Abstract. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. In this note for a $(S, \omega)$-Armendariz ring $R$ we study some properties of skew generalized power series ring $R[[S, \omega]]$. In particular, among other results, we show that for a $S$-compatible $(S, \omega)$-Armendariz ring $R$, $\alpha(R[[S, \omega]]) = \alpha(R)[[S, \omega]] = \text{Nil}^*(R)[[S, \omega]]$, where $\alpha$ is a radical in a class of radicals which includes the Wedderburn, lower nil, Levitzky and upper nil radicals. We also show that several properties, including the symmetric, reversible, $ZC_n$, zip and 2-primal property, transfer between $R$ and the skew generalized power series ring $R[[S, \omega]]$, in case $R$ is $S$-compatible $(S, \omega)$-Armendariz.

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1. Introduction

Let $(S, \leq)$ be a partially ordered set. Then $(S, \leq)$ is called artinian if every strictly decreasing sequence of elements of $S$ is finite and $(S, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of $S$ is finite. A monoid $S$ equipped with an order $\leq$ is called an ordered monoid if for any $s_1, s_2, t \in S$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $t s_1 \leq t s_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and $t s_1 < t s_2$, then $(S, \leq)$ is said to be strictly ordered. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. For $s \in S$, let $\omega_x$ denote the image of $s$ under $\omega$. Let $A$ be the set of all functions $f : S \to R$ such that the support $\text{supp}(f) = \{ s \in S : f(s) \neq 0 \}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x, y) \in X_s(f, g)} f(x)\omega_x(g(y))$$

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(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, \( A \) becomes a ring, called the ring of skew generalized power series with coefficients in \( R \) and exponents in \( S \), denoted by \( R[[S, \omega, \leq]] \) (or by \( R[[S, \omega]] \) when there is no ambiguity concerning the order) (for more details see [14]).

Special cases of the skew generalized power series construction include polynomial rings \( (S = \mathbb{N} \cup \{0\} \) with usual addition, and trivial \( \omega \) and \( \omega_1 \)), skew polynomial ring \( R[x; \alpha] \) for some \( \alpha \in \text{End}(R) \) \( (S = \mathbb{N} \cup \{0\} \) with usual addition, trivial \( \omega \) and \( \omega_1 = \alpha \)), skew Laurent polynomial ring \( R[x, x^{-1}; \alpha] \) for some \( \alpha \in \text{End}(R) \) \( (S = \mathbb{N} \cup \{0\} \) with usual addition, trivial \( \omega \) and \( \omega_1 = \alpha \)), skew Laurent series ring \( R[[x, x^{-1}; \alpha]] \) for some \( \alpha \in \text{End}(R) \) \( (S = \mathbb{Z} \) with usual addition, usual order and \( \omega_1 = \alpha \)), skew monoid rings (trivial \( \omega \)), skew power series ring \( R[[x; \alpha]] \) for some \( \alpha \in \text{End}(R) \) \( (S = \mathbb{N} \cup \{0\} \) with usual addition, usual order and \( \omega_1 = \alpha \)), skew Laurent series ring \( R[[x, xe^{-1}; \alpha]] \) a totally ordered group and trivial \( \omega \) the Mal’cev-Neumann construction \((S, \geq) \) a totally ordered group; see [10], p. 230).

A ring \( R \) is called Armendariz if whenever polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( a_ib_j = 0 \) for each \( i, j \). The study of Armendariz rings was initiated by Rege and Chhawchharia [15].

W. Cortez in [5], G. Marks et al. in [13] by introducing the following definition, unified and generalized the notion of Armendariz rings and its generalizations to skew generalized power series rings. Let \( R \) be a ring and \( \alpha : R \rightarrow R \) be a ring endomorphism. We denote \( R[x; \alpha] \) the skew polynomial ring whose elements are the polynomials over \( R \), the addition is defined as usual and the multiplicative subject to the relation \( xa = \alpha(a)x \) for any \( a \in R \). The Armendariz property was extended to skew polynomial rings by Hong et al. in [7]. For an endomorphism \( \alpha \) of a ring \( R \), \( R \) is called an \( \alpha \)-skew Armendariz ring if for polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x; \alpha], f(x)g(x) = 0 \) implies \( a_i\alpha^i(b_j) = 0 \) for each \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \) (note that by W. Cortez in [5], \( \alpha \)-skew Armendariz ring is called \( SA_1 \)). In [11], Z. Liu, extended the Armendariz notion to monoid rings. For a ring \( R \) and monoid \( S, R \) is called an Armendariz ring relative to monoid \( S \) if for elements \( f(x) = a_1s_1 + a_2s_2 + \cdots + a_ns_n \) and \( g(x) = b_1t_1 + b_2t_2 + \cdots + b_mt_m \) of the monoid ring \( R[S], f \circ g = 0 \) implies \( a_is_j = 0 \) for all \( i, j \).

G. Marks et al. in [13] by introducing the following definition, unified and generalized the notion of Armendariz rings and its generalizations to skew generalized power series rings. Let \( R \) be a ring, \((S, \leq) \) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. A ring \( R \) is called \((S, \omega)\)-Armendariz if whenever \( f_g = 0 = f, g \in R[[S, \omega]], \) then \( f(s)\omega_s(g(t)) = 0 \) for all \( s, t \in S \) [13, Definition 2.1]. For each \( r \in R \) and \( s \in S \), let \( c_r, e_s \in R[[S, \omega]] \) defined by

\[
c_r(x) = \begin{cases} 
r & \text{if } x = 1 \\
0 & \text{if } x \in S\setminus\{1\} \end{cases} \\
e_s(x) = \begin{cases} 
1 & \text{if } x = s \\
0 & \text{if } x \in S\setminus\{s\} \end{cases}.
\]

A ring \( R \) is called linearly \((S, \omega)\)-Armendariz if for all \( s \in S\setminus\{1\} \) and \( a_0, a_1, b_0, b_1 \in R \), whenever \( (c_{a_0} + c_{a_1}e_s)(c_{b_0} + c_{b_1}e_s) = 0 \) in \( R[[S, \omega]] \), then \( a_0b_0 = a_0b_1 = a_1e_s(b_0) = a_1e_s(b_1) = 0 \) in \( R[[S, \omega]] \), then \( a_0b_0 = a_0b_1 = a_1e_s(b_0) = a_1e_s(b_1) = 0 \) in \( R[[S, \omega]] \). Each \((S, \omega)\)-Armendariz ring \( R \) is linearly \((S, \omega)\)-Armendariz, but a linearly \((S, \omega)\)-Armendariz ring \( R \) need not be \((S, \omega)\)-Armendariz. G. Marks et al. in [13] introduced and investigated the notion of \((S, \omega)\)-Armendariz ring and studied some property of this class of rings. In this note we continue study of this class of rings.
The Wedderburn radical (i.e., the largest nilpotent ideal in $R$), the lower nil radical (i.e., the intersection of all the prime ideals in $R$), the Levitzky radical (i.e., sum of all locally nilpotent ideals), the upper nil radical (i.e., sum of all nil ideals), the set of all nilpotent elements of $R$ and the sum of all nil left ideals of $R$ (which coincides with the sum of all nil right ideals of $R$) is denoted by $N_0(R)$, $\text{Nil}_\omega(R)$, $L$-$\text{rad}(R)$, $\text{Nil}^*(R)$, and $A(R)$, respectively. An endomorphism $\alpha$ of a ring $R$ is called compatible if for all $a, b \in R$, $ab = 0$ if and only if $\alpha(ab) = 0$. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \longrightarrow \text{End}(R)$ a monoid homomorphism. A ring $R$ is called $S$-compatible If $\omega_s$ is compatible for each $s \in S$.

G. Marks et al. in [13] proved that for strictly ordered monoid $(S, \leq)$ and monoid homomorphism $\omega : S \longrightarrow \text{End}(R)$, if $R$ is $(S, \omega)$-Armendariz and $\omega_s$ is compatible for some $s \in S \setminus \{1\}$, then:

(i) for all $a, b \in R$ and $x \in \text{Nil}(R)$, $ab = 0$ implies $axb = 0$;
(ii) $\text{Nil}(R)$ is a (nonunital) subring of $R$;
(iii) $N_0(R) = \text{Nil}_\omega(R) = L$-$\text{rad}(R) = \text{Nil}^*(R) = A(R)$. In particular, the Köthe problem has a positive solution in the class of $(S, \omega)$-Armendariz rings.

In this note we first show that this result is correct for the class of linearly $(S, \omega)$-Armendariz rings. We also show that if $R$ is locally finite linearly $(S, \omega)$-Armendariz and $\omega_s$ is compatible for some $s \in S \setminus \{1\}$, then $R$ is 2-primal (i.e. its prime radical contains every nilpotent elements). For the class of $S$-compatible $(S, \omega)$-Armendariz rings, we also prove that: $N_0(R[[S, \omega]]) = \text{Nil}_\omega(R[[S, \omega]]) = L$-$\text{rad}(R[[S, \omega]]) = \text{Nil}^*(R[[S, \omega]]) = A(R[[S, \omega]])$. In particular, we show that $R[[S, \omega]]$ satisfies the Köthe’s conjecture for this class of rings.

In section 3, we compute radicals of skew generalized power series ring $R[[S, \omega]]$, in case $R$ is $S$-compatible $(S, \omega)$-Armendariz. We show that for a $S$-compatible $(S, \omega)$-Armendariz ring $R$, $\text{Nil}_\omega(R[[S, \omega]]) = \text{Nil}_\omega(R)[[S, \omega]]$. By using this fact we prove that for a $S$-compatible $(S, \omega)$-Armendariz ring $R$, $R[[S, \omega]]$ is 2-primal if and only if $R$ is.

In section 4, we show that for a $S$-compatible $(S, \omega)$-Armendariz ring $R$, $R[[S, \omega]]$ is right zip if and only if $R$ is. We also prove that if $R$ is a $(S, \omega)$-Armendariz ring and $\omega_s$ is an automorphism for each $s \in S$, then $R[[S, \omega]]$ is right zip if and only if $R$ is. These theorems extend and unify several known results.

2. Prime radical of skew generalized power series rings

In this section we study the radicals of a linearly $(S, \omega)$-Armendariz ring $R$ and radicals of skew generalized power series ring $R[[S, \omega]]$, when $R$ is $(S, \omega)$-Armendariz.

Proposition 2.1. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \longrightarrow \text{End}(R)$ a monoid homomorphism. If $R$ is linearly $(S, \omega)$-Armendariz and $\omega_s$ is compatible for some $s \in S \setminus \{1\}$, then for all $a, b \in R$ and $x \in \text{Nil}(R)$, $ab = 0$ implies $axb = 0$.

Proof. Let $a, b \in R$ and $x \in \text{Nil}(R)$ such that $ab = x^n = 0$ for some positive integer $n$. The case $n = 1$ is clear. Assume $n \geq 2$. First we show that $ax^{n-1}b = 0$. Since $\omega_s$ is compatible then $ax^{n-1} \omega_s(x^{n-1} \omega_s(b)) = 0$. Thus in $R[[S, \omega]]$ we have $(e_a + e_{-ax^{n-1}}e_s)(e_b + e_{x^{n-1} \omega_s(b)}e_s) = 0$ and since $R$ is linearly $(S, \omega)$-Armendariz,
then \( ax^{n-1} \omega_s(b) = 0 \). Thus \( ax^n b = 0 \), since \( \omega_s \) is compatible. Now assume that \( ab = x^n = 0 \) and \( n = 2^k \), for some positive integer \( k \). Since \( ab = (x^{2^{k-1}})^2 = 0 \), we have \( ax^{2^{k-1}} b = 0 \). Now take \( p = c_a + c_{-ax^{2^{k-2}}} e_s, q = c_b + c_{ax^{2^{k-2}} \omega_s(b)} e_s \in R[[S, \omega]] \).

Since \( \omega_s \) is compatible, we have \( pq = 0 \) and so \( ax^{2^{k-2}} \omega_s(b) = 0 \). Hence \( ax^{2^{k-2}} b = 0 \). By the same argument we see that \( (c_a + c_{-ax^{2^{k-3}}} e_s)(c_b + c_{ax^{2^{k-3}} \omega_s(b)} e_s) = 0 \). So \( ax^{2^k - 3} \omega_s(b) = 0 \) and hence \( ax^{2^{k-3}} b = 0 \). By continuing \( k \) times in this way, we have \( ax^{2^{k-4}}(b) = 0 \). Hence \( ax^2 b = 0 \). Thus we have \( (c_a + c_{-ax} e_s)(c_b + c_{ax} \omega_s(b)} e_s) = 0. So \( ax \omega_s(b) = 0 \) and hence \( axb = 0 \).

**Corollary 2.2.** Let \( R \) be a ring, \((S, \leq) \) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is linearly \((S, \omega)\)-Armendariz and \( \omega_s \) is compatible for some \( s \in S \setminus \{1\} \), then:

(i) \( N \ell(R) \) is a (nonunital) subring of \( R \);

(ii) \( N_0(R) = N \ell(R) \) is an Armendariz ring. In particular, the Köthe problem has a positive solution in the class of \( S \)-compatible linearly \((S, \omega)\)-Armendariz rings.

**Proof.** Using Proposition 2.1, the proof is similar to that of [13, Proposition 4.5].

It is a natural question that, whether \( N \ell^*(R) = N \ell(R) \), for an \((S, \omega)\)-Armendariz ring? The answer is negative, by the following example.

**Example 2.3.** ([9, Example 14]) Let \( F \) be a field and \( A = F[a, b, c] \) be the free algebra of polynomials with zero constant terms in noncommuting indeterminates \( a, b, c \) over \( F \). Let \( I \) be the ideal of \( R = A + F \), generated by \( ac, cc \) and \( crc \), for all \( r \in A \). Then \( R/I \) is a semiprime Armendariz ring but \( N \ell(R) \neq 0 \).

Next we show that under a certain condition, a linearly \((S, \omega)\)-Armendariz ring will be 2-primal.

A ring is said to be **abelian** if all its idempotent elements are central. A ring is called **locally finite** if every finite subset in it generates a finite semigroup multiplicatively. Finite rings are clearly locally finite, and an algebraic closure of a finite field is locally finite but not finite.

**Theorem 2.4.** Let \( R \) be a ring, \((S, \leq) \) a nontrivial strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is linearly \((S, \omega)\)-Armendariz, \( \omega_s \) is compatible for some \( s \in S \setminus \{1\} \) and \( R \) is locally finite, then \( N_0(R) = N \ell_s(R) = L \text{-rad}(R) = N \ell^*(R) = A(R) = N \ell(R) \).

**Proof.** By Corollary 2.2, \( N_0(R) = N \ell_s(R) = L \text{-rad} R = N \ell^*(R) = A(R) \). So it is enough to show that \( N \ell^*(R) = N \ell(R) \). Let \( Z(R) \) be the center of \( R \). We first show that, for each \( r \in R \) there exists a positive integer \( m_r \) such that \( r^{m_r} \in Z(R) \). Let \( r \in R \). Since \( R \) is locally finite there exist positive integers \( m, k \geq 1 \), such that \( r^m = r^{m+k} \). So we have \( r^m = r^{m+k} = r^{m+2k} = \ldots = r^{m+mk} = r^{m(k+1)} \).

Thus we have \( r^{mk} = r^{(k-1)m+mk} = (r^{(k-1)m}r^{mk}) = r^{2mk} = (r^{km})^2 \). Hence \( r^{km} \) is an idempotent element of \( R \). By [13, Proposition 4.9], \( R \) is abelian, and so \( r^{km} \in Z(R) \). Now suppose \( a \in N \ell(R) \), so \( a^n = 0 \) for some positive integer.
n. To show that \( a \in \text{Nil}^*(R) \), it is enough to show that the ideal \( (a) \) generated by \( a \) in \( R \) is a nil ideal. Since by Corollary 2.2, \( \text{Nil}^*(R) \) is a subring of \( R \), then it is enough to show that for each \( r,s \in R \), \( sar \) is nilpotent. By the argument above there exists a positive integer \( m \) such that \( (rs)^m \in \mathbb{Z}(R) \). Since \( a^n = 0 \), so we have \( a(rs)^ma^{-1} = 0 \). Hence \( a(rs)(rs)^{m-1}a^{-1} = 0 \). Since \( a^n = 0 \) then \( a(rs)a(rs)^{m-2}a^{-1} = 0 \), by Proposition 2.1. So \( a(rs)a(rs)^{m-2}a^{-1} = 0 \). Again by Proposition 2.1, \( a(rs)a(rs)^{m-2}a^{-1} = 0 \). Continuing in this process, we get \( a(rs)a(rs)a...a(rs)a^{-1} = 0 \), and hence \( (ars)^m a^{-1} = 0 \). Since \( (rs)^m \in \mathbb{Z}(R) \), we have \( (ars)^m a^m a^{-1} = 0 \). After \( n \)-times doing this in this way, we have \( (ars)^m (ars)^m ... (ars)^{m^a} = 0 \) Thus we get \( (ars)^m(n-1)+1 = 0 \) and hence \( s(ar)^{m(n-1)+1} = 0 \). Therefore we have \( (sar)^{m(n-1)+2} = 0 \), and the result follows.

**Lemma 2.5.** Let \( R \) be a ring, \( (S, \leq) \) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is \( S \)-compatible \( (S, \omega) \)-Armendariz then for each \( f_1, \ldots, f_n \in R[[S, \omega]] \), \( f_1f_2 \cdots f_n = 0 \) implies \( f_1(s_1)f_2(s_2) \cdots f_n(s_n) = 0 \), where \( s_i \in S \) for each \( i \).

**Proof.** The proof is by induction on \( n \). By definition the result is true for \( n = 2 \). Assume that the result is true for all \( m < n \). Let \( f_1f_2 \cdots f_n = 0 \), and \( s_1 \in S \). Since \( f_1(s_1)f_2 \cdots f_n = 0 \), by definition \( (f_1(s_1))(f_2 \cdots f_n)s(s) = 0 \) for each \( s \in S \). Since \( R \) is \( S \)-compatible, \( ef_1(s_1)f_2 \cdots f_n = 0 \) and \( ef_1(s_1)f_2f_3 \cdots f_n = 0 \). By the induction hypothesis, for each \( s_i \in S \) with \( 2 \leq i \leq n \), \( ef_1(s_1)f_2 \cdots f_{n+1}(s_n) = 0 \). Since \( R \) is \( S \)-compatible, \( f_1(s_1)f_2(s_2) \cdots f_n(s_n) = 0 \) and the result follows.

**Proposition 2.6.** Let \( R \) be a ring, \( (S, \leq) \) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is \( S \)-compatible \( (S, \omega) \)-Armendariz, then for all \( f, g \in R[[S, \omega]] \) and \( h \in \text{Nil}^*(R[[S, \omega]]) \), \( fg = 0 \) implies \( fhg = 0 \).

**Proof.** Assume that \( h^\ell = 0 \) for some positive integer \( \ell \). By Lemma 2.5, \( (h(s_1))^\ell = 0 \) for each \( s_1 \in S \). Since \( fg = 0 \) and \( R \) is \( S \)-compatible \( (S, \omega) \)-Armendariz, for each \( s_2, s_3 \in S \) we have \( f(s_2)g(s_3) = 0 \). Hence by Proposition 2.1, for each \( s_1, s_2, s_3 \in S \) we have \( f(s_2)h(s_1)g(s_3) = 0 \). Since \( R \) is \( S \)-compatible, \( fhg = 0 \) and the result follows.

**Theorem 2.7.** Let \( R \) be a ring, \( (S, \leq) \) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is \( S \)-compatible \( (S, \omega) \)-Armendariz, then:

(i) \( \text{Nil}^*(R[[S, \omega]]) \) is a (nonunital) subring of \( R[[S, \omega]] \);

(ii) \( N_0(R[[S, \omega]]) = \text{Nil}_0(R[[S, \omega]]) = L\text{-rad}(R[[S, \omega]]) = \text{Nil}^*(R[[S, \omega]]) = A(R[[S, \omega]]) \). In particular, if \( R \) is \( S \)-compatible \( (S, \omega) \)-Armendariz, then \( R[[S, \omega]] \) satisfies the Köthe’s conjecture.

**Proof.** Using Proposition 2.6, the proof is similar to that of [13, Proposition 4.5].

A ring \( R \) is said to be Dedekind finite, if \( ab = 1 \) implies \( ba = 1 \), for each \( a, b \in R \).

**Lemma 2.8.** Every abelian ring is Dedekind finite.
Proof. If for \(a, b \in R\), \(ab = 1\), then we have \((ba)^2 = baba = ba\). So \(1 = a(ba)b = (ab)(ba)\). Thus \(ba = 1\).

G. Marks et al. in [13] showed that if \(S\) is nontrivial strictly ordered monoid and \(R\) is linearly \((S, \omega)\)-Armendariz, then \(R\) is abelian. They also proved that if \(R\) is \((S, \omega)\)-Armendariz, then \(R[[S, \omega]]\) is abelian. By using this fact we have the following corollary:

**Corollary 2.9.** Let \(R\) be a ring, \((S, \leq)\) a strictly ordered monoid and \(\omega : S \rightarrow \text{End}(R)\) a monoid homomorphism.

(i) If \(S\) is nontrivial and \(R\) is linearly \((S, \omega)\)-Armendariz, then \(R\) is Dedekind finite;

(ii) If \(R\) is \((S, \omega)\)-Armendariz, then \(R[[S, \omega]]\) is Dedekind finite.

### 3. 2-primal skew generalized power series rings

The classes of rings under consideration are defined as follows. A ring \(R\) is called symmetric if for all \(a, b, c \in R\) we have \(abc = 0\) implies that \(acb = 0\). A ring \(R\) is called reversible if for all \(a, b \in R\) we have \(ab = 0\) implies that \(ba = 0\). A ring \(R\) is called semicommutative if \(ab = 0\) implies \(aRb = 0\) for each \(a, b \in R\). A ring \(R\) is called 2-primal if \(\text{Nil}_*(R) = \text{Nil}(R)\). A ring \(R\) is called reduced if it has no nonzero nilpotent.

Note that every reduced ring is symmetric and every symmetric ring is reversible and every reversible ring is 2-primal. But the converse is not true in general.

In [9, Example 2], C. Huh, Y. Lee and A. Smoktunowicz showed that, there is a semicommutative ring \(R\) such that the polynomial ring \(R[x]\) is not semicommutative. G. Marks et al. in [13] proved that for a ring \(R\), strictly ordered monoid \((S, \leq)\) which is artinian, narrow, unique product (a.n.u.p., see [13, Definition 4.11]) and monoid homomorphism \(\omega : S \rightarrow \text{End}(R)\), if \(R\) is \(S\)-compatible and \((S, \omega)\)-Armendariz, then \(R[[S, \omega]]\) is semicommutative if and only if \(R\) is.

G.F. Birkenmeier, H.E. Heatherly, and E.K. Lee in [3, Proposition 2.6] proved that the 2-primal condition is inherited by ordinary polynomial extensions. In [12] G. Marks investigated conditions on ideals of a 2-primal ring \(R\) that will ensure that a skew polynomial ring \(R[x; \alpha]\) be 2-primal.

In this section, we first show that, if \(R\) is a \(S\)-compatible \((S, \omega)\)-Armendariz ring, then \(\text{Nil}_*(R[[S, \omega]]) = \text{Nil}_*(R)[[S, \omega]]\). Then we prove that for a \(S\)-compatible \((S, \omega)\)-Armendariz ring \(R\), \(R\) is 2-primal if and only if \(R[[S, \omega]]\) is 2-primal. We also show that symmetric, reversible and \(ZC_n\) properties transfer between \(R\) and the skew generalized power series ring \(R[[S, \omega]]\), in case \(R\) is \(S\)-compatible \((S, \omega)\)-Armendariz.

**Theorem 3.1.** Let \(R\) be a ring, \((S, \leq)\) a strictly ordered monoid and \(\omega : S \rightarrow \text{End}(R)\) a monoid homomorphism. If \(R\) is \(S\)-compatible \((S, \omega)\)-Armendariz, then \(\text{Nil}_*(R[[S, \omega]]) = \text{Nil}_*(R)[[S, \omega]]\).

**Proof.** Let \(a \in \text{Nil}_*(R)\), then by Corollary 2.2, \(RaR\) is a nilpotent ideal in \(R\). Since \(R\) is \(S\)-compatible, for each \(s \in S\), \(R\omega_s(a)R\) is a nilpotent ideal of \(R\) and so \(\omega_s(a) \in \text{Nil}_*(R)\). Thus \(\text{Nil}_*(R)\) is \(S\)-invariant and so \(\text{Nil}_*(R)[[S, \omega]]\) is an ideal of \(R[[S, \omega]]\).
exists a positive integer \( g \in f \). By Theorem 2.7, \( R[[S, \omega]]fR[[S, \omega]] \) is a nilpotent ideal of \( R[[S, \omega]] \) and so by using Lemma 2.5, for each \( s \in S \), \( Rf(s)R \) is a nilpotent ideal of \( R \). Then \( f(s) \in \text{Nil}_s(R) \), for each \( s \in S \) and so \( f \in \text{Nil}_s(R)[[S, \omega]] \). Now let \( f \in \text{Nil}_s(R)[[S, \omega]] \), for each \( s \in S \), \( f(s) \in \text{Nil}_s(R) \). By Corollary 2.2, there exists a positive integer \( n \) such that for each \( s \in S \), \((Rf(s)R)^n = 0 \). Since \( R \) is \( S \)-compatible, then for each \( g, h \in R[[S, \omega]] \), \((gh)^n = 0 \). By the first part of the proof, we know that if \( g \in \text{Nil}_s(R[[S, \omega]]) \), then \( g(s) \in \text{Nil}_s(R) \), for each \( s \in S \). So \( f \in \text{Nil}_s(R[[S, \omega]]) \) and the result follows.

**Corollary 3.2.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. If \( R \) is \( S \)-compatible \((S, \omega)\)-Armendariz, then

\[
N_0(R[[S, \omega]]) = N_0(R)[[S, \omega]] = \text{Nil}_s(R[[S, \omega]]) = \text{Nil}_s(R)[[S, \omega]] = \text{Nil}_s(R)[[S, \omega]] = \text{Nil}_s(R)[[S, \omega]].
\]

**Corollary 3.3.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. Suppose that \( R \) is \( S \)-compatible and \((S, \omega)\)-Armendariz. Then \( R[[S, \omega]] \) is 2-primal if and only if \( R \) is.

**Proof.** Assume that \( R \) is 2-primal and \( f \in \text{Nil}(R[[S, \omega]]) \). So for some positive integer \( m \), \( f^m = 0 \), and by Lemma 2.5, \( (f(s))^m = 0 \) for each \( s \in S \). Hence \( f(s) \in \text{Nil}(R) \) for each \( s \in S \). Since \( R \) is 2-primal, then \( f(s) \in \text{Nil}_s(R) \) for each \( s \in S \). Thus \( f \in \text{Nil}_s(R)[[S, \omega]] = \text{Nil}_s(R)[[S, \omega]] \), by Theorem 3.1. Therefore \( R[[S, \omega]] \) is 2-primal. Conversely, suppose that \( R[[S, \omega]] \) is 2-primal, then \( R \) is 2-primal by [3, Proposition 2.2].

In [1] S.A. Amitsur asked if \( R \) is a nil ring, whether the polynomial ring \( R[x] \) is nil? A. Smoktunowicz answers in negative by an example in [16]. But for \( S \)-compatible \((S, \omega)\)-Armendariz rings we have the following result.

**Corollary 3.4.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. If \( R \) is a nil \( S \)-compatible \((S, \omega)\)-Armendariz ring, then \( R[[S, \omega]] \) is a nil ring.

So by the above result we see that Amitsur’s question is true for Armendariz rings.

By Anderson and Camillo [2], a ring \( R \) satisfies \( ZC_n \), if for \( a_1, a_2, \ldots, a_n \in R \) with \( a_1a_2 \cdots a_n = 0 \) it implies that \( a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)} = 0 \) for each \( \sigma \in S_n \) and \( n \geq 2 \), where \( S_n \) denotes the permutation group on \( n \) letters.

**Proposition 3.5.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. If \( R \) is \( S \)-compatible \((S, \omega)\)-Armendariz, then:

(i) \( R \) satisfies \( ZC_n \) if and only if \( R[[S, \omega]] \) satisfies \( ZC_n \);

(ii) \( R \) is symmetric if and only if \( R[[S, \omega]] \) is symmetric;

(iii) \( R \) is reversible if and only if \( R[[S, \omega]] \) is reversible.

**Proof.** (i) Assume that \( R \) satisfies \( ZC_n \) and that \( f_1f_2 \cdots f_n = 0 \) with \( f_1, f_2, \ldots, f_n \in R[[S, \omega]] \). Since \( R \) is \( S \)-compatible \((S, \omega)\)-Armendariz, \( f_1(s_1)f_2(s_2) \cdots f_n(s_n) = 0, \)
for each $s_i \in S$, by Lemma 2.5. Since $R$ satisfies $ZC_n$, for each $\sigma \in S_n$ we have $f_{\sigma(1)}(s_{\sigma(1)})f_{\sigma(2)}(s_{\sigma(2)}) \cdots f_{\sigma(n)}(s_{\sigma(n)}) = 0$. Since $R$ is $S$-compatible, $f_{\sigma(1)}f_{\sigma(2)} \cdots f_{\sigma(n)} = 0$, and the result follows.

The proof of (ii) and (iii) is similar to the proof of (i).

C. Huh et al. in [9, Proposition 16] proved that locally finite Armendariz rings are semicommutative and especially finite Armendariz rings are semicommutative. We generalized this result for $(S, \omega)$-Armendariz rings.

Recall that a monoid $S$ is aperiodic if for any $s \in S \setminus \{1\}$ and positive integers $m \neq n$ we have $s^m \neq s^n$.

**Lemma 3.6.** Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $R$ is $(S, \omega)$-Armendariz and $\omega_s$ is compatible for some $s \in S \setminus \{1\}$, then for all $a, b, x \in R$ and integer $n \geq 1$, $ab = ax^n b = 0$ implies $axb = 0$.

**Proof.** Let $a, b, x \in R$ and integer $n \geq 1$ such that $ab = ax^n b = 0$ and let $s \in S \setminus \{1\}$ such that $\omega_s$ is compatible. Then $ax\omega_s(x)\omega_s(x) \cdots \omega_{s^{n-1}}(x)b = 0$, since $\omega_s$ is compatible. As suggested by the proof of [13, Lemma 4.2 (iii)], let $f = c_2 e_4 \in R[[S, \omega]]$. Then $c_0 f^n c_0 = 0$, since $ax\omega_s(x)\omega_s(x) \cdots \omega_{s^{n-1}}(x)b = 0$. Thus $c_0 (1-f)(1+f+f^2+\cdots+f^{n-1})c_0 = c_0 (1-f^n)c_0 = 0$. Since $R$ is $(S, \omega)$-Armendariz and $S$ is aperiodic by [13, Lemma 4.2 (ii)], then $0 = [c_0 (1-f)](1),[(1+f+f^2+\cdots+f^{n-1})c_0](s) = ax\omega_s(b)$. Since $\omega_s$ is compatible, $axb = 0$ and the result follows.

By using the idea similar to one used by C. Huh et al. in [9, Proposition 16], we proved the following proposition.

**Proposition 3.7.** Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $R$ is locally finite $(S, \omega)$-Armendariz and $\omega_s$ is compatible for some $s \in S \setminus \{1\}$, then $R$ is semicommutative.

**Proof.** Let $a, b, r \in R$ with $ab = 0$. By the proof of Theorem 2.4, there exists a positive integer $n$ such that $r^n$ is an idempotent element of $R$. By [13, Proposition 4.10], $R$ is abelian, then $ab = ar^n b = 0$. Hence by Lemma 3.6, $arb = 0$ and the result follows.

By using [13, Theorem 4.1] we have the following corollary.

**Corollary 3.8.** Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $R$ is locally finite $S$-compatible $(S, \omega)$-Armendariz, then $R[[S, \omega]]$ is semicommutative.

4. Skew generalized power series zip rings

A ring $R$ is called right zip provided that if the right annihilator $r_R(X)$ of a subset $X$ of $R$ is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for a left ideal $L$ of $R$ with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. $R$ is zip if it is right and left zip. The
concept of zip rings initiated by Zelmanowitz [17]. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring. Extensions of zip rings were studied by several authors. Faith in [6] proved that if \( R \) is a commutative zip ring and \( G \) is a finite abelian group, then the group ring \( R[G] \) of \( G \) over \( R \) is zip. Cedó in [4] gives an example of a zip ring \( R \) for which \( R[x] \) is not zip. He also showed that there exists a right zip ring \( R \) and finite group \( G \) such that the group ring \( R[G] \) is not right zip. Faith in [6] raised the following questions:

(1) When does \( R \) being a right zip ring imply \( R[x] \) being right zip?

(2) When does \( R \) being a right zip ring imply \( R[G] \) being right zip when \( G \) is a finite group?

Hong et al. in [8] proved that \( R \) is a right zip ring if and only if \( R[x] \) is a right zip ring when \( R \) is an Armendariz ring. In [5], W. Cortes studied the relationship between right zip property of \( R \) and skew polynomial extensions over \( R \) by using the skew versions of Armendariz rings and generalized some results of Hong et al.

In this section we show that for a ring \( R \), strictly ordered monoid \((S,\leq)\) and monoid homomorphism \( \omega : S \rightarrow \text{End}(R) \), if \( R \) is \( S \)-compatible \((S,\omega)\)-Armendariz, then \( R[[S,\omega]] \) is right zip if and only if \( R \) is right zip.

**Proposition 4.1.** Let \( R \) be a ring, \((S,\leq)\) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R[[S,\omega]] \) is a right zip ring and either of the following conditions holds:

(i) \( \omega_1 \) is compatible; or

(ii) \( \omega_1 \) is automorphism.

Then \( R \) is a right zip ring.

**Proof.** Suppose that \( A = R[[S,\omega]] \) is a right zip ring and \( \omega_1 \) is compatible. Let \( U \subseteq R \) with \( r_U(U) = 0 \) and \( U' = \{c_r | r \in U \} \). If \( f \in r_A(U') \), then \( c_u f = 0 \) for each \( u \in U \). Thus for each \( s \in S \) and each \( u \in U \), \( c_u \omega_1(f(s)) = 0 \). Since \( \omega_1 \) is compatible, \( uf(s) = 0 \) and so for each \( s \in S \), \( f(s) \in r_R(U) = 0 \). Therefore \( f = 0 \) and so \( r_A(U') = 0 \). Since \( R[[S,\omega]] \) is right zip, there exists a finite set \( V' \subseteq U' \) such that \( r_A(V') = 0 \). Now let \( V = \{v | r \in U, c_r \in V' \} \). \( V \) is a finite subset of \( U \). Now we show that \( r_V(V) = 0 \). Let \( r \in r_R(V) \). Since \( \omega_1 \) is compatible then \( c_r \in r_A(V') = 0 \) and so \( r = 0 \). Hence \( r_V(V) = 0 \) and the result follows. Now assume that \( A = R[[S,\omega]] \) is a right zip ring and \( \omega_1 \) is automorphism. Let \( U \subseteq R \) with \( r_U(U) = 0 \) and \( U' = \{c_{\omega_1(r)} | r \in U \} \). If \( f \in r_A(U') \), then \( c_{\omega_1(s)} f = 0 \) for each \( s \in S \) and each \( u \in U \). Thus for each \( s \in S \) and each \( u \in U \), \( (c_{\omega_1(s)} \omega_1(f(s)) = 0 \) and so \( \omega_1(uf(s)) = 0 \). Since \( \omega_1 \) is automorphism, for each \( s \in S \), \( f(s) \in r_R(U) = 0 \) and so \( r_A(U') = 0 \). Since \( R[[S,\omega]] \) is right zip, there exists a finite set \( V' \subseteq U' \) such that \( r_A(V') = 0 \). Now let \( V = \{v | r \in U, c_{\omega_1(r)} \in V' \} \). \( V \) is a finite subset of \( U \). Now we show that \( r_V(V) = 0 \). Let \( r \in r_R(V) \) then for each \( v \in V \), \( vr = 0 \). Thus \( c_{\omega_1(v)} c_r = 0 \), for each \( v \in V \) and so \( c_r \in r_A(V') = 0 \). Hence \( r_V(V) = 0 \) and the result follows.

**Theorem 4.2.** Let \( R \) be a ring, \((S,\leq)\) a strictly ordered monoid and \( \omega : S \rightarrow \text{End}(R) \) a monoid homomorphism. If \( R \) is a right zip \( S \)-compatible, \((S,\omega)\)-Armendariz ring, then \( R[[S,\omega]] \) is a right zip ring.
Proof. Suppose that $R$ is a right zip ring and put $A = R[[S, \omega]]$. Let $U \subseteq R[[S, \omega]]$ with $r_A(U) = 0$. Now let $V = \{ f(s) | f \in U, s \in S \}$. If $r \in r_R(V)$, then $f(s)r = 0$ for any $f \in U$ and $s \in S$. Since $R$ is $S$-compatible, $fcr = 0$ for any $f \in U$.

Then $c_r \in r_A(U) = 0$ and hence $r_R(V) = 0$. Since $R$ is right zip, there exists a finite set $V_0 \subseteq V$ such that $r_R(V_0) = 0$. For each $v \in V_0$, there exists $g_v \in U$ such that $v = g_v(s)$, for some $s \in S$. Let $U_0$ be a minimal subset of $U$ such that $g_v \in U_0$ for each $v \in V_0$. Then $U_0$ is a nonempty finite subset of $U$.

Let $V_1 = \{ f(s) | f \in U_0, s \in S \}$. Then $V_0 \subseteq V_1$ and so $r_R(V_1) \subseteq r_R(V_0) = 0$. If $f \in r_A(U_0)$, then $gf = 0$ for any $g \in U_0$. Since $R$ is $S$-compatible $(S, \omega)$-Armendariz, then $f(s) \in r_R(V_1) = 0$ for each $s \in S$, and so $f = 0$. Hence $r_A(U_0) = 0$ and the result follows.

Theorem 4.3. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. If $R$ is a right zip $(S, \omega)$-Armendariz ring and $\omega_s$ is automorphism for each $s \in S$, then $R[[S, \omega]]$ is a right zip ring.

Proof. Suppose that $R$ is a right zip ring and put $A = R[[S, \omega]]$. Let $U \subseteq R[[S, \omega]]$ with $r_A(U) = 0$. Now let $V = \{ f(s) | f \in U, s \in S \}$. If $r \in r_R(V)$, then $\omega^{-1}(f(s))r = 0$ for each $f \in U$ and $s \in S$. Thus for each $f \in U$ and $s \in S$, $f(s)\omega_s(r) = 0$ and so $fcr = 0$. Then $c_r \in r_A(U) = 0$ and hence $r_R(V) = 0$.

Since $R$ is right zip, there exists a finite set $V_0 \subseteq V$ such that $r_R(V_0) = 0$. For each $v \in V_0$, there exists $g_v \in U$ such that $v = \omega^{-1}(g_v(s))$, for some $s \in S$. Let $U_0$ be a minimal subset of $U$ such that $g_v \in U_0$ for each $v \in V_0$. Then $U_0$ is a nonempty finite subset of $U$.

Let $V_1 = \{ \omega^{-1}(f(s)) | f \in U_0, s \in S \}$. Then $V_0 \subseteq V_1$ and so $r_R(V_1) \subseteq r_R(V_0) = 0$. If $f \in r_A(U_0)$, then $gf = 0$ for any $g \in U_0$. Since $R$ is $(S, \omega)$-Armendariz, then $g(s)\omega_s(f(t)) = 0$ for each $s, t \in S$. Thus $\omega^{-1}(g(s))f(t) = 0$ for each $s, t \in S$ and so for each $t \in S$, $f(t) \in r_R(V_1) = 0$. Then $r_A(U_0) = 0$ and the result follows.

The skew power series ring is denoted by $R[[x; \alpha]]$, whose elements are the series $\sum_{i=0}^{\infty} a_i x^i$ for some $a_i \in R$ and non-negative integers $i$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. Suppose that $\alpha$ is an endomorphism of $R$. A ring $R$ satisfies $SA2'$ if for $f = \sum_{i=0}^{\infty} a_i x^i, g = \sum_{i=0}^{\infty} b_i x^i \in R[[x; \alpha]], fg = 0$ implies that $a_i \alpha^i(b_j) = 0$ for each $i \geq 0$ and $j \geq 0$ [5, Definition 2.2]. Note that $SA2'$ is special case of $(S, \omega)$-Armendariz ring (let $S = \mathbb{N} \cup \{0\}$ under addition, with its natural linear order and $\omega$ is determined by $\omega(1) = \alpha$).

Corollary 4.4. Let $R$ be a ring. Then:

(i) [8, Theorem 11] If $R$ is Armendariz, then $R$ is right zip if and only if $R[x]$ is.

(ii) [5, Theorem 2.8] Suppose that $\alpha$ is an automorphism of $R$ and $R$ satisfies SA1'. Then the following conditions are equivalent:

a) $R$ is a right zip ring;

b) $R[x; \alpha]$ is a right zip ring;

c) $R[x, x^{-1}; \alpha]$ is a right zip ring.

(iii) Suppose that $R$ is $\alpha$-compatible $\alpha$-skew Armendariz. Then the following
conditions are equivalent:
  a) $R$ is a right zip ring;
  b) $R[x;\alpha]$ is a right zip ring.
(iv) [5, Theorem 2.8] Suppose that $\alpha$ is an automorphism of $R$ and $R$ satisfies $SA2'$. Then the following conditions are equivalent:
  a) $R$ is a right zip ring;
  b) $R[[x;\alpha]]$ is a right zip ring;
  c) $R[[x,x^{-1};\alpha]]$ is a right zip ring.
(v) Suppose that $R$ is $\alpha$-compatible and $R$ satisfies $SA2'$. Then the following conditions are equivalent:
  a) $R$ is a right zip ring;
  b) $R[[x;\alpha]]$ is a right zip ring;
  c) $R[[x,x^{-1};\alpha]]$ is a right zip ring.
(vi) Let $S$ be a monoid and suppose that $R$ is an Armendariz ring relative to monoid $S$. Then the following conditions are equivalent:
  a) $R$ is a right zip ring;
  b) $R[S]$ is a right zip ring.

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