. semi-open) subsets of $X$ s.t., for every $x \in X$, there exists an $\alpha$-open subset of $X$ containing $x$ such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of semi-closed (resp. preclosed) subsets of $X$ with empty intersection there exists a decreasing sequence $\{G_n\}$ of preopen (resp. semi-open) subsets of $X$ s.t., $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $\{F_n\}$ be a decreasing sequence of semi-closed (resp. preclosed) subsets of $X$ with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-open (resp. preopen) subsets of $X$. By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ s.t., every $V_n$ is an $\alpha$-open subset of $X$ and $\alpha Cl(V_n) \subseteq F_n^c$. By setting $G_n = (\alpha Cl(V_n))^c$, we obtain a decreasing sequence of $\alpha$-open subsets of $X$ with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-open (resp. preopen) subsets of $X$, we set for $n \in \mathbb{N}$, $F_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of semi-closed (resp. preclosed) subsets of $X$ with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of preopen (resp. semi-open) subsets of $X$ s.t., $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets $W_n$ of $X$ in the following manner:

$W_1$ is an $\alpha$-open subset of $X$ s.t., $G_1^c \subseteq W_1$ and $\alpha Cl(W_1) \cap F_1 = \emptyset$.

$W_2$ is an $\alpha$-open subset of $X$ s.t., $\alpha Cl(W_1) \cup G_2^c \subseteq W_2$ and $\alpha Cl(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.1, $W_1$ exists.)

Then since $\{G_n : n \in \mathbb{N}\}$ is a covering for $X$, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for $X$ consisting of $\alpha$-open subsets of $X$. Moreover, we have

(i) $\alpha Cl(W_n) \subseteq W_{n+1}$,

(ii) $G_n^c \subseteq W_n$,

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \alpha Cl(W_{n-1})$.

Then since $\alpha Cl(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $\alpha$-open subsets of $X$ and covers $X$. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$S_1 \cap H_1, \quad S_1 \cap H_2, \quad S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3, \quad S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4,$$

$$\vdots$$
These sets are $\alpha$-open subsets of $X$, cover $X$ and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is an $\alpha$-open subset of $X$ containing $x$ that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i + 1\}$ refines $\{H_n : n \in \mathbb{N}\}$ s.t., its elements are $\alpha$-open subsets of $X$, and for every point in $X$ we can find an $\alpha$-open subset of $X$ containing the point that intersects only finitely many elements of that refinement.

**Corollary 3.5.** If every two disjoint semi-closed and preclosed subsets of $X$ can be separated by $\alpha$-open subsets of $X$, and in addition, every countable covering of semi-open (resp. preopen) subsets of $X$ has a refinement that consists of preopen (resp. semi-open) subsets of $X$ s.t., for every point of $X$ we can find an $\alpha$-open subset containing that point s.t., it intersects only a finite number of refining members then $X$ has the weakly $\alpha$-insertion property for $(pc, sc)$ (resp. $(sc, pc)$).

**Proof.** Since every two disjoint sets semi-closed and preclosed can be separated by $\alpha$-open subsets of $X$, therefore by Corollary 3.4, $X$ has the weak $\alpha$-insertion property for $(pc, sc)$ and $(sc, pc)$. Now suppose that $f$ and $g$ are real-valued functions on $X$ with $g < f$, s.t., $g$ is $pc$ (resp. $sc$), $f$ is $sc$ (resp. $pc$) and $f - g$ is $sc$ (resp. $pc$). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is $sc$ (resp. $pc$), hence $A(f - g, 3^{-n+1})$ is a semi-closed (resp. preclosed) subset of $X$.

Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of semi-closed (resp. preclosed) subsets of $X$ and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preopen (resp. semi-open) subsets of $X$ s.t., $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 3.2, $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of semi-closed (resp. preclosed) and preclosed (resp. semi-closed) subsets of $X$ can be completely separated by $\alpha$-continuous functions. Hence by Theorem 2.2, there exists an $\alpha$-continuous function $h$ defined on $X$ s.t., $g < h < f$, i.e., $X$ has the weakly $\alpha$-insertion property for $(pc, sc)$ (resp. $(sc, pc)$).

**ACKNOWLEDGEMENT**

This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan).

**REFERENCES**