VIRTUAL GORENSTEINNESS OVER GROUP ALGEBRAS

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Abstract. Let $\Gamma$ be a finite group and $\Lambda$ be any artin algebra. It is shown that the group algebra $\Lambda \Gamma$ is virtually Gorenstein if and only if $\Lambda \Gamma'$ is virtually Gorenstein, for all elementary abelian subgroups $\Gamma'$ of $\Gamma$. Moreover, assume that $\Gamma'$ is a subgroup of $\Gamma$ such that the triple $(\Gamma, \Gamma', \Lambda)$, satisfies Moore’s condition, then $\Lambda \Gamma$ is virtually Gorenstein whenever $\Lambda \Gamma'$ is so.

1. Introduction

Let $\Lambda$ be an artin algebra. The algebra $\Lambda$ is called left (resp. right) Gorenstein if $\Lambda$ viewed as a left (resp. right) module has finite injective dimension. Note that it is an open problem whether or not a left Gorenstein algebra is right Gorenstein. $\Lambda$ is called Gorenstein if it is both left and right Gorenstein. The problem to understand the Gorenstein left-right symmetry, which is referred as Gorenstein symmetry Conjecture (see [6, Conjecture (13)]), provided a motivation for studying the class of virtually Gorenstein algebras which has been introduced in [13]. We recall from [11] that an algebra $\Lambda$ is said to be virtually Gorenstein if for every $\Lambda$-module $X$, the functor $\text{Ext}^i_{\Lambda}(X, -)$ vanishes for all $i > 0$ on all Gorenstein injective $\Lambda$-modules if and only if $\text{Ext}^i_{\Lambda}(-, X)$ vanishes for all $i > 0$ on all Gorenstein projective $\Lambda$-modules. It is known that if $\Lambda$ is virtually Gorenstein, then the Gorenstein Symmetry Conjecture is true for $\Lambda$; see [12, Theorem 11.4]. Virtually Gorenstein algebras provide a natural enlargement of the class of Gorenstein algebras giving at the same time a homological generalization of algebras of finite representation type and more generally of algebras of finite Cohen-Macaulay type. We would like to stress that all artin algebras are “locally”, that is, at the finitely generated level, virtually Gorenstein [12]. However, in [11] an example of artin algebra which is not virtually Gorenstein is presented. The main result of [11] provides a remarkable characterization of virtually Gorenstein algebras in terms of finitely generated modules. Precisely, it is shown in [11, Theorem 1] that $\Lambda$ being virtually Gorenstein is equivalent to saying that $\text{Thick}(\text{proj} \Lambda \cup \text{inj} \Lambda)$, the smallest thick subcategory of $\text{mod} \Lambda$ containing $\text{proj} \Lambda \cup \text{inj} \Lambda$, is functorially finite (i.e. both contravariantly and covariantly finite). Here, $\text{mod} \Lambda$ denotes the class of all finitely generated (left) $\Lambda$-modules, $\text{proj} \Lambda$ (resp. $\text{inj} \Lambda$) denotes the full subcategory of $\text{mod} \Lambda$ which consists of all projective (resp. injective) $\Lambda$-modules. As a corollary of this result, one immediately deduces that virtual Gorensteinness is left-right symmetric; see also [12, Theorem 8.7].

Elementary abelian subgroup induction plays a crucial role in cohomology and representation theory of finite groups (see [3], [14], [16], [17], [27]). Roughly speaking, the results say that important cohomological properties hold for a group ring $RG$, $\Gamma$ finite
and $R$ an arbitrary ring with identity, if and only if they hold for $RT'$ where $T'$ runs over all elementary abelian subgroups of $T$. It is shown in [17] that if $M$ is any module over $RT'$ then it is weakly projective (projective) if and only if it is weakly projective (projective) over all subrings $RT'$ where $T'$ runs over all elementary abelian subgroups of $T$. Moreover, it is known that if $M$ is any module over an arbitrary $RT$-module, $T$ is finite and $R$ is an arbitrary ring, then a given element $x$ of the cohomology ring $\text{Ext}_{RT}(M, M)$ (with Yoneda’s product) is nilpotent if and only if its restriction to $\text{Ext}_{RT'}(M, M)$ is nilpotent where $T'$ runs over all elementary abelian subgroups of $T$; see [16] and [26]. These important results exhibit the role of the elementary abelian subgroups. In this direction, we investigate virtual Gorensteinness over group algebras. Indeed, it is shown that virtual Gorensteinness over $T$ can be determined by its elementary abelian subgroups. Precisely, our main result in this context is as follows:

**Theorem 1.1.** Let $T$ be a finite group and $\Lambda$ be any artin algebra. Then $\Lambda T$ is a virtually Gorenstein algebra if and only if $\Lambda T'$ is virtually Gorenstein for every elementary abelian subgroup $T'$ of $T$.

Our second task in this paper is to generalize the above theorem to infinite groups. Let $R$ be an associative ring with identity. Inspired by the definition of virtually Gorenstein algebra, we say that $R$ is a virtually Gorenstein ring provided $\text{GP}(R)^\perp = \perp \text{GI}(R)$, where $\text{GP}(R)$ and $\text{GI}(R)$ denote the subcategories of Gorenstein projective and Gorenstein injective modules respectively and the symbol $\perp$ refers to the $\text{Ext}_R^1$-orthogonal classes. We study the descent and ascent of virtual Gorensteinness between $T$ and its subgroups of finite index. It is proved that virtual Gorensteinness carries over from $T$ to such subgroups, but the converse is true in certain circumstances. Actually, we establish the sequel result.

**Theorem 1.2.** Let $T$ be an $HF$-group, $T'$ be its subgroup of finite index and $R$ be any ring with identity. Assume that the triple $(T, T', R)$ satisfies Moore’s condition, that is, for all $x \in (T - T')$, at least one of the following holds:

1. there is an integer $n$ such that $1 \neq x^n \in T'$.
2. $\text{ord}(x)$ is finite and invertible in $R$.

Moreover, assume that any Gorenstein projective (resp. Gorenstein injective) $RT$-module is also Gorenstein projective (resp. Gorenstein injective) over $R$. Then, $RT$ is a virtually Gorenstein ring if and only if so is $RT'$.

Restricting the above theorem to finite groups, yields the following.

**Corollary 1.3.** Let $T$ be a finite group, $T'$ be its subgroup and $\Lambda$ be any artin algebra. Assume that the triple $(T, T', \Lambda)$ satisfies Moore’s condition. Then, $\Lambda T$ is a virtually Gorenstein algebra if and only if so is $\Lambda T'$.

Throughout the paper, $T$ is a group, $R$ is an associative ring with identity and $RT$ is the group algebra (of $T$ over $R$), in fact $RT$ is the ring $R \otimes \mathbb{Z}T$. We also fix that $\Lambda$ is an artin algebra. If $T$ assumed to be finite, then $\Lambda T$ will be an artin algebra. All modules are supposed to be left modules unless otherwise stated. Also, $\text{pd}_{\mathbb{Z}T}M$ stands for the projective dimension of a module $M$ over a group ring $\mathbb{Z}T$.

2. Preliminaries

In this section, we recall basic definitions and fundamental facts for later use.
2.1. Orthogonal classes. Let \( \mathcal{X} \) be a class of objects in \( \text{Mod} R\Gamma \), the category of all left \( R\Gamma \)-modules. The left orthogonal of \( \mathcal{X} \) in \( \text{Mod} R\Gamma \), denoted by \( \perp \mathcal{X} \), is defined by
\[
\perp \mathcal{X} = \{ M \in \text{Mod} R\Gamma \mid \text{Ext}^i_R(M, X) = 0, \text{ for all } X \in \mathcal{X} \text{ and all } i > 0 \}.
\]
The right orthogonal of \( \mathcal{X} \) in \( \text{Mod} R\Gamma \) is defined similarly.

2.2. Let \( \Gamma \) be an arbitrary group and \( \Gamma' \) be a subgroup of \( \Gamma \). Since \( R\Gamma \) is a free \( R\Gamma' \)-module, any projective \( R\Gamma \)-module is also projective over \( R\Gamma' \). Consequently, for any \( R\Gamma \)-module \( M \), one has the inequality \( \text{pd}_{R\Gamma'} M \leq \text{pd}_{R\Gamma} M \). Moreover, analogues to [15, Proposition VIII 2.4 (a)], one may show that the equality holds if \( \text{pd}_{R\Gamma} M < \infty \) and \( \Gamma' \) is of finite index in \( \Gamma \).

2.3. Let \( \Gamma \) be a group and \( \Gamma' \) be a subgroup of \( \Gamma \) of finite index. Let \( M \) be a left \( R\Gamma' \)-module. Then a verbatim pursuit of the proof of [15, Proposition III 5.9], gives rise the isomorphism \( R\Gamma \otimes_{R\Gamma'} M \cong \text{Hom}_{R\Gamma'}(R\Gamma, M) \), as left \( R\Gamma \)-modules. One should note that the left hand side is a left \( R\Gamma \)-module, since \( R\Gamma \) is an \( R\Gamma \)-\( R\Gamma' \)-bimodule. However, the case for the right hand side follows from \( R\Gamma' \)-\( R\Gamma \)-bimodule structure of \( R\Gamma \).

2.4. Gorenstein modules. An \( R\Gamma \)-module \( M \) is said to be Gorenstein projective if it is a syzygy of some exact sequence of projective \( R\Gamma \)-modules
\[
\cdots \rightarrow T_2 \rightarrow T_1 \rightarrow T_0 \rightarrow T_{-1} \rightarrow \cdots
\]
which remains exact after applying the functor \( \text{Hom}_{R\Gamma}(\cdot, P) \), for any projective \( R\Gamma \)-module \( P \). The exact sequence \( T_\bullet \) is called a totally acyclic complex of projectives. Gorenstein injective modules are defined dually. The class of all Gorenstein projective and Gorenstein injective \( R\Gamma \)-modules will be denoted by \( \text{GP}(R\Gamma) \) and \( \text{GI}(R\Gamma) \), respectively. The reader is advised to look at [20] for the basic properties of these modules.

Remark 2.5. Gorenstein projective modules, which are a refinement of projective modules, were defined by Enochs and Jenda in [19]. This concept even goes back to Auslander and Bridger [4], who introduced the G-dimension of a finitely generated module \( M \) over a two-sided noetherian ring; and then Avramov, Martisinkovsky and Reiten proved that \( M \) is Gorenstein projective if and only if the G-dimension of \( M \) is zero (see also the remark following Theorem (4.2.6) in [18], for the historical information).

Example 2.6. The following are examples of Gorenstein projective and injective modules.

(i) Every projective (resp. injective) \( R\Gamma \)-module is Gorenstein projective (resp. injective).

(ii) Let \( \Gamma' \) be a subgroup of \( \Gamma \) and \( M \) be a Gorenstein projective (resp. injective) \( R\Gamma' \)-module. Then \( R\Gamma \otimes_{R\Gamma'} M \) (resp. \( \text{Hom}_{R\Gamma'}(R\Gamma, M) \)) is a Gorenstein projective (resp. injective) \( R\Gamma \)-module.

Proof. (i) The result follows from the definition.

(ii) To prove the assertion for Gorenstein projective module, one only needs to apply the same argument which have been used in [8, Example 2.1 (c)]. The case for Gorenstein injective modules can be obtained dually. \( \square \)

The class \( H_\mathfrak{F} \) was defined by Kropholler in [25] as the smallest class of groups which contains the class of finite groups and whenever a group \( \Gamma \) admits a finite dimensional contractible \( \Gamma \)-CW-complex with stabilizes in \( H_\mathfrak{F} \), then \( \Gamma \) is in \( H_\mathfrak{F} \). It is worth pointing
out that $H\mathfrak{F}$ is a very large class which is extension closed and contains all countable linear and countable soluble groups.

2.7. Let $\Gamma$ be a group, $\Gamma'$ be a subgroup of $\Gamma$ and $M$ be a Gorenstein projective $R\Gamma$-module. It is worth noting that, unlike the projectivity, we do not know that whether or not $M$ is Gorenstein projective as an $R\Gamma'$-module. In the next result, we impose mild assumptions on $M$ and $\Gamma'$ ensuring the Gorenstein projectivity of $M$ over $R\Gamma'$. We also point out that the same statement can be obtained dually for Gorenstein injective modules.

**Proposition 2.8.** Let $\Gamma$, $\Gamma'$ and $M$ be as above. Let, additionally, $\Gamma'$ be an $H\mathfrak{F}$-group and $M$ be Gorenstein projective as an $R\Gamma$-module. Then $M$ is a Gorenstein projective $R\Gamma'$-module.

**Proof.** The proof is actually follows the same lines in [8, Lemma 4.4]. In fact, the only thing which we need to check that $M$ is Gorenstein projective as an $R\Gamma'$-module. But, this indeed follows from Theorem 2.9 of [10]. The proof then is complete. $\square$

**Remark 2.9.** Let $\Gamma$ be a group and $\Gamma'$ be an arbitrary subgroup of $\Gamma$. Then by using the adjointness of Hom and $\otimes$ in conjunction with Example 2.6, one may deduce that $\text{GP}(R\Gamma) \perp \subseteq \text{GP}(R\Gamma') \perp$ and $\perp \text{GI}(R\Gamma) \subset \perp \text{GI}(R\Gamma')$.

3. RESULTS AND PROOFS

We begin this section by the following result which says that virtual Gorensteinness descends from a group $\Gamma$ to its subgroups of finite index.

**Proposition 3.1.** Let $\Gamma$ be a group and $\Gamma'$ be a subgroup of $\Gamma$ of finite index. If $R\Gamma$ is a virtually Gorenstein ring, then so is $R\Gamma'$.

**Proof.** Assume that $M$ is an $R\Gamma'$-module which belongs to $\text{GP}(R\Gamma') \perp$. It should be noted that, by virtue of [15, III 3.6], we infer that the functor $\text{Hom}_{R\Gamma'}(R\Gamma, -)$ is right adjoint to the restriction functor, and according to 2.3, it takes projective $R\Gamma'$-modules to projective $R\Gamma$-modules. These facts indeed yield that, every Gorenstein projective $R\Gamma$-module is also Gorenstein projective over $R\Gamma'$. Now take an arbitrary Gorenstein projective $R\Gamma$-module $X$. Using the adjoinness of Hom and $\otimes$ together with 2.3, gives rise to the following isomorphisms;

$$\text{Ext}^i_{R\Gamma}(X, R\Gamma \otimes_{R\Gamma'} M) \cong \text{Ext}^i_{R\Gamma'}(R\Gamma \otimes_{R\Gamma} X, M) \cong \text{Ext}^i_{R\Gamma'}(X, M),$$

implying that $R\Gamma \otimes_{R\Gamma'} M \in \text{GP}(R\Gamma') \perp$. Hence, the hypothesis made on $R\Gamma$ induces that $\text{Hom}_{R\Gamma'}(R\Gamma, M) \cong R\Gamma \otimes_{R\Gamma'} M \in \perp \text{GI}(R\Gamma')$. So, in order to show that $\text{GP}(R\Gamma') \perp \subset \perp \text{GI}(R\Gamma')$, it suffices to prove that for each $R\Gamma$-module $M \in \perp \text{GI}(R\Gamma')$, it also belongs to $\perp \text{GI}(R\Gamma')$ as well, because $M$ is an $R\Gamma'$-direct summand of $R\Gamma \otimes_{R\Gamma'} M$; see [15, III 3.4]. For this purpose, let $X$ be a Gorenstein injective $R\Gamma'$-module. Then for a given $R\Gamma$-module $M \in \text{GI}(R\Gamma')$, one has the following isomorphism;

$$\text{Ext}^i_{R\Gamma}(M, \text{Hom}_{R\Gamma'}(R\Gamma, X)) \cong \text{Ext}^i_{R\Gamma'}(R\Gamma \otimes_{R\Gamma} M, X) \cong \text{Ext}^i_{R\Gamma'}(M, X),$$

implying that for any $i > 0$, the left hand side vanishes, thanks to Example 2.6. Consequently, the right hand side also vanishes, as required. $\square$
Lemma 3.2. Let \( \Gamma \) be a group and \( \Gamma' \) be its subgroup of finite index such that the index of \( \Gamma' \) in \( \Gamma \) is invertible in \( R \). Assume that \( R\Gamma' \) is a virtually Gorenstein ring. Then so is \( R\Gamma \).

Proof. Suppose that \( M \in \text{GP}(R\Gamma)^\perp \) is arbitrary. Argue as in the proof of the above proposition to deduce that \( M \in \text{\perp GI}(R\Gamma') \) and so \( \text{Hom}_{R\Gamma'}(R\Gamma, M) \in \text{\perp GI}(R\Gamma) \). In addition, since the index of \( \Gamma' \) in \( \Gamma \) is invertible in \( R \), we infer that \( M \) is a \( R\Gamma \)-direct summand of \( \text{Hom}_{R\Gamma'}(R\Gamma, M) \). This, in particular, implies that \( M \in \text{\perp GI}(R\Gamma) \). The inclusion \( \text{\perp GI}(R\Gamma) \subseteq \text{GP}(R\Gamma)^\perp \) can be proved similarly. So we are done. \( \square \)

Remark 3.3. Let \( \Gamma \) be a finite group and \( \Lambda \) be a Gorenstein artin algebra. Since the functor \( \text{Hom}_{\Lambda}(\Lambda \Gamma, -) \) carries left (resp. right) injective \( \Lambda \)-modules to right (resp. left) injective \( \Lambda \)-modules, applying this functor to an injective resolution of \( \Lambda \) and using \( \Lambda \)-isomorphisms \( \text{Hom}_{\Lambda}(\Lambda \Gamma, \Lambda) \cong \Lambda \otimes_{\Lambda} \Lambda \cong \Lambda \), yields that the artin algebra \( \Lambda \Gamma \) is Gorenstein, as well. However, we do not know whether the same statement is true if one replaces Gorensteinness with virtual Gorensteinness. It follows from the above lemma that this is the case whenever the order of \( \Gamma \) is invertible in \( \Lambda \).

Let \( \Gamma \) be a finite group and \( p \) be a prime integer dividing the order of \( \Gamma \). Recall that a Sylow \( p \)-subgroup of \( \Gamma \) is a maximal \( p \)-subgroup of \( \Gamma \). It is known that such subgroups exist for every prime divisor of the order of \( \Gamma \) and in general, they are not unique (but any two such are conjugate). However, when \( \Gamma \) is abelian, Sylow \( p \)-subgroup is unique. By a Sylow subgroup, we mean a Sylow \( p \)-subgroup, for some prime integer \( p \) dividing the order of \( \Gamma \).

Theorem 3.4. Let \( \Gamma \) be a finite group and \( \Lambda \) be any artin algebra. The following statements are equivalent:

1. \( \Lambda \Gamma \) is a virtually Gorenstein algebra.
2. \( \Lambda \Gamma' \) is virtually Gorenstein for all Sylow subgroups \( \Gamma' \) of \( \Gamma \).

Proof. \( (1 \Rightarrow 2) \). This follows from Proposition 3.1. 
\( (2 \Rightarrow 1) \). Suppose that \( M \in \text{\perp GI}(\Lambda \Gamma) \) and \( X \) is an arbitrary Gorenstein projective \( \Lambda \Gamma \)-module. We must show that \( \text{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \). Taking \( \Gamma' = 1 \), we have \( \text{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \) and hence, by [17, Lemma 3.1], \( \text{Ext}^i_{\Lambda \Gamma}(X, M) \cong \text{H}^i(\Gamma, \text{Hom}_{\Lambda}(X, M)) \). Assuming \( \Gamma' \) is an arbitrary Sylow subgroup of \( \Gamma \), the hypothesis enforces that, \( \text{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \) and consequently, \( \text{H}^i(\Gamma', \text{Hom}_{\Lambda}(X, M)) = 0 \), for all \( i > 0 \). Hence, we may invoke [29, Corollary 9.90 (iii)] and conclude that \( \text{H}^i(\Gamma', \text{Hom}_{\Lambda}(X, M)) = 0 \) implying \( \text{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \), as required. Likewise, one can show the inclusion \( \text{GP}(\Lambda \Gamma)^\perp \subseteq \text{\perp GI}(\Lambda \Gamma) \). So the proof is complete. \( \square \)

Recall that a \( \mathbb{Z}\Gamma \)-module \( M \) is said to be cohomologically trivial provided \( \hat{\text{H}}^i(\Gamma', M) = 0 \), for all \( i \in \mathbb{Z} \) and all subgroups \( \Gamma' \) of \( \Gamma \), where \( \hat{\text{H}}^i(\Gamma', -) \) denotes the doubly infinite Tate cohomology of \( \Gamma' \). We refer the reader to [15] and also [28] for a detailed discussion on cohomologically trivial modules.

We are now in a position to present the proof of Theorem 1.1, which is stated in the introduction.

Proof of Theorem 1.1. We only need to prove the ‘if’ part; the ‘only if’ part follows from Proposition 3.1. According to Theorem 3.4, we may assume that \( \Gamma \) is a \( p \)-group,
for some prime integer \( p \). Suppose that \( M \in {}^\perp \mathrm{GI}(\Lambda \Gamma) \) and \( X \) is an arbitrary Gorenstein projective \( \Lambda \Gamma \)-module. We would like to show that \( \mathrm{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \), for all \( i > 0 \). Let \( \Gamma' \) be an elementary abelian subgroup of \( \Gamma \). As we have seen previously, \( M \in {}^\perp \mathrm{GI}(\Lambda \Gamma') \) and \( X \in \mathrm{GP}(\Lambda \Gamma') \). So, by the hypothesis, \( \mathrm{Ext}^i_{\Lambda \Gamma'}(X, M) = 0 \) for all \( i > 0 \). Consequently, in view of Proposition 3.1, one has \( \mathrm{Ext}^i_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \). Now, by invoking Lemma 3.1 of [17] one may conclude that \( \mathrm{Ext}^i_{\Lambda \Gamma'}(X, M) \cong \mathrm{H}^i(\Gamma', \mathrm{Hom}_\Lambda(X, M)) = 0 \), for all \( i > 0 \). In particular, \( \mathrm{H}^i(\Gamma', \mathrm{Hom}_\Lambda(X, M)) = 0 \) for all \( i > 0 \). Hence, by making use of [15, Theorem VI.8.7], we infer that \( \mathrm{Hom}_\Lambda(X, M) \), with diagonal action, is a cohomologically trivial \( \Lambda \Gamma' \)-module. Thus, Theorem VI.8.12 of [15] yields that \( \mathrm{pd}_{\mathrm{Z}^{\infty}}(\mathrm{Hom}_\Lambda(X, M)) \leq 1 \), for all elementary abelian subgroups \( \Gamma' \) of \( \Gamma \). Now apply Corollary 1.1 of [17] in order to conclude that \( \mathrm{pd}_{\mathrm{Z}^{\infty}}(\mathrm{Hom}_\Lambda(X, M)) < \infty \). Especially, \( \mathrm{H}^1(\Gamma, \mathrm{Hom}_\Lambda(X, M)) \cong \mathrm{Ext}^1_{\Lambda \Gamma}(X, M) = 0 \) for all \( i > 0 \), as needed. Similarly, one can verify that \( \mathrm{GP}(\Lambda \Gamma^0) \subseteq {}^\perp \mathrm{GI}(\Lambda \Gamma) \). Hence, \( \Lambda \Gamma \) is virtually Gorenstein, as required. \( \square \)

As an immediate consequence of the above theorem in conjunction with [15, Proposition VI.9.5], we include the following result.

**Corollary 3.5.** Let \( \Gamma \) be a finite group such that every Sylow subgroup of \( \Gamma \) is cyclic or generalized quaternion group. Then \( \Lambda \Gamma \) is virtually Gorenstein if and only if \( \Lambda \mathbb{Z}_p \) is virtually Gorenstein for all prime integers \( p \) dividing the order of \( \Gamma \).

**Remark 3.6.** Let \( \Gamma \) be any group and \( \Gamma' \) be its subgroup of finite index. Following Aljadeff [2], we say that the triple \( (\Gamma, \Gamma', R) \) satisfies Moore’s condition if for all \( x \in (\Gamma - \Gamma') \) at least one of the following holds:

1. there is an integer \( n \), such that \( 1 \neq x^n \in \Gamma' \).
2. \( \text{ord}(x) \) is finite and invertible in \( R \).

In 1976, J. Moore posed the following conjecture which concerns a criterion for modules over group rings to be projective.

**Moore’s conjecture** (see [17]). Let \( \Gamma \) be a group, \( \Gamma' \) be a subgroup of \( \Gamma \) of finite index and \( R \) be any ring with identity. Assume that Moore’s condition holds for the triple \( (\Gamma, \Gamma', R) \). Then every \( RT \)-module \( M \) which is projective over \( RT' \) is projective over \( RT \) as well.

This conjecture then has been the subject of several expositions. Recently, it is shown by Aljadeff and Meir that Moore’s conjecture is valid for groups which belong to Kropholler’s hierarchy \( \text{LH}_3 \); see [1].

**Proposition 3.7.** Let \( R \) be a virtually Gorenstein ring and \( \Gamma \) be an infinite cyclic group. Let every Gorenstein projective (resp. Gorenstein injective) \( RT \)-module is also Gorenstein projective (resp. Gorenstein injective) over \( R \). Then \( RT \) is a virtually Gorenstein ring.

**Proof.** Assume that \( M \in {}^\perp \mathrm{GI}(R \Gamma) \) and \( X \) is an arbitrary Gorenstein projective \( R \Gamma \)-module. We must show that \( \mathrm{Ext}^i_{R \Gamma}(X, M) = 0 \), for all \( i > 0 \). For this purpose, one should note that, according to Remark 2.9, \( M \in {}^\perp \mathrm{GI}(R) \). So, in view of the hypothesis, \( \mathrm{Ext}^i_R(X, M) = 0 \), for all \( i > 0 \). Consequently, Lemma 3.1 of [17], gives rise the isomorphism \( \mathrm{Ext}^i_{R \Gamma}(X, M) \cong \mathrm{H}^i(\Gamma, \mathrm{Hom}_R(X, M)) \) implying \( \mathrm{Ext}^i_{R \Gamma}(X, M) = 0 \) for all \( i > 2 \), since \( \mathrm{pd}_{\mathrm{Z}^{\infty}} \mathbb{Z} = 1 \). Hence, it remains to show that \( \mathrm{Ext}^1_{R \Gamma}(X, M) \). To do this, take the
following short exact sequence of $\RG$-modules;

\[ 0 \to X \to P \to X' \to 0, \]

whereas $P$ is projective and $X'$ is Gorenstein projective. Hence, by applying the functor $\Hom_{\RG}(-, M)$ to this sequence, one obtains the isomorphism $\Ext^i_{\RG}(X, M) \cong \Ext^i_{\RG}(X', M) = 0$. Therefore, $\GI(R\Gamma) \subseteq \GP(R\Gamma)^\perp$. The inverse inclusion can be obtained similarly. So, the proof is complete. \hfill \Box

**Proof of Theorem 1.2.** In view of Proposition 3.1, we only need to show the ‘if’ part. For this purpose, we proceed by induction on the ordinal number $\alpha$ such that the group $\Gamma$ belongs to $\mathbb{H}_\alpha \mathcal{F}$. If $\alpha = 0$, then $\Gamma$ is a finite group. Since $\RG'$ is virtually Gorenstein, $R$ is also virtually Gorenstein. Let $p$ be a prime integer and $\Gamma''$ be an elementary abelian $p$-subgroup of $\Gamma$. If $p$ is invertible in $R$, then it follows from Lemma 3.2 that $\RG''$ is virtually Gorenstein. Moreover, if $p$ is not invertible in $R$, then the hypothesis implies that $\Gamma''$ is contained in $\Gamma'$ and hence $\RG''$ is virtually Gorenstein, thanks to Proposition 3.1. Now the claim follows directly from Theorem 1.1. Next assume that the result is true for $\Gamma \in \mathbb{H}_\beta \mathcal{F}$, for all $\beta < \alpha$ and let $\Gamma \in \mathbb{H}_\alpha \mathcal{F}$. Suppose that $M \in \GI(R\Gamma)$ and $X$ is a Gorenstein projective $\RG$-module. We would like to show that $\Ext^i_{\RG}(X, M) = 0$, for all $i > 0$. Since $\Gamma \in \mathbb{H}_\alpha \mathcal{F}$, there is an exact sequence of $\RG$-modules

\[ 0 \to C_r \to \cdots \to C_0 \to R \to 0, \]

where each $C_i$ is a direct sum of modules of the form $R[\Gamma/H_j]$ whereas $H_j$, for each $j$, is a subgroup of $\Gamma$ and $H_j \in \mathbb{H}_\beta \mathcal{F}$, $\beta < \alpha$. Applying the functor $\Hom_R(-, M)$ to this sequence, gives the following exact sequence of $\RG$-modules:

\[ 0 \to M \to \Hom_R(C_0, M) \to \cdots \to \Hom_R(C_r, M) \to 0. \]

One should observe that, for any $i$, one has the following $\RG$-isomorphisms;

\[ \Hom_R(C_i, M) \cong \Hom_R(\oplus R[\Gamma/H_j], M) \cong \prod \Hom_R(R[\Gamma/H_j], M) \cong \prod \Hom_R(\RG \otimes_{RH_j} R, M) \cong \prod \Hom_{RH_j}(\RG \oplus \Hom(R, M)) \cong \prod \Hom_{RH_j}(\RG, M). \]

By invoking induction hypothesis in conjunction with Proposition 2.8 and Remark 2.9, one may obtain that for any $i > 0$,

\[ \Ext^i_{\RG}(X, \prod \Hom_{RH_j}(\RG, M)) \cong \prod \Ext^i_{RH_j}(X, M) = 0, \]

implying $\Ext^i_{\RG}(X, M) = 0$, for all $i > r$. Now, take the exact sequence of $\RG$-modules;

\[ 0 \to X \to P_0 \to \cdots \to P_r \to X' \to 0, \]

whereas, for each $j$, $P_j$ is projective and $X'$ is Gorenstein projective. As we have seen just above, $\Ext^i_{\RG}(X', M) = 0$. Hence, the result follows from the isomorphism $\Ext^i_{\RG}(X, M) \cong \Ext^i_{\RG}(X, M)$, for all $i > 0$. The inclusion $\GI(R\Gamma) \subseteq \GI(R\Gamma)^\perp$ holds true in a similar way. Then the proof is completed. \hfill \Box

As a direct consequence of the Theorem 1.2, we include the following result.

**Corollary 3.8.** Let $\Gamma$ be a finite group, $\Gamma'$ be its subgroup and $\Lambda$ be any artin algebra. Assume that the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore’s condition. Then, $\Lambda\Gamma$ is a virtually Gorenstein algebra if and only if so is $\Lambda\Gamma'$. 

Recall from [11] that a subcategory $\mathcal{X}$ of $\text{mod} \Lambda \Gamma$ is contravariantly finite if every finitely generated $\Lambda \Gamma$-module $C$ has a right $\mathcal{X}$-approximation $X \to C$, that is, $X \in \mathcal{X}$ and the induced map $\text{Hom}_{\Lambda \Gamma}(X', X) \to \text{Hom}_{\Lambda \Gamma}(X', C)$ is surjective for every $X' \in \mathcal{X}$. Covariantly finite subcategories are defined dually.

Combining Corollary 3.8 and [11, Theorem 1] yields the following result.

**Corollary 3.9.** Let $\Gamma$ be a finite group and $\Gamma'$ be a subgroup of $\Gamma$. Assume that $\text{Thick}(\text{proj} \Lambda \Gamma' \cup \text{inj} \Lambda \Gamma')$ is contravariantly finite. Then $\text{Thick}(\text{proj} \Lambda \Gamma \cup \text{inj} \Lambda \Gamma)$ is also contravariantly finite provided the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore’s condition.

**Example 3.10.** Let $R$ be a commutative Artin algebra and $\Gamma$ be a finite $p$-group, for some prime integer $p$. Assume that $p.1_R$ does not belong to $p$, where $p$ is a prime ideal of $R$. So assuming $R_p$ is virtually Gorenstein, one deduces that the same is true for $R_p \Gamma$, because of the fact that the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore’s condition.

**Remark 3.11.** As we have mentioned in the introduction, virtually Gorenstein algebras are a common generalization of algebras of finite representation type and of finite Cohen-Macaulay type; see [12, Example 8.4]. Assuming $\Gamma$ is a finite group and $\Gamma'$ is its subgroup, Lemma VI 3.1 of [6] yields that finite representation type and finite Cohen-Macaulay type ascend and descend between $\Lambda \Gamma$ and $\Lambda \Gamma'$, if the index $\Gamma'$ in $\Gamma$ is invertible in $\Lambda$. However, Theorem 3.8 provides a weaker criterion that guarantees the ascent and descent of virtual Gorensteinness between $\Lambda \Gamma$ and $\Lambda \Gamma'$.

For a ring $R$, the little finitistic dimension, $\text{fin dim}_R$, is defined as the supremum of the projective dimensions attained on the category of all finitely generated left $R$-modules having finite projective dimension. The big finitistic dimension, $\text{Fin dim}_R$, is defined correspondingly on the category of arbitrary left $R$-modules of finite projective dimension. It is well known that these dimensions may be infinite. Moreover, they do not coincide in general; see [22]. It is important to mention that there is a tie connection between the Gorensteinness of artinian algebra $\Lambda$ and the finiteness of $\text{fin dim} \Lambda$. Precisely, a left Gorenstein algebra $\Lambda$ is right Gorenstein if and only if $\text{fin dim} \Lambda$ is finite; see [5].

We end this paper by showing that finiteness of finitistic dimensions carries over from $\Gamma$ to $\Gamma'$ and vice versa, whenever, $\Gamma'$ is a subgroup of finite index.

**Proposition 3.12.** Let $R$ be any ring, $\Gamma$ be a group and $\Gamma'$ be a subgroup of finite index. Then, one has the equalities $\text{Fin dim}_R \Gamma' = \text{Fin dim}_R \Gamma$ and $\text{fin dim}_R \Gamma' = \text{fin dim}_R \Gamma$.

**Proof.** We only prove the first equality, the proof of the second one follows the same lines. To do this, we first show that $\text{Fin dim}_R \Gamma' \leq \text{Fin dim}_R \Gamma$. If $\text{Fin dim}_R \Gamma = \infty$, there is nothing to prove. So assume that $\text{Fin dim}_R \Gamma$ is finite, say $t$. Take an arbitrary $\Gamma'$-module $M$ with finite projective dimension. According to the $\Gamma$-isomorphism $\Gamma \otimes_{\Gamma'} \Gamma' \cong \Gamma$, we conclude that $\Gamma \otimes_{\Gamma'} M$ has finite projective dimension as an $\Gamma$-module. Hence, our assumption yields that $\text{pd}_{\Gamma}(\Gamma \otimes_{\Gamma'} \Gamma' \otimes M) \leq t$ and so $\text{2.2}$ induces that $\text{pd}_{\Gamma'}(\Gamma \otimes_{\Gamma'} \Gamma' \otimes M) \leq t$. Now $M$ being an $\Gamma'$-direct summand of $\Gamma \otimes_{\Gamma'} M$ implies that $\text{pd}_{\Gamma'} M \leq t$ and consequently, $\text{Fin dim}_R \Gamma' \leq \text{Fin dim}_R \Gamma$. To that end, we may assume that $\text{Fin dim}_R \Gamma'$ is finite. In view of $\text{2.2}$, we have that for any $\Gamma$-module $M$ of finite projective dimension, the equality $\text{pd}_{\Gamma'} M = \text{pd}_{\Gamma} M$ holds true. This, in turn, deduces the claim. \[\square\]

**Remark 3.13.** Let $\Lambda$ be an Artin algebra. In [9] Bass posed two dimension conjectures: the first one asserts that the little and big finitistic dimension of $\Lambda$ are equal
is, $\text{Fin dim } \Lambda = \text{fin dim } \Lambda$ and the second one, which is called the finitistic dimension conjecture, says that $\text{fin dim } \Lambda$ is finite. Because of [30], the first conjecture does not hold in general. However, the finitistic dimension conjecture is still open now. Some of the known cases in which the finitistic dimension conjecture holds are the radical cubed zero case [21], algebras of representation dimension at most three [24] and algebras in which the category of modules of finite projective dimension is contravariantly finite in mod$\Lambda$ [5]. It is worth pointing out that, according to the above proposition, the validity of finitistic dimension conjecture ascends and descends between $\Lambda$ and $\Lambda\Gamma$, provided $\Gamma$ is a finite group. In particular, the finiteness of finitistic dimension carries over from a finite group $\Gamma$ to its subgroups and vice versa. Moreover, if $\text{Fin dim } \Lambda = \text{fin dim } \Lambda$, then the same is true for $\Lambda\Gamma$. Precisely, the validity of the first conjecture stated above, carries over from $\Lambda$ to $\Lambda\Gamma$.

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References


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