TOTAL ACYCLICITY FOR COMPLEXES OF REPRESENTATIONS OF QUIVERS

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Abstract. We study total acyclicity for complexes of projective and injective representations of quivers. A classification is given for such complexes in terms of associated vertex-complexes. When the base ring or the quiver is nice enough, this classification is used to prove the existence of Gorenstein projective precovers in the category of representations of quivers. Furthermore, we exploit this local description to obtain some criteria for the category of representations of a quiver to be Gorenstein or virtually Gorenstein. If \( A \) is an artin algebra it is proved that, for an arbitrary quiver \( Q \), the representation category \( \text{Rep}(Q,A) \) is virtually Gorenstein whenever \( A \) is virtually Gorenstein. A description of Gorenstein projective and Gorenstein injective representations of quivers over general rings is also provided.

1. INTRODUCTION

Let us begin with some notation.

- \( Q \) is a quiver, i.e. a directed graph, \( R \) is an associative ring with unity, \( \text{Mod}R \) is the category of all (left) \( R \)-modules, and \( \text{Rep}(Q,R) \) is the category of representations of \( Q \) by \( R \)-modules and \( R \)-homomorphisms.
- For an additive category \( \mathcal{A} \), \( \text{Ch}(\mathcal{A}) \) denotes the category of complexes and chain maps over \( \mathcal{A} \). The homotopy category of \( \mathcal{A} \) will be denoted by \( \mathbb{K}(\mathcal{A}) \).
- If \( \mathcal{A} \) is the category \( \text{Mod}R \) (\( \text{Rep}(Q,R) \)) or one of the subcategories \( \text{Prj}R \) (\( \text{Prj}Q \)) and \( \text{Inj}R \) (\( \text{Inj}Q \)) respectively of projective and injective \( R \)-modules (representations), then the corresponding homotopy categories will be denoted by \( \mathbb{K}(R) \) (\( \mathbb{K}(Q) \)), \( \mathbb{K}(\text{Prj}R) \) (\( \mathbb{K}(\text{Prj}Q) \)), and \( \mathbb{K}(\text{Inj}R) \) (\( \mathbb{K}(\text{Inj}Q) \)).

Let \( \mathfrak{X} \) be a full subcategory of an additive category \( \mathcal{A} \). A complex \( E \in \text{Ch}(\mathfrak{X}) \) is said to be \( \mathfrak{X} \)-totally acyclic provided it is both \( \text{Hom}_{\mathcal{A}}(-,X) \) and \( \text{Hom}_{\mathcal{A}}(X,-) \)-exact for any object \( X \in \mathfrak{X} \). The homotopy category of totally acyclic complexes of projective (injective) \( R \)-modules is denoted by \( \mathbb{K}_{\text{tac}}(\text{Prj}R) \) (\( \mathbb{K}_{\text{tac}}(\text{Inj}R) \)) and the corresponding notation for a quiver \( Q \) is \( \mathbb{K}_{\text{tac}}(\text{Prj}Q) \) (\( \mathbb{K}_{\text{tac}}(\text{Inj}Q) \)). An \( R \)-module \( G \) is called Gorenstein projective (Gorenstein injective) if it is a syzygy of a totally acyclic complex of projective (injective) \( R \)-modules.

Totally acyclic complexes of modules are known to reflect some nice properties of the base ring \( R \); see for instance [14] and also [20], where it is shown that if \( R \) is a commutative noetherian ring of finite Krull dimension, then the necessary and sufficient condition for \( R \) to be Gorenstein is that every acyclic complex of projective \( R \)-modules is totally acyclic. On the other hand, Gorenstein projective and injective modules play a particular role in the so-called

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Recall that "Auslander’s Last Theorem" asserted that any finitely generated module over a local Gorenstein ring has a Gorenstein projective cover. This result has been extended by Enochs, et.al. [13] to the case of local Cohen-Macaulay rings which admit a dualizing module and modules of finite $G$-dimension.

Also, if $R$ is a noetherian ring with a dualizing complex, it follows from an argument by Jørgensen [15] that the inclusion functor $K_{tac}(\text{Prj}R) \to K(\text{Prj}R)$ has a right adjoint and then every $R$-module has a Gorenstein projective precover. This result has been extended in [20] to the case of noetherian rings of finite Krull dimension.

One of the purposes of this paper is to provide a non-commutative version of these results. An appropriate and wider framework where these lines of thoughts seem to be applicable is the category of representations of a quiver over an arbitrary associative ring. In this respect, we point out that the project of studying representations of arbitrary quivers over arbitrary rings was initiated by Enochs and his school through a series of papers; see for instance [7], [9], [11] and [12]. The importance and motivations of these studies goes back, mainly, to the fact that quivers usually provide a very nice, accessible, and more intuitive framework to introduce some notions which very often leads to simplifications in proofs. For example, we mention the paper [8] where it is shown that the category of quasi coherent sheaves over a scheme is equivalent to the category of a particular type of representations of a quiver.

Hence in this paper, we study the homotopy categories of totally acyclic complexes of projective and injective representations of quivers. We will first present a description for such complexes in terms of the complexes obtained in each vertex; see Theorems 3.1.3 and 3.1.6. Using that description we will be able to extend the above mentioned results to the category of representations of certain quivers; in particular, we show that if $R$ is noetherian of finite Krull dimension and $Q$ is left rooted, then the inclusion functor $K_{tac}(\text{Prj}Q) \to K(\text{Prj}Q)$ has a right adjoint. This implies that, in this case, Gorenstein projective representations form a precovering class in $\text{Rep}(Q, R)$. We also seek for conditions under which every acyclic complex of projective (or injective) representations of quivers is totally acyclic and show how this property may characterize Gorensteinness of the representation category $\text{Rep}(Q, R)$; see Theorem 3.3.5. Moreover, describing totally acyclic complexes of projective and injective representations leads directly to a proof for the virtually Gorensteinness of $\text{Rep}(Q, \Lambda)$ whenever $\Lambda$ is a virtually Gorenstein artin algebra. We also take the advantage of this local description to give a classification for Gorenstein injective and Gorenstein projective representations of quivers over arbitrary rings.

2. PRELIMINARIES

Recall that a quiver $Q$ is a directed graph $(V, E)$, where $V$ and $E$ are respectively the sets of vertices and arrows of $Q$, endowed with a couple of functions $s$ and $t$ which assign respectively to any arrow $a$ of $Q$ its origin and terminal vertices. If $R$ is a ring and $Q$ is a quiver, a representation $\mathcal{X}$ of $Q$ by $R$-modules and $R$-homomorphisms is obtained by associating to any vertex $v$ an $R$-module $\mathcal{X}_v$ and to any arrow $a : v \to w$ an $R$-homomorphism $\mathcal{X}_a : \mathcal{X}_v \to \mathcal{X}_w$. If $\mathcal{X}$ and $\mathcal{Y}$ are two representations of $Q$, then a morphism $f : \mathcal{X} \to \mathcal{Y}$ is determined by a family $\{f_v\}_{v \in V}$ of $R$-homomorphisms in such a way that for any arrow $a : v \to w$, the commutativity condition $\mathcal{Y}_a f_v = f_w \mathcal{X}_a$ holds. For a given quiver $Q$ and a ring $R$, the representations of $Q$ and the morphisms between them form a category which is denoted by $\text{Rep}(Q, R)$. This is a Grothendieck category with enough projectives. It is known that if $Q$ is finite, i.e. $V$ and $E$ are both finite, then this category is equivalent to...
the category of $RQ$-modules where $RQ$ is the path ring of $Q$. For any two representations $X$ and $Y$ of $Q$, we write $\text{Hom}_Q(X, Y)$ for the abelian group of morphisms from $X$ to $Y$ in the category $\text{Rep}(Q, R)$.

Let $Q$ be an arbitrary quiver and $Q'$ be its subquiver. The restriction functor $e^{Q'} : \text{Rep}(Q, R) \rightarrow \text{Rep}(Q', R)$ which, by definition, restricts any representation of $Q$ to the vertices of $Q'$ is known to possess right and left adjoints; for instance, we remind the reader of the argument given in [2]. Because these adjoints are of particular importance in this work, we prefer to point out the formulas explicitly. For any $X \in \text{Rep}(Q', R)$, if $v \in V(Q')$ then

$$e^{Q'}_\rho(X)_v = X_v \oplus \prod_{s(\alpha) = v \atop t(\alpha) \in Q' \atop a \notin Q'} X_{t(\alpha)}$$

and if $v \notin V(Q')$, then

$$e^{Q'}_\rho(X)_v = \prod_{s(\alpha) = v \atop t(\alpha) \in Q' \atop a \notin Q'} X_{t(\alpha)}.$$

Similarly, for any $Y \in \text{Rep}(Q', R)$, if $v \in V(Q')$ then

$$e^{Q'}_\lambda(Y)_v = Y_v \oplus \bigoplus_{s(\alpha) \in Q' \atop t(\alpha) = v \atop a \notin Q'} Y_{s(\alpha)}$$

and if $v \notin V(Q')$, then

$$e^{Q'}_\lambda(Y)_v = \bigoplus_{s(\alpha) \in Q' \atop t(\alpha) = v \atop a \notin Q'} Y_{s(\alpha)},$$

where the symbols $e^{Q'}_\lambda$, $e^{Q'}_\rho$, $\alpha$ and $a$ are reserved respectively for the left adjoint functor, right adjoint functor, paths and arrows of $Q$. A familiar, and frequently used, example of this adjoints arises when one looks at the single subquiver $\{v\} \subseteq Q$. The adjoints are then denoted respectively by $e^v_\rho$ and $e^v_\lambda$. Throughout the paper, we will freely use the fact that these right (left) adjoints preserve injective (projective) objects.

The other key point in this context is that the restriction functor

$$e^{Q'} : \text{Rep}(Q, R) \rightarrow \text{Rep}(Q', R)$$

naturally extends to a functor $k^{Q'}$ between the corresponding homotopy categories and that this extension also possesses left and right adjoints, denoted respectively by $k^{Q'}_\lambda$ and $k^{Q'}_\rho$. Moreover, these adjoints are nothing but the natural extensions of the functors $e^{Q'}_\lambda$ and $e^{Q'}_\rho$.

Let $Q$ be a quiver and set

$$V_1 = \{v \in V : \exists a \in E : s(a) = v\}.$$ 

Suppose that $\alpha$ is an ordinal number and that we have defined $V_\gamma$ for every ordinal $\gamma < \alpha$. Let

$$V_\alpha = \{v \in V \setminus \cup_{\gamma < \alpha} V_\gamma : \exists a \in E \setminus \{a : t(a) \in \cup_{\gamma < \alpha} V_\gamma\} \text{ such that } s(a) = v\}.$$
Definition 2.1. A quiver $Q = (V, E)$ is called right rooted if there exists an ordinal $\beta$ such that $V = \bigcup_{\gamma \leq \beta} V_\gamma$. We denote by $\mu(Q)$ the least ordinal number $\beta$ for which $V$ can be written in this way.

In fact, we are adapting this definition from [9] where it is declared that a quiver $Q$ is right rooted if and only if it contains no path of the form $\bullet \rightarrow \bullet \rightarrow \cdots$. We also note that left rooted quivers have been defined, in a dual sense, for instance in [7]. In fact, a quiver $Q$ is said to be left rooted provided it contains no path of the form $\cdots \rightarrow \bullet \rightarrow \bullet$. For a left rooted $Q$, $\mu(Q)$ has the same meaning as that for right rooted ones.

A nice classification for injective and projective representations respectively over right and left rooted quivers has been given in [9] and [7]. For future use, we include one of them and refer to the other one to [7, Theorem 3.1].

Theorem 2.2. [9, Theorem 4.2] Let $Q$ be a right rooted quiver. A representation $\mathcal{X}$ of $Q$ is injective if and only if for any vertex $v$,

(i) $\mathcal{X}_v$ is an injective $R$-module,

(ii) The $R$-homomorphism $\mathcal{X}_v \rightarrow \prod_{s(a) = v} \mathcal{X}_{t(a)}$ is a splitting epimorphism.

For a right (or left) rooted quiver $Q$ and an ordinal number $\delta$, let $Q^{\delta}$ be the subquiver of $Q$ whose set of vertices is $V(Q^{\delta}) = \bigcup_{\gamma \leq \delta} V_\gamma$. For simplicity we use the symbols $e_x^\delta, e_a^\delta, e_x^\lambda, k^\delta, k_x^\delta$, and $k_a^\delta$ to denote the restriction functor, its adjoints and their corresponding extensions.

Triangulated And Homotopy Categories. Let $\mathcal{T}$ be a triangulated category with suspension functor $\Sigma$ which admits arbitrary coproducts. We refer to [21] for axioms and basic properties. An object $X \in \mathcal{T}$ is said to be compact provided that $\operatorname{Hom}_\mathcal{T}(X, -)$ commutes with coproducts. By definition, a set $S$ of objects of $\mathcal{T}$ generates it if for any non-zero object $X$ of $\mathcal{T}$, there exists a non-zero map $\Sigma^n S \rightarrow X$ for some $S \in S$ and $n \in \mathbb{Z}$.

Our use of triangulated categories lies mainly in the case of homotopy categories and derived categories; recall that for an additive category $\mathcal{A}$, the homotopy category $\mathbb{K}(\mathcal{A})$ of $\mathcal{A}$ is defined by the following data: the objects are all complexes over $\mathcal{A}$ and the morphisms are homotopy equivalence classes of the chain maps in $\operatorname{Ch}(\mathcal{A})$. In case of homotopy categories arising from the category $\operatorname{Rep}(Q, R)$, the term $R$ is always omitted because it does not imply any ambiguity. Note also that for a quiver $Q$, the derived category of $\operatorname{Rep}(Q, R)$ which is denoted by $\mathbb{D}(Q)$, is obtained by formally inverting all quasi-isomorphisms in $\mathbb{K}(Q)$.

3. Main Results

3.1. Total Acyclicity. Let $Q = (V, E)$ be a quiver, $w \in V$ and $\mathcal{X} \in \operatorname{Rep}(Q, R)$. Moreover, let $Q(w)$ denote the set of all paths starting from $w$. Any element of $\prod_{v \in V} e_v^w(\mathcal{X}_v)_w$ is of the form $(\alpha_p)_{p \in Q(w)}$ where $\alpha_p \in \mathcal{X}_{t(p)}$. If $p = a_m \ldots a_1$ is a path with $s(p) = w$ and $t(p) = v$ for some $v \in V$, let $f_w : \mathcal{X}_w \rightarrow \prod_{v \in V} e_v^w(\mathcal{X}_v)_w$ be induced from $F_p = \mathcal{X}(a_m) \ldots \mathcal{X}(a_1)$. It is straightforward to see that $f_w$ defines a monomorphism $f : \mathcal{X} \rightarrow \prod_{v \in V} e_v^w(\mathcal{X}_v)$.

On the other hand, any element of $\prod_{a} (e_v^s(a)(\mathcal{X}_{t(a)}))_w$ can be viewed as $(\beta_q)_{q \in Q(w)}$ where $q$ is either the trivial path on $w$, or $q = ap$ for some arrow $a$ with $s(p) = w$. It is also easy to check that this gives an epimorphism $g : \prod_{v \in V} e_v^w(\mathcal{X}_v) \rightarrow \prod_{a} (e_v^s(a)(\mathcal{X}_{t(a)}))$ of representations defined, for the component corresponding to $q = ap$, as $g_w(\alpha_q) = \alpha_q - \mathcal{X}_w(\alpha_p)$, and $g_w(\alpha_{e_w}) = 0$. 
Hence, for any representation $X$ of an arbitrary quiver $Q$, there exists an exact sequence

$$0 \to X \xrightarrow{f} \prod_v e^v_p(X_v) \xrightarrow{g} \prod_a e^a_p(X_{t(a)}) \to 0 \quad (1).$$

We note that one may find a similar short exact sequence, in a dual vein, in [19]. Explicitly, if $X$ is a representation of an arbitrary quiver $Q$ then, using an argument similar to that above, we get an exact sequence

$$0 \to \bigoplus_a e^{t(a)}_\lambda(X_{s(a)}) \to \bigoplus_v e^v_\lambda(X_v) \to X \to 0 \quad (2).$$

In fact the above sequences behave as natural as they can be extended to complexes. Let for instance $X$, $Y$ and $Z$ be representations of an arbitrary quiver $Q$ and let $f : X \to Y$, $g : Y \to Z$ be morphisms of representations. Clearly, there are natural morphisms

$$\bigoplus_v e^v_\lambda(X_v) \to \bigoplus_v e^v_\lambda(Y_v)$$

and

$$\bigoplus_a e^{t(a)}_\lambda(X_{s(a)}) \to \bigoplus_a e^{t(a)}_\lambda(Y_{s(a)})$$

which are induced from the $R$-homomorphisms $f_v : X_v \to Y_v$ for every vertex $v$. If $gf = 0$, then the fact that $e^v_\lambda$ is an additive functor implies that the composition of any two induced consequent maps is zero. Also it is completely straightforward to insure that, indeed, the required diagrams commute.

For any complex $X$ of representations of a quiver $Q$ we have the following short exact sequences.

$$0 \to X \xrightarrow{f} \prod_v k^v_p(X_v) \xrightarrow{g} \prod_a k^{t(a)}_p(X_{t(a)}) \to 0 \quad (3)$$

and

$$0 \to \bigoplus_a k^{t(a)}_\lambda(X_{s(a)}) \to \bigoplus_v k^v_\lambda(X_v) \to X \to 0 \quad (4)$$

Throughout the paper, we will refer to the above sequences by calling them (1), (2), (3) and (4).

Let $Q$ be a quiver and $Q'$ be a subquiver of $Q$. Consider the following two combinatorial properties in which $p, q$ are paths and $a$ is an arrow in $Q$.

† For any $p = aq$ in $Q$ whose end points belong to $Q'$, the arrow $a$ also belongs to $Q'$.

†† For any $p = qa$ in $Q$ whose end points belong to $Q'$, the arrow $a$ also belongs to $Q'$.

An immediate consequence of the definitions of $e^Q_p$ and $e^Q_v$ in the previous section is that whenever $Q'$ satisfies condition †, then for any representation $X$ of $Q'$, we have $e^Q_p e^{Q'}_p(X) = X$ and if $Q'$ satisfies condition ††, then for any representation $Y$ of $Q'$, we have $e^Q_v e^{Q'}_v(Y) = Y$.

In the following lemma we will see how these easy observations may aid us to transfer total acyclicity from subquivers to the original quiver and vice versa.

**Lemma 3.1.1.** Let $Q'$ be a subquiver of a quiver $Q$, $E \in \mathbb{K}(\text{Inj}Q')$ and $P \in \mathbb{K}(\text{Prj}Q')$.

(i) Assume that $Q$ is right rooted and $Q'$ satisfies †. Then $k^E_p(E) \in \mathbb{K}_{tac}(\text{Inj}Q)$ if and only if $E \in \mathbb{K}_{tac}(\text{Inj}Q')$.

(ii) Assume $Q$ is left rooted and $Q'$ satisfies ††. Then $k^P_v(P) \in \mathbb{K}_{tac}(\text{Prj}Q)$ if and only if $P \in \mathbb{K}_{tac}(\text{Prj}Q')$.

**Proof.** (i) Suppose first that $k^E_p(E) \in \mathbb{K}_{tac}(\text{Inj}Q)$. It suffices to show that for any injective representation $E$ of $Q'$ and any integer $i$, the sequence

$$\text{Hom}_Q(E, E^{i-1}) \to \text{Hom}_Q(E, E^i) \to \text{Hom}_Q(E, E^{i+1})$$

is exact if $E \in \mathbb{K}_{tac}(\text{Inj}Q')$. Let $f : E \to E'$ be a morphism, then $f$ is injective if and only if $f \in \mathbb{K}_{tac}(\text{Inj}Q')$. Thus it is sufficient to show that $f$ is injective. This is clear because $k^E_p(E) \in \mathbb{K}_{tac}(\text{Inj}Q)$ implies that $k^E_p(E)$ is exact for any $E \in \mathbb{K}_{tac}(\text{Inj}Q)$.

(ii) Suppose $k^P_v(P) \in \mathbb{K}_{tac}(\text{Prj}Q)$. It suffices to show that $k^P_v(P)$ is exact for any $P \in \mathbb{K}_{tac}(\text{Prj}Q)$. Let $g : P \to P'$ be a morphism, then $g$ is injective if and only if $g \in \mathbb{K}_{tac}(\text{Prj}Q')$. Thus it is sufficient to show that $g$ is injective. This is clear because $k^P_v(P) \in \mathbb{K}_{tac}(\text{Prj}Q)$ implies that $k^P_v(P)$ is exact for any $P \in \mathbb{K}_{tac}(\text{Prj}Q)$.
is exact. Note that, by the above paragraph, \( \mathcal{E} = e^{Q'} e^{Q'}_p (\mathcal{E}) \). The result now follows from the following commutative diagram.

\[
\begin{array}{cccc}
\text{Hom}_{\mathcal{Q}}(e^{Q'}_p (\mathcal{E}), e^{Q'}_p (E^{i-1})) & \rightarrow & \text{Hom}_{\mathcal{Q}}(e^{Q'}_p (\mathcal{E}), e^{Q'}_p (E^i)) & \rightarrow & \text{Hom}_{\mathcal{Q}}(e^{Q'}_p (\mathcal{E}), e^{Q'}_p (E^{i+1})) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\text{Hom}_{\mathcal{Q}}(\mathcal{E}, E^{i-1}) & \rightarrow & \text{Hom}_{\mathcal{Q}}(\mathcal{E}, E^i) & \rightarrow & \text{Hom}_{\mathcal{Q}}(\mathcal{E}, E^{i+1})
\end{array}
\]

The reverse direction can be settled in the same way. Just note that by Theorem 2.2, for any injective representation \( \mathcal{E} \) of \( \mathcal{Q} \), \( e^{Q'} (\mathcal{E}) \) is an injective representation of \( Q' \). The next part is dual. \( \square \)

**Remark 3.1.2.** Even though we will use Lemma 3.1.1 later on, for the following theorem it is not needed in this generality. That is, we only need to know that if \( E (P) \) is a totally acyclic complex of injective (projective) \( R \)-modules, then for any \( v \), \( k^v(E) (k^v(P)) \) is also totally acyclic. This follows by the argument of Lemma 3.1.1 without any condition on \( Q \). Furthermore, the converse is true if \( Q \) is right (left) rooted (more generally, if it has no oriented cycle). The other point that must be highlighted is that each of the assumptions in statements (i) and (ii) of the above lemma is used in the proof of only one direction of the claims.

**Theorem 3.1.3.** Let \( R \) be a ring and \( \mathcal{Q} \) be a quiver. Assume \( E, P \) are complexes respectively of injective and projective representations of \( \mathcal{Q} \). Then

(i) If \( E_v \in K_{\text{tac}}(\text{Inj} R) \), for any vertex \( v \), then \( E \in K_{\text{tac}}(\text{Inj} \mathcal{Q}) \).

(ii) If \( P_v \in K_{\text{tac}}(\text{Prj} R) \), for any vertex \( v \), then \( P \in K_{\text{tac}}(\text{Prj} \mathcal{Q}) \).

**Proof.** (i) Note that since \( E \in K(\text{Inj} \mathcal{Q}) \), the sequence (3), when applied for \( E \), splits in each degree and the terms \( \prod_v k^v(E_v) \) and \( \prod_v k^0(\text{Inj} \mathcal{Q})(E_v) \) belong to \( K_{\text{tac}}(\text{Inj} \mathcal{Q}) \) by the assumption and Remark 3.1.2. Consequently, \( E \) itself belongs to \( K_{\text{tac}}(\text{Inj} \mathcal{Q}) \). The proof of (ii) is similar. \( \square \)

Our aim in the sequel is to prove the converse of the preceding theorem. To this end, we prove the following lemma.

**Lemma 3.1.4.** Suppose that \( \mathcal{Q} \) is a quiver and \( \delta \) is an ordinal number.

(i) If \( \mathcal{Q} \) is right rooted, \( \delta < \mu(\mathcal{Q}) \) and \( E \in K_{\text{tac}}(\text{Inj} \mathcal{Q}) \), then \( k^\delta(E) \) belongs to \( K_{\text{tac}}(\text{Inj} \mathcal{Q}) \).

(ii) If \( \mathcal{Q} \) is left rooted, \( \delta < \mu(\mathcal{Q}) \) and \( P \in K_{\text{tac}}(\text{Prj} \mathcal{Q}) \), then \( k^\delta(P) \) belongs to \( K_{\text{tac}}(\text{Prj} \mathcal{Q}) \).

**Proof.** (i) Let \( E \in K_{\text{tac}}(\text{Inj} \mathcal{Q}) \). Extend the unit morphism \( E \to k(E) \) to a triangle

\[
E' \rightarrow E \rightarrow k(E) \rightarrow \cdot
\]

in \( K(\text{Inj} \mathcal{Q}) \) (Note that in view of Theorem 2.2, \( k(E) \in K(\text{Inj} \mathcal{Q}) \)). Since the functors \( k \) and \( k^\delta \) are exact, \( k^\delta(E) \) is an exact complex. We claim that \( E' \) is totally acyclic. For denote by \( \mathcal{Q}' \) the subquiver of \( \mathcal{Q} \) with the set of vertices \( V \setminus \bigcup_{v \leq \delta} V_v \). Then \( E' = k^{\mathcal{Q}'}(E') \). By virtue of Lemma 3.1.1, the claim is proved if we show that \( k^{\mathcal{Q}'}(E') \in K_{\text{tac}}(\text{Inj} \mathcal{Q}') \); see Remark 3.1.2. Observe that for any injective representation \( \mathcal{E}' \) of \( \mathcal{Q}' \), \( \text{Hom}_{K(\text{Inj} \mathcal{Q}')} (e^{\mathcal{Q}'}_p (\mathcal{E}'), k^\delta(E)) = 0. \)
Now apply the homological functor $\text{Hom}_{K(\text{Inj}\mathcal{Q})}(e^Q_v(E'), -)$ to the above triangle and use total acyclicity of $E$ to deduce that $\text{Hom}_{K(\text{Inj}\mathcal{Q})}(e^Q_v(E'), E') = 0$. Hence the identifications

\[ \text{Hom}_{K(\text{Inj}\mathcal{Q}')}^Q(E', k^Q(E')) \cong \text{Hom}_{K(\text{Inj}\mathcal{Q})}^Q(e^Q_v(E'), k^Q(E')) = \text{Hom}_{K(\text{Inj}\mathcal{Q})}^Q(e^Q_v(E'), E') \]

show that $E' \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$. Consequently $k^Q(E)$ is totally acyclic. Dualize this argument to prove (ii).

\[ \square \]

Lemma 3.1.5. Let $\mathcal{Q}$ be a quiver.

(i) If $\mathcal{X}$ is a representation of $\mathcal{Q}$ such that $\mathcal{X}_v$ is an injective $R$-module for all $v$, then $\text{id}\mathcal{X} \leq 1$.

(ii) If $\mathcal{Y}$ is a representation of $\mathcal{Q}$ such that $\mathcal{Y}_v$ is a projective $R$-module for all $v$, then $\text{pd}\mathcal{Y} \leq 1$.

Proof. These follow respectively from the exact sequences (1) and (2).

\[ \square \]

Theorem 3.1.6. Let $\mathcal{Q}$ be a quiver, $E \in K(\text{Inj}\mathcal{Q})$ and $P \in K(\text{Prj}\mathcal{Q})$.

(i) Assume $\mathcal{Q}$ is right rooted or the base ring $R$ is noetherian. If $E \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$, then $E_v \in K_{\text{tac}}(\text{Inj}R)$ for all $v$.

(ii) Assume $\mathcal{Q}$ is left rooted or the base ring $R$ is left perfect and right coherent. If $P \in K_{\text{tac}}(\text{Prj}\mathcal{Q})$, then $P_v \in K_{\text{tac}}(\text{Prj}R)$ for all $v$.

Proof. We just give a proof for (i). Assume first that $R$ is noetherian. Then by Lemma 3.1.5, for any injective $R$-module $I$ and $v$, the representation $e^Q(I)$ has finite injective dimension. It is easy to verify that the functor $\text{Hom}_{\mathcal{Q}}(e^Q(I), -)$ leaves $E$ exact because it remains exact under applying $\text{Hom}_{\mathcal{Q}}(E, -)$ for any injective representation $E$ of $\mathcal{Q}$. Now the adjoint isomorphism $\text{Hom}_{\mathcal{Q}}(e^Q(I), E') \cong \text{Hom}_{R}(I, E'_v)$, for any integer $i$, gives the desired result.

Let then $\mathcal{Q}$ be right rooted. By transfinite induction on $\mu(\mathcal{Q})$, assume first that $V_{\mu(\mathcal{Q})} = \emptyset$. Then by Lemma 3.1.4, for any ordinal number $\delta < \mu(\mathcal{Q})$, $k^Q_\delta(E) \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$. It is not difficult to verify that $Q^\delta$ satisfies statement $\dagger$. So by Lemma 3.1.1, $k^Q(E) \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$. Induction hypothesis then implies that for any $v \in V(Q^\delta)$, $k^Q(E)_v \in K_{\text{tac}}(\text{Inj}R)$. Note that for any such $v$, $k^Q(E)_v = E_v$.

Suppose next that $V_{\mu(\mathcal{Q})} \neq \emptyset$. An argument similar to that above yields that for any $v \in \bigcup_{\gamma < \mu(\mathcal{Q})} V_{\gamma}$, $E_v \in K_{\text{tac}}(\text{Inj}R)$. Hence $\prod_{\gamma \in \mu(\mathcal{Q})} k^Q_{\delta(\gamma)}(E_{\delta(\gamma)}) \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$. In fact, this happens because the vertices of $V_{\mu(\mathcal{Q})}$ do not appear in this product. Therefore the exact sequence (3) applied for $E$ gives that $\prod_{\gamma \in \mu(\mathcal{Q})} k^Q_{\delta(\gamma)}(E_v) \in K_{\text{tac}}(\text{Inj}\mathcal{Q})$. This has $k^Q_{\delta}(E_v)$, for any $v \in V_{\mu(\mathcal{Q})}$, as a direct summand. The proof is completed by applying Remark 3.1.2.

To prove (ii) notice that since $R$ is left perfect and right coherent, then by [6], $\text{Prj}(R)$ is closed under taking arbitrary products. Then dualize the argument of (i).

\[ \square \]

3.2. Existence of adjoint functors. Let $\mathcal{Q}$ be a quiver. As mentioned in the introduction, regardless of its own significance, the existence of a right adjoint to the inclusion $K_{\text{tac}}(\text{Prj}\mathcal{Q}) \hookrightarrow K(\text{Prj}\mathcal{Q})$ is of particular importance because it directly gives the existence of Gorenstein projective precovers. In the sequel we will apply the results of the previous subsection to deduce the existence of such an adjoint.

Prior to this, let us write down a few lines concerning the existence of a left adjoint to the inclusion $K_{\text{tac}}(\text{Inj}\mathcal{Q}) \hookrightarrow K(\text{Inj}\mathcal{Q})$. The results in the preceding subsection seem, at the first
Let us briefly refine the arguments given in [2]. We just sketch a proof since it follows the lines of [2, Theorem 3.5]. For any vertex \( v \), let \( k^v_\lambda \) be the left adjoint to the restriction functor \( k^v : \mathcal{K}(Q) \to \mathcal{K}(R) \). Since it preserves quasi-isomorphisms, it generalizes to a functor \( k^v : \mathbb{D}(R) \to \mathbb{D}(Q) \). Let then \( S \) be a compact generating set for \( \mathbb{D}(R) \) (Note that for any ring \( R \), \( \mathbb{D}(R) \) is compactly generated). It follows, from definitions, that for any \( M \in S \), \( k^v_\lambda(M) \) is compact. On the other hand, a complex \( X \) of representations is exact if and only if \( X_v \) is exact for all vertex \( v \). Hence the set
\[
S' = \{ k^v_\lambda(M) \mid M \in S, v \in V \}
\]
is a compact generating set for \( \mathbb{D}(Q) \).

For the sake of completeness of this discussion, we include the following corollary which may be proved using a standard argument.

**Corollary 3.2.2.** Let \( Q \) be a noetherian quiver. Then every representation of \( Q \) has a Gorenstein injective preenvelope.

Let us come back to the inclusion \( \mathbb{K}_{\text{dac}}(\text{Prj}Q) \to \mathbb{K}(\text{Prj}Q) \). The following theorem due to Margolis will be useful. We note that it has been proved originally in the category of spectra; see [18, § 7] and also [17, § 6] for its proof. Recall also that if \( T \) is a triangulated category with suspension functor \( \Sigma \), an additive functor \( H : T \to \text{Ab} \) is said to be homological provided it sends triangles in \( T \) to long exact sequences in \( \text{Ab} \), where \( \text{Ab} \) denotes the category of Abelian groups. The kernel of \( H \) is then the full subcategory of \( T \) consisting of all objects \( X \) such that \( H(\Sigma^n(X)) = 0 \), for all integers \( n \).

**Theorem 3.2.3.** Let \( T \) be a compactly generated triangulated category. If \( H \) is a coproduct-preserving homological functor \( T \to \text{Ab} \), the inclusion \( \ker(H) \to T \) admits a right adjoint.

**Lemma 3.2.4.** Let \( Q \) be an arbitrary quiver and \( R \) be a ring such that the homotopy category \( \mathbb{K}(\text{Prj}R) \) is compactly generated. Then \( \mathbb{K}(\text{Prj}Q) \) will also be compactly generated.

**Proof.** We briefly refine the arguments given in [2]. Let \( P \in \mathbb{K}(\text{Prj}Q) \) be such that for any vertex \( v \) the complex \( P_v \), as an element in \( \mathbb{K}(\text{Prj}R) \), is contractible. Deduce that the complexes \( \bigoplus_a k^{(a)}_\lambda(P_{v(a)}) \) and \( \bigoplus_v k^v_\lambda(P_v) \) are contractible. Furthermore, since \( P \in \mathbb{K}(\text{Prj}Q) \), the exact sequence (4), applied for \( P \), splits in each degree showing that \( P \) itself, is contractible. Conversely, if \( P \in \mathbb{K}(\text{Prj}Q) \) is contractible, it follows just from definition that \( P \) is contractible in each vertex. Now if \( S \) is a compact generating set for \( \mathbb{K}(\text{Prj}R) \), apply the argument given in [2, Theorem 3.5] to prove that the set \( \{ k^v_\lambda(M) \mid M \in S, v \in V \} \) will do the job. \( \square \)
So, the main theorem of this subsection is

**Theorem 3.2.5.** Let $R$ be a commutative ring and $Q$ be a quiver. Suppose either of the following situations is satisfied.

(i) $R$ is noetherian of finite Krull dimension and $Q$ is left rooted.

(ii) $R$ is artinian.

Then the inclusion $\mathbb{K}_{tac}(\text{Prj}Q) \longrightarrow \mathbb{K}(\text{Prj}Q)$ admits a right adjoint.

**Proof.** Assume (i) takes place. Set $I = \oplus_p E(R/p)$, where $p$ runs over all prime ideals of $R$, and let for any $X \in \mathbb{K}(\text{Prj}Q)$, $\Gamma(X) = H^0(X) \oplus H^0(X')$, where $H^0$ is the 0-th homology functor and for any vertex $v$ of $Q$, $X_v = X_v \otimes I$. Obviously this defines a homological functor $\Gamma : \mathbb{K}(\text{Prj}Q) \longrightarrow \text{Ab}$. Note that an object $X$ of $\mathbb{K}(\text{Prj}Q)$ belongs to the kernel of $\Gamma$ if and only if it is acyclic and for any $v$ and $i$, $H^i(X_v \otimes I) = 0$. But Lemma 4.3 of [20] implies that $X \in \ker \Gamma$ if and only if for any vertex $v$, $X_v$ is a totally acyclic complex of projective $R$-modules. Hence, by Theorems 3.1.3 and 3.1.6, $\ker \Gamma = \mathbb{K}_{tac}(\text{Prj}Q)$. One the other hand $\mathbb{K}(\text{Prj}R)$ is a compactly generated homotopy category, by [22, Proposition 7.14]. Thus we deduce from 3.2.4 that $\mathbb{K}(\text{Prj}Q)$ is also compactly generated. This, in conjunction with Theorem 3.2.3, gives the required result.

If the hypothesis of (ii) is satisfied, then $\mathbb{K}(\text{Prj}R)$ is again compactly generated. On the other hand, it is known that artinian rings are perfect, see e.g. [1, Corollary 28.8]. Then iterate the above argument along with Theorems 3.1.3 and 3.1.6.

Now the corollary below follows by applying the argument given by Jørgensen in [15].

**Corollary 3.2.6.** Let $R$ and $Q$ satisfy either (i) or (ii) of Theorem 3.2.5. The class of Gorenstein projective representations is then precovering in $\text{Rep}(Q, R)$. In particular, if $R$ is commutative noetherian of finite Krull dimension and $Q$ is finite, then every $RQ$-module has a Gorenstein projective precover.

### 3.3. Gorensteinness of $\text{Rep}(Q, R)$

In this section, we study total acyclicity in connection with the Gorenstein property. More precisely, we consider the situation in which every acyclic complex of projective (or injective) representations of certain quivers happens to be totally acyclic and show that this, beautifully, describes Gorensteinness of the category $\text{Rep}(Q, R)$, hence generalizing a theorem by Iyengar and Krause [14] and another one by Murfet and Salarian [20] to the category of representations of a quiver. First we need to recall some facts about Gorenstein categories.

Assume that $\mathcal{A}$ is a Grothendieck category with enough projectives. We say that an object $X$ of $\mathcal{A}$ is of finite Gorenstein projective (Gorenstein injective) dimension, and indicate it by $\text{Gpd}_\mathcal{A} X < \infty$ ($\text{Gid}_\mathcal{A} X < \infty$), if there exists a projective (injective) resolution of $X$ with a Gorenstein projective (Gorenstein injective) syzygy (cosyzygy). $\mathcal{A}$ is called a Gorenstein category if there exists an integer $n$ such that for any object $X$ of $\mathcal{A}$, $\text{Gpd}_\mathcal{A} X < n$ and $\text{Gid}_\mathcal{A} X < n$. Note that this definition has been motivated by [10, Theorem 2.28].

Let $\mathcal{A}$ be a Grothendieck category with enough projectives. We will need the following homological dimensions of $\mathcal{A}$.

$$\text{FID}(\mathcal{A}) = \sup \{ \text{id}(X) \mid X \text{ is an object of } \mathcal{A} \text{ and } \text{id}(X) < \infty \},$$

$$\text{gGid}(\mathcal{A}) = \sup \{ \text{Gid}(X) \mid X \text{ is an object of } \mathcal{A} \},$$

where $\text{id}(X)$ is the injective dimension of $X$. 
The notions of FPD(\(\mathcal{A}\)) and glGpd(\(\mathcal{A}\)) are defined similarly. It is known (see e.g. [10, Theorem 2.28]) that if the category \(\mathcal{A}\) is Gorenstein, then all these dimensions coincide and their common value is called the global dimension \(\text{gld}(\mathcal{A})\) of \(\mathcal{A}\).

Recall that it follows from [9, Proposition 6.5] that if \(Q\) is a right rooted quiver, then \(\text{FID}(\text{Rep}(Q,R)) \leq \text{FID}(R) + 1\). This can be extended to arbitrary quivers.

**Proposition 3.3.1.** Let \(R\) be a ring and \(Q\) be an arbitrary quiver. Then

\[\text{FID}(\text{Rep}(Q,R)) \leq \text{FID}(R) + 1.\]

The equality holds if \(Q\) is not a discrete quiver, that is, has at least one arrow. A similar statement is also true for FPD.

**Proof.** If \(\text{FID}(R)\) is infinite, then the first part is clear; so let \(\text{FID}(R) = n < \infty\). Then, for any representation \(\mathcal{X}\) of \(Q\) with finite injective dimension, the argument given in [9, Proposition 6.5] gives an exact sequence

\[0 \to \mathcal{X} \to \mathcal{E}^0 \to \cdots \to \mathcal{E}^{n-1} \to \mathcal{E}^n \to 0\]

such that for \(0 \leq i \leq n - 1\), \(\mathcal{E}^i\) is injective and \(\mathcal{E}_v^n\) is an injective \(R\)-module for any \(v\). Thus it follows from Lemma 3.1.5 that \(\text{id}(\mathcal{E}^n) \leq 1\); i.e. \(\text{id}(\mathcal{X}) \leq n + 1\).

Let then \(Q\) be non-discrete and \(v\) be a vertex which has at least one incoming arrow \(b\). If \(\text{FID}(\text{Rep}(Q,R))\) is infinite, then the desired equality holds trivially. So let \(\text{FID}(\text{Rep}(Q,R))\) be finite and \(C\) be an \(R\)-module of finite injective dimension. Then the constant representation \(C\), in which all the vertices are represented by \(C\) and the arrows are represented by the identity, also has finite injective dimension. Thus, by the assumption, \(\text{id}(C) \leq \text{FID}(\text{Rep}(Q,R))\) which implies \(\text{id}(C) \leq \text{FID}(\text{Rep}(Q,R))\), that is, \(\text{FID}(R)\) should be finite. So let \(0 \leq \text{FID}(R) = n < \infty\) and take an \(R\)-module \(M\) with \(\text{id}(M) = n\). We will show that \(\text{id}(S_v(M)) = n + 1\) where \(S_v(M)\) is the representation of \(Q\) in which the vertex \(v\) is represented by \(M\) and all other ones are represented by zero. Assume to the contrary that \(\text{id}(S_v(M)) \leq n\) and let

\[0 \to S_v(M) \to \mathcal{E}^0 \to \cdots \to \mathcal{E}^n \to 0\]

be an injective resolution. If \(s(b) = w\), then we get the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & \mathcal{E}_w^0 & \to & \cdots & \to & \mathcal{E}_w^{n-1} & \to & \mathcal{E}_w^n & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & \prod_{s(a) = w} \mathcal{E}_{t(a)}^0 & \to & \cdots & \to & \prod_{s(a) = w} \mathcal{E}_{t(a)}^{n-1} & \to & \prod_{s(a) = w} \mathcal{E}_{t(a)}^n & \to & 0
\end{array}
\]

where \(a\) stands for an arrow. Use Theorem 2.2 to observe that the map \(\alpha\) is a split epimorphism. Therefore \(\text{id}(M) < n\) which is a contradiction. \(\square\)

We want to show that Gorensteinness may carry over from \(\text{Mod}R\) to the category of representations of a quiver. First we need the following lemma.

**Lemma 3.3.2.** Let \(M\) be an \(R\)-module and \(Q\) be a quiver. If \(M\) is a Gorenstein injective (Gorenstein projective) \(R\)-module then for any vertex \(v\), \(e_v^\rho(M)\) (\(e_v^\rho(M)\)) is a Gorenstein injective (Gorenstein projective) representation of \(Q\).

**Proof.** Let \(E \in \mathbb{K}_{\text{tac}}(\text{Inj}R)\) be such that \(M = \text{Ker}E\). Then by Remark 3.1.2, \(k^\rho(E) \in \mathbb{K}_{\text{tac}}(\text{Inj}Q)\) and \(e_v^\rho(M) = \text{Ker}E_v^\rho(E)\). Hence \(e_v^\rho(M)\) is Gorenstein injective. The other claim is similar. \(\square\)
Theorem 3.3.3. Let $R$ be a ring and $Q$ be a non-discrete quiver. If $\text{Mod}R$ is a Gorenstein category then so is $\text{Rep}(Q, R)$. If this is the case, then $\text{gld}(\text{Rep}(Q, R)) = \text{gld}(R) + 1$.

Proof. Use Lemma 3.3.2 to adapt the proof of Proposition 3.3.1. Note that the sequence $(1)$ implies $\text{Gid}(X) \leq 1$ whenever $X$ is a representation of $Q$ with $X_v$ Gorenstein injective $R$-module for all $v$. 

Lemma 3.3.4. Let $R$ be a commutative noetherian ring of finite Krull dimension. Then $\mathbb{K}_{\text{ac}}(\text{Inj}R) = \mathbb{K}_{\text{tac}}(\text{Inj}R)$ if and only if $R$ is Gorenstein.

Proof. $(\Rightarrow)$ Let $E$ be an injective cogenerator of $\text{Mod}R$ and $P \in \mathbb{K}_{\text{ac}}(\text{Prj}R)$. Then $\text{Hom}_R(P, E)$ belongs to $\mathbb{K}_{\text{ac}}(\text{Inj}R) = \mathbb{K}_{\text{tac}}(\text{Inj}R)$. Therefore $\text{Hom}_R(I, \text{Hom}_R(P, E))$ is exact for any injective $R$-module $I$ and so $P \otimes_R I$ is exact. Hence $P \in \mathbb{K}_{\text{tac}}(\text{Prj}R)$ by [20, Lemma 4.21]. Then apply [20, Corollary 4.32] to see that $R$ is Gorenstein. The other direction follows from [14, Corollary 5.5].

The main theorem of this subsection is the following.

Theorem 3.3.5. Let $R$ be a commutative noetherian ring of finite Krull dimension and $Q$ be a noetherian quiver. Consider the following statements.

(i) $R$ is Gorenstein.
(ii) $\mathbb{K}_{\text{ac}}(\text{Inj}Q) = \mathbb{K}_{\text{tac}}(\text{Inj}Q)$.
(iii) $\mathbb{K}_{\text{ac}}(\text{Prj}Q) = \mathbb{K}_{\text{tac}}(\text{Prj}Q)$.
(iv) $\text{Rep}(Q, R)$ is a Gorenstein category.

Then $(i) \iff (ii) \iff (iv)$. Moreover if $R$ is artinian or $Q$ is left rooted, then all the statements are equivalent.

Proof. We know from the previous subsection that the above categories are indeed triangulated. Suppose that $\text{Rep}(Q, R)$ is a Gorenstein category and take the representation $e^*_v(R)$ for some vertex $v$ of $Q$. Since it is a projective representation, it is of finite injective dimension. In particular, $R$, as a direct summand of $\bigoplus_{Q(v,v)} R = (e^*_v(R))_v$, has finite injective dimension. That is, $R$ is a Gorenstein ring. Observe also that if $E \in \mathbb{K}_{\text{ac}}(\text{Inj}R)$ is such that $k^v(E) \in \mathbb{K}_{\text{tac}}(\text{Inj}Q)$ for some $v$, then $E$ is a direct summand of $k^v(E)_v$ and therefore will be totally acyclic by 3.1.6. In view of these easy observations, combine Theorems 3.1.3, 3.1.6, 3.3.3, and Lemma 3.3.4 to get the first assertion. To prove the second assertion use, in addition, [20, Corollary 4.32].

3.4 Virtually Gorensteinness of $\text{Rep}(Q, R)$. Let us recall that the notion of a virtually Gorenstein algebra was first defined in [5] as a generalization of the notion of Gorenstein algebras. It is known [3] that over any artin algebra $\Lambda$, $\text{Gp}(\Lambda)^{\perp} = \text{Gi}(\Lambda)$, where $\text{Gp}(\Lambda)$ and $\text{Gi}(\Lambda)$ denote the subcategories of finitely generated Gorenstein projective and Gorenstein injective modules respectively and the symbol $^{\perp}$ refers to the Ext$^1_{\Lambda}$-orthogonal classes. However, there are examples of artin algebras for which $\text{GP}(\Lambda)^{\perp} \neq \text{GI}(\Lambda)$ where $\text{GP}(\Lambda)$ and $\text{GI}(\Lambda)$ are the subcategories of Gorenstein projective and Gorenstein injective modules; see [4]. Therefore, by definition, an artin algebra $\Lambda$ is said to be virtually Gorenstein provided $\text{GP}(\Lambda)^{\perp} = \text{GI}(\Lambda)$.

Let then $R$ be a ring and $Q$ be a quiver. Inspired by the above definition, we say that $\text{Rep}(Q, R)$ is virtually Gorenstein provided $\text{GP}(Q)^{\perp} = \text{GI}(Q)$, where $\text{GP}(Q)$ and $\text{GI}(Q)$
Let denote respectively the subcategories of Gorenstein projective and Gorenstein injective representations of $Q$. The purpose of this subsection is to check if virtually Gorensteinness carries over from $R$ to $\text{Rep}(Q,R)$. The following proposition is fundamental.

**Proposition 3.4.1.** Let $R$ be a ring, $Q$ be a quiver and $P \in \mathbb{K}(Q)$. Then

(i) If $R$ is left perfect and right coherent or $Q$ is left rooted, then $P \in \mathbb{K}_{\text{tac}}(\text{Prj}Q)^{\perp}$ if and only if for any vertex $v$, $P_v \in \mathbb{K}_{\text{tac}}(\text{Prj}R)^{\perp}$.

(ii) If $R$ is noetherian or $Q$ is right rooted, then $P \in \mathbb{K}_{\text{tac}}(\text{Inj}Q)$ if and only if for any vertex $v$, $P_v \in \mathbb{K}_{\text{tac}}(\text{Inj}R)$.

**Proof.** (i) Let $P \in \mathbb{K}_{\text{tac}}(\text{Prj}Q)^{\perp}$ and $Y \in \mathbb{K}_{\text{tac}}(\text{Prj}R)$. Then by Remark 3.1.2 for any vertex $v$, $k_v^X(Y) \in \mathbb{K}_{\text{tac}}(\text{Prj}Q)$ and we have

$$0 = \text{Hom}_{\mathbb{K}(Q)}(k_v^X(Y), P) = \text{Hom}_{\mathbb{K}(R)}(Y, P_v).$$

Conversely, if for any $v$, $P_v \in \mathbb{K}_{\text{tac}}(\text{Prj}R)^{\perp}$ and $X \in \mathbb{K}_{\text{tac}}(\text{Prj}Q)$ we infer from Theorem 3.1.6 that for any vertex $w$,

$$0 = \text{Hom}_{\mathbb{K}(R)}(X_v, P_w) = \text{Hom}_{\mathbb{K}(Q)}(k_w^X(X_v), P).$$

Now the exact sequence (4), when applied for $X$, splits in each degree so that it fits into a triangle

$$\bigoplus_a k_v^{t(a)}(X_{s(a)}) \longrightarrow \bigoplus_a k_v^X(X_v) \longrightarrow X \rightarrow$$

in $\mathbb{K}(\text{Prj}Q)$. One may complete the proof by applying the homological functor $\text{Hom}_{\mathbb{K}(\text{Prj}Q)}(-, P)$ to this triangle. Part (ii) can be proved similarly. 

**Corollary 3.4.2.** Let $R$ be a ring, $Q$ be a quiver and $M \in \text{Rep}(Q, R)$.

(a) If $R$ is left perfect and right coherent or $Q$ is left rooted, then $M \in \text{GP}(Q)^{\perp}$ if and only if for any vertex $v$, $\mathcal{M}_v \in \text{GP}(R)^{\perp}$.

(b) If $R$ is noetherian or $Q$ is right rooted, then $M \in \text{GI}(Q)$ if and only if for any vertex $v$, $\mathcal{M}_v \in \text{GI}(R)$.

The following theorem is a direct consequence of Corollary 3.4.2 and the definition of virtually Gorenstein artin algebras given above.

**Theorem 3.4.3.** Assume that $\Lambda$ is a virtually Gorenstein artin algebra and that $Q$ is an arbitrary quiver. Then $\text{Rep}(Q, \Lambda)$ is virtually Gorenstein. In particular if $Q$ is finite and has no oriented cycle, then $\Lambda Q$ is a virtually Gorenstein artin algebra.

#### 3.5. Gorenstein Representations

Gorenstein injective (Gorenstein projective) representations of quivers have been classified in [9] in case the base ring $R$ is Gorenstein. Therefore it would be of interest to know if this classification is valid without any assumption on $R$. Our characterization of totally acyclic complexes then provides a useful tool.

**Theorem 3.5.1.** Let $Q$ be a quiver and $M \in \text{Rep}(Q, R)$.

(a) If $Q$ is right rooted, then $M$ is Gorenstein injective if and only if for any vertex $v$,

(i) $\mathcal{M}_v$ is a Gorenstein injective $R$-module.

(ii) The $R$-homomorphism $f_v : \mathcal{M}_v \to \prod_{s(a)=v} \mathcal{M}_{t(a)}$ is an epimorphism whose kernel is Gorenstein injective.

(b) If $Q$ is left rooted, then $M$ is Gorenstein projective if and only if for any vertex $v$,
(i) $\mathcal{M}_v$ is a Gorenstein projective $R$-module.
(ii) The $R$-homomorphism $g_v : \oplus_{t(a)=v} \mathcal{M}_{s(a)} \to \mathcal{M}_v$ is a monomorphism whose cokernel is Gorenstein injective.

Proof. Let us prove statement (a); statement (b) is similar. Assume $\mathcal{M}$ is Gorenstein injective. There exists an epimorphism $\mathcal{X} \to \mathcal{M} \to 0$ with $\mathcal{X}$ injective which leads in the commutative diagram

$$
\begin{array}{ccc}
\mathcal{X}_v & \xrightarrow{\varphi_v} & \mathcal{M}_v \\
\downarrow \psi_v & & \downarrow \\
\prod_{s(a)=v} \mathcal{X}_{t(a)} & \xrightarrow{} & \prod_{s(a)=v} \mathcal{M}_{t(a)} \\
\end{array}
$$

Hence $f_v$ is an epimorphism by Theorem 2.2. To prove (i), take $E \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{Q})$ such that $\mathcal{M} = \ker \partial_E^0$. By Theorem 3.1.6, for any $v$, $E_v \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{R})$. Hence $\mathcal{M}_v$ is Gorenstein injective since $\mathcal{M}_v = \ker \partial_E^0$. On the other hand, for any $v$, there exists a degree-wise split, exact sequence

$$0 \to E^v \to E_v \to \prod_{s(a)=v} E_{t(a)} \to 0$$

which implies $E^v \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{Q})$. Finally the second part of (ii) holds true because $f_v = \ker \partial_0^{E^w}$.

For the converse, we use transfinite induction to construct complexes $I^\alpha \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{Q}^\alpha)$, for any ordinal $\alpha \leq \mu(\mathcal{Q})$, satisfying the following conditions.

1. For any $v \in V(\mathcal{Q}^\alpha)$, $I^\alpha_v \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{R})$.
2. For any $v \in V(\mathcal{Q}^\alpha)$, $\mathcal{M}_v = \ker \partial_0^{I^\alpha_v}$.
3. $\forall \beta < \alpha \leq \mu(\mathcal{Q})$, $\forall v \in V(\mathcal{Q}^\beta)$, $I^\beta_v = I^\alpha_v$.

By statement (i), for any $v \in V_1$, there exists $I^v \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{R})$ such that $\mathcal{M}_v = \ker \partial_0^{I^v}$. Then from definition of $V_1$ we may construct $I^1 \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{Q}^1)$ as required. Let $\alpha$ be an ordinal and $w \in V_\alpha$. Then for any arrow $a$ starting from $w$, there exists an ordinal $\beta < \alpha$ such that $t(a) \in V_\beta$. By assumption, there exists an exact sequence $0 \to \ker f_w \to \mathcal{M}_w \to \prod_{s(a)=w} \mathcal{M}_{t(a)} \to 0$. Hence from condition (ii) and the induction hypothesis, we may construct a complex $I^w \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{R})$ such that $\mathcal{M}_w = \ker \partial_0^{I^w}$. In fact $I^w$ in degree $j$ is just the direct sum of the modules which appear in degree $j$ of the complexes corresponding to the end terms of this sequence. Set $I^w_0 = I^w$ and use the natural projections to represent the arrows of $\mathcal{Q}^\alpha$. Having done this procedure, one obtains an exact complex $I$ of injective representations such that for any vertex $v$, $I_v \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{R})$ and $\mathcal{M} = \ker \partial_0^I$. Theorem 3.1.3 then implies $I \in \mathbb{K}_{\text{tac}}(\text{Inj} \mathcal{Q})$ so $\mathcal{M}$ is Gorenstein injective.

Example 3.5.2. It is worth noting that in view of Theorem 3.1.6, the 'only if' direction of the first (second) part of Theorem 3.5.1 holds true for arbitrary quivers when one chooses $R$ to be noetherian (left perfect and right coherent). However, we provide an example showing that the 'if' part is not true even if the base ring is good enough. Let $k$ be a field and $\mathcal{Q}$ be the loop quiver. Then $\text{Rep}(\mathcal{Q}, k) \simeq k[x] - \text{Mod}$. Denote by $\mathcal{L}$ the representation $k \rightarrow k$ which corresponds to $k$ as a trivial $k[x]$-module. Clearly $\mathcal{L}$ satisfies the local descriptions of Theorem 3.5.1. But it is neither Gorenstein projective nor Gorenstein injective. For, the sequences (1) and (2) show that $k$, when considered as a representation, has finite injective and projective...
dimension. Therefore it is Gorenstein projective (Gorenstein injective) if and only if it is projective (injective). But clearly $k$ is neither projective nor injective as a $k[x]$-module.

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