NEW IMPLEMENTATION OF MLBIE METHOD FOR HEAT
CONDUCTION ANALYSIS IN FUNCTIONALLY GRADED
MATERIALS

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ABSTRACT. The meshless local boundary integral equation (MLBIE) method with an efficient technique to deal with the time variable are presented in this article to analyze the transient heat conduction in continuously nonhomogeneous functionally graded materials (FGMs). In space, the method is based on the local boundary integral equations and the moving least squares (MLS) approximation of the temperature and heat flux. In time, again the MLS approximates the equivalent Volterra integral equation derived from the heat conduction problem. It means that, the MLS is used for approximation in both time and space domains, and we avoid using the finite difference discretization or Laplace transform methods to overcome the time variable. Finally the method leads to a single generalized Sylvester equation rather than some (many) linear systems of equations. The method is computationally attractive, which is shown in couple of numerical examples for a finite strip and a hollow cylinder with an exponential spatial variation of material parameters.

Keywords. Meshless local boundary integral equation (MLBIE) method; Moving least squares (MLS) approximation; Heat conduction problem; Functionally graded materials (FGM).

1. INTRODUCTION

Functionally graded materials (FGMs) are those in which the composition and the volume fraction of FGM constituents vary gradually, giving a non-uniform microstructure with continuously graded macro-properties (e.g. specific heat, conductivity, density). For instance, one face of a structural component (plate or cylinder) may be an engineering ceramic that can resist severe thermal loading, and the other face may be a metal to maintain structural strength and toughness [1, 2]. In recent
years FGMs have drawn considerable interest [3]. These FGMs are expected to be highly heat resistant materials that can be used under high temperature and high temperature gradient conditions. In this case, it is important to investigate the temperature distributions in the FGMs.

Due to the high mathematical complexity of the initial boundary value problems, analytical approaches for the thermo-mechanics of FGMs are restricted to simple geometry and boundary conditions. Thus, the transient heat conduction analysis in FGM demands accurate and efficient numerical methods [4].


There are also many works on meshless methods for solving some other time-dependent problems. For example see [11, 12, 13, 14, 15] and etc. Meshless methods are based on approximation in terms of scattered data [16]. These methods are becoming popular, due to their high adaptivity and a low cost to prepare input data for numerical analysis. The moving least squares (MLS) is one of the scattered data approximation methods, which has been used successfully to approximate the trial space in many meshless methods. For instance we can mention the MLBIE method [17, 18]. This method is based on local sub-domains, rather than a global domain, and requires neither domain elements nor background cells in either the approximation or the integration. Hence it is a truly weak-based meshless method. During these recent years, MLBIE methods have been widely used to solve many steady-state and time-dependent engineering problems.

The authors of all above-mentioned papers employed either the Laplace transform or the time discretization method to overcome the time variable. In this paper, we propose an interesting meshless local boundary integral equation (MLBIE) method which employs the MLS to approximate the quantities in both time
and space domains. Although the technique is presented for the MLBIE method, it can be easily extended to other MLS based meshless methods.

This paper is organized as follows. In Section 2, MLS approximation is discussed. In Section 3, the governing equations of heat conduction problem in FGMs are converted to an algebraic system of equations using MLBIE method. In Section 4, numerical results for a finite stirp and a hallow cylinder are presented.

2. MLS APPROXIMATION

The MLS as an approximation method has been introduced by Lancaster and Salkauskas [19] inspired by the pioneer work of Shepard [20]. Since the numerical approximations of MLS are based on a cluster of scattered nodes instead of interpolation on elements, there have been many meshless methods based on the MLS method for the numerical solution of differential equations in recent years.

Given data values \( \{ u_j \}_{j=1}^{N} \) at centers \( \{ x_1, \ldots, x_N \} \), the MLS method produces a function \( \hat{u} \) that approximates \( u \) in a weighted square sense. Let \( \mathbb{P}_m^d \) be the space of polynomial of degree \( m \) in \( \mathbb{R}^d \), and let \( B = \{ p_1, p_2, \ldots, p_Q \} \) be any polynomial basis of \( \mathbb{P}_m^d \), where \( Q = \binom{m+d}{d} \). The MLS approximation \( \hat{u}(x) \) of \( u(x) \), \( \forall x \in \Omega \), can be defined as

\[
\hat{u}(x) = p^T(x)a(x), \quad \forall x \in \Omega,
\]

where \( p^T(x) = [p_1(x), p_2(x), \ldots, p_Q(x)] \) and \( a(x) \) is a vector with components \( a_j(x) \), \( j = 1, 2, \ldots, Q \), which are functions of the space coordinates \( x \). In this paper we suppose that the basis of \( \mathbb{P}_m^d \) is a complete monomial basis of order \( Q \).

The coefficient vector \( a(x) \) is determined by minimizing a weight discrete square norm, which is defined as

\[
I(x) = \sum_{j=1}^{N} w(x, x_j) (p^T(x_j)a(x) - u_j)^2, \tag{2.1}
\]

where \( w(x, x_j) \) is the weight function associated with the node \( j \). In MLS, we are interested in local continuous weight function \( w \). To be more precise we choose a continuous function \( \phi : [0, \infty) \rightarrow [0, \infty) \) with

- \( \phi(r) > 0 \), \( 0 \leq r < 1 \),
- \( \phi(r) = 0 \), \( r \geq 1 \),

and define

\[
w_\delta(x, y) = \phi \left( \frac{\|x - y\|_2}{\delta} \right),
\]

for \( \delta > 0 \) as a weight function. We further define the set of indices

\[
J(x) = J(x, \delta, X) = \{ j \in \{ 1, 2, \ldots, N \} : \|x - x_j\|_2 \leq \delta \},
\]

of centers contained in the interior of closed ball \( B(x, \delta) \) of radius \( \delta \) around \( x \).
If we define the matrices $P$ and $W$ as

\[ P := (p_\ell(x_j)), \quad 1 \leq \ell \leq Q, \quad j \in J(x), \]
\[ W(x) := (\delta_{jk}w_\ell(x,x_k))_{j,k \in J(x)}, \]
the stationarity point of $I$, in equation (2.1), with respect to $a(x)$ leads to

\[ \varphi(x) = p(x)A^{-1}B, \quad (2.2) \]

and

\[ \hat{u}(x) = \sum_{j=1}^{N} \varphi_j(x)u(x_j), \quad (2.3) \]

where $A(x) = PW(x)P^T, B(x) = PW(x)$ and $\varphi(x) = [\varphi_1(x), ..., \varphi_N(x)]$, providing $\varphi_j(x) = 0$ for $j \notin J(x)$. The functions $\varphi_j(x)$ are called the shape functions of the MLS approximation, corresponding to nodal point $x_j$. If $w(x,x_j) \in C^k(\Omega)$, then $\varphi_j(x) \in C^k(\Omega)$ [21, 22].

The matrix $A$ is of size $Q \times Q$. The MLS approximation is well-defined only when the moment matrix $A$ is non-singular. It can be seen that this is the case if and only if the rank of $P$ equals $Q$ ($P$ should be a full-rank matrix). A necessary condition is $|J(x)| \geq Q$, i.e. there should be at least $Q$ points in the support of weight function $w_\ell$ (the domain of definition of MLS approximation), but this condition is not sufficient. A necessary and sufficient condition is the $P^d_m$-unisolvency condition which is defined on set \{ $x_1, ..., x_N$ \} and depends on the geometric shape of $\Omega$. For more precise details see [21] and Chapters 3 and 4 of [22].

The partial derivatives of $\hat{u}(x)$ are obtained as:

\[ \hat{u}_i(x) = \sum_{j=1}^{N} \varphi_{j,i}(x)u(x_j), \quad x \in \Omega, \quad (2.4) \]

where

\[ \varphi_{j,i}(x) = \sum_{k=1}^{Q} (p_{k,i}[A^{-1}B]_{kj} + p_k[A^{-1}B]_{ji} + A^{-1}_{ji}B_{kj}), \]

where $A^{-1}_{ji} = (A^{-1})_{ji}$ represents the derivative of the inverse of $A$ with respect to $i$-th component of $x$, which is given by

\[ A^{-1}_{ji} = -A^{-1}A_jA^{-1}. \]

In the MLS and other scattered data approximation methods two quantities fill distance and separation distance are important to measure the quality of centers and derive the rate of convergence. For a set of points $X = \{x_1, x_2, ..., x_N\}$ in a bounded domain $\Omega \subseteq \mathbb{R}^d$ the fill distance is defined to be

\[ h_{X,\Omega} = \sup_{x \in \Omega} \min_{1 \leq j \leq N} \|x - x_j\|_2, \]
and the separation distance is defined by
\[ q_{X} = \frac{1}{2} \min_{i \neq j} \| x_i - x_j \|_2. \]

A set \( X \) of data sites is said to be \textit{quasi-uniform} with respect to a constant \( c_{qu} > 0 \) if
\[ q_{X} \leq h_{X,\Omega} \leq c_{qu} q_{X}. \]  

(2.5)

The error analysis of MLS approximation can be found in [23, 21, 22]. If \( \Omega \) satisfies an interior cone condition, then there exist constants \( C \), such that for all \( u \in C^{m+1}(\Omega^*) \) and all quasi-uniform \( X \subset \Omega \), we have
\[ \| u - \hat{u} \|_{\infty} \leq C h_{X,\Omega}^{m+1} |u|_{C^{m+1}(\Omega^*)}, \]  

(2.6)

where \( \Omega^* \) is the closure of \( \bigcup_{x \in \Omega} B(x, C h_{X,\Omega}) \) with \( h_{X,\Omega} \leq h_0 \) and \( h_0 \) depends on \( \Omega \).

The semi-norm in the right hand side is defined by
\[ |u|_{C^k(\Omega^*)} := \max_{|\beta| = k} \|D^\beta u\|_{L^\infty(\Omega)}. \]


3. Transient heat conduction in FGM

The thermal heat conduction problem in the FGM is described by the following governing equation [5]
\[ \frac{1}{\alpha(x)} \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + \frac{1}{\kappa(x)} \nabla \kappa(x) \cdot \nabla u(x, t) + \frac{1}{\kappa(x)} f(x, t), \]

(3.1)

where \( x \in \Omega \) and \( 0 \leq t \leq t_F \) denote the time and the space variables, respectively, and \( t_F \) is the final time. The initial and boundary conditions are
\[ u(x, 0) = u_0(x), \quad x \in \Omega, \]  

(3.2)

\[ u(x, t) = \pi(x, t), \quad x \in \Gamma_u, \quad 0 \leq t \leq t_F, \]  

(3.3)

\[ \kappa(x) \frac{\partial u}{\partial n}(x, t) = \tau(x, t), \quad x \in \Gamma_q, \quad 0 \leq t \leq t_F. \]  

(3.4)

In (3.1)-(3.4), \( \Delta \) represents the Laplacian operator, \( u(x, t) \) is the temperature field, \( \kappa(x) \) and \( \alpha(x) \) stand for the thermal conductivity and diffusivity, respectively, and \( f(x, t) \) is the density of the body heat sources. Moreover \( n \) is the unit outward
normal to the boundary $\Gamma$, $\pi$ and $\eta$ are specified values on the Dirichlet boundary $\Gamma_u$ and Neumann boundary $\Gamma_q$ where $\Gamma = \Gamma_u \cup \Gamma_q$.

To employ the MLS approximation for the time domain, we do the following: from (3.1) and (3.2) we have
\[
\frac{1}{\alpha} u(x,t) = \int_0^t \left[ \Delta u(x,\tau) + \frac{1}{\kappa} \nabla \kappa \cdot \nabla u(x,\tau) \right] d\tau + \frac{1}{\kappa} f(x,\tau) d\tau + \frac{1}{\alpha} u_0(x),
\] keeping in mind $\alpha := \alpha(x)$ and $\kappa := \kappa(x)$ are some functions of spatial variable.

First the MLS approximation is written in respect to the time variable $t$. The idea comes up from the MLS based method for Fredholm and Volterra integral equations presented by Mirzaei and Dehghan [24]. Here the Volterra type is compatible.

Consider $F$ distinct points $T = \{t_1, t_2, ..., t_F\}$ in the time domain $[0, t_F]$, with the fill distance $h_T, [0, t_F]$. It is better $t_F$ be included because in various cases we need the solution at the final time. According to (2.3), replacing $x$ by $t$ and $N$ by $F$, the univariate MLS approximation to equation (3.5) in respect to the time variable $t$, after imposing at $t = t_k$ for $1 \leq k \leq F$, is
\[
\frac{1}{\alpha} \sum_{\ell=1}^{F} \varphi(t_k)u(x,t_\ell) - \sum_{\ell=1}^{F} \left( \int_0^{t_k} \varphi(\tau) d\tau \right) \left( \Delta u(x,t_\ell) + \frac{1}{\kappa} \nabla \kappa \cdot \nabla u(x,t_\ell) \right) = \frac{1}{\kappa} \int_0^{t_k} f(x,\tau) d\tau + \frac{1}{\alpha} u_0(x).
\]

The integrations over $[0, t_k]$ can be done by converting this interval to interval $[0, 1]$ using the following linear transformation
\[
\tau(t, \theta) = t \theta.
\]

Therefore, if we set
\[
E_{k,\ell} = \varphi(t_k), \quad 1 \leq k, \ell \leq F, \quad G_{k,\ell} = \int_0^{t_k} \varphi(\tau) d\tau = t_k \int_0^1 \varphi(t_k, \theta) d\theta = t_k \sum_{j=1}^M \varphi(t_k \theta_j) \omega_j, \quad 1 \leq k, \ell \leq F,
\]
\[
u(x) = [u(x,t_1), ..., u(x,t_F)],
\]
\[
f(x) = [f_1(x), ..., f_F(x)],
\]
\[
f_k(x) = t_k \sum_{j=1}^M f(x,t_k \theta_j) \omega_j, \quad 1 \leq k \leq F,
\]
\[
u_0(x) = [u_0(x), ..., u_0(x)]_{1 \times F},
\]
where $\{\theta_j, \omega_j\}_{j=1}^M$ is a $M$-point Gauss quadrature (or others) formula in $[0, 1]$, then we have
\[
\frac{1}{\alpha} E u^T(x) - G \left( \Delta u^T(x) + \frac{1}{\kappa} [\nabla \kappa \cdot \nabla u(x)]^T \right) = \frac{1}{\kappa} f^T(x) + \frac{1}{\alpha} u_0^T(x).
\]
In addition from Dirichlet boundary condition (3.3), we have
\[ F \sum_{\ell=1}^{F} \phi_\ell(t_k) u(x,t_\ell) = \Pi(x,t_k), \quad 1 \leq k \leq F, \quad x \in \Gamma_u. \]

If we set \( \Pi(x) = [\Pi(x,t_1), \ldots, \Pi(x,t_F)] \), then
\[ E u^T(x) = \Pi^T(x), \quad x \in \Gamma_u. \]  \( (3.9) \)

Moreover, from the Neumann boundary condition (3.4), we have
\[ F \sum_{\ell=0}^{F} \phi_\ell(t_k) \frac{\partial u}{\partial n}(x,t_\ell) = \frac{1}{\kappa} q(x,t_k), \quad 1 \leq k \leq F, \quad x \in \Gamma_q \]
and if we set \( q(x) = [q(x,t_1), \ldots, q(x,t_F)] \), then
\[ E \frac{\partial u^T}{\partial n}(x) = \frac{1}{\kappa} q^T(x), \quad x \in \Gamma_q. \]  \( (3.10) \)

Equations (3.8)-(3.10) construct a time-free system of boundary value problems which can be solved by a proper method. We are trying to apply the MLBIE method, presented by Zhu et al [17], to this system of equations. Thus the MLS approximation in respect to the space variable \( x \) should be applied. According to the MLBIE process, first, the local weak form is written on local sub-domains \( \Omega_y \subset \Omega \) where \( \partial \Omega_y = L_y \cup \Gamma_y \), and \( \Gamma_y \) is a part of sub-domain’s boundary which has intersection with global boundary \( \Gamma \) over which the Neumann boundary conditions are applied. The local sub-domains could be of any geometric shape and size. For simplicity they are taken to be \( B(y,\sigma) \cap \Omega \). It is clear that for every internal node \( y \), \( \Gamma_y \sigma = \emptyset \) and \( L_y = \partial \Omega_y \). In MLBIE method the solution of the governing equation (3.8) in two dimensional space can be found in a second weak form via a modified fundamental solution (companion solution [17])
\[ u^*(x,y) = \frac{1}{2\pi} \ln \frac{\sigma}{r} \]
corresponds to the Poisson equation
\[ \Delta u^*(x,y) + \delta(x,y) = 0, \]
where \( \delta(x,y) \) is the Dirac delta function, \( r \) is the distance of the field and source points, i.e., \( r = \|x - y\|_2 \) and \( \sigma \) is the radius of sub-domains.

By the use of the Gauss divergence theorem to the weak form of Equation (3.8) and then applying the Neumann boundary condition (3.10), we obtain an integral
lies on the smooth boundary \( y \) if that the value \( \varphi \) we use the notation \( X \) centers \( x \) replacing \( x \) using the MLS approximation \([22]\). For simplicity suppose that \( \beta(y) \) is the interior angle at point \( y \). The symbol \( f \) denotes the Cauchy Principle Value (CPV) integral. The right hand side of Equation (3.11) is a known vector of size \( F \times 1 \), say \( r(y) \).

Note that neither Lagrange multipler nor penalty parameters are introduced in the local weak form, because the Dirichlet boundary conditions are imposed directly using the MLS approximation. Hence, for \( y \in \Gamma_u \), Equation (3.9) is repeated replacing \( x \) by \( y \).

Now consider a quasi-uniform set \( X = \{x_1, x_2, \ldots, x_N\} \) in \( \Omega \) with the fill distance \( h_{X,\Omega} \). The set \( X \) should also satisfy the unisolvency condition to have a well-defined MLS approximation \([22]\). For simplicity suppose that \( x_j \in \Gamma_u \) for \( 1 \leq j \leq N_1 \) and \( x_j \in \Gamma_q \cup \text{int}(\Omega) \) for \( N_1 + 1 \leq j \leq N \). Thanks to the MLS approximations (2.3) and (2.4), the vectors \( u \) and its derivatives can be written in terms of values of \( u \) at centers \( X \). To make difference between the MLS approximation in time and space, we use the notation \( \phi_j(x) \) instead of \( \varphi_j(x) \) for shape functions in (2.3) and (2.4).

Thus equation (3.9) can be approximated by

\[
E \left( \sum_{j=1}^{N} \phi_j(y) u^T(x_j) \right) = u^T(y), \quad y \in \Gamma_u. \tag{3.12}
\]

and equation (3.11) by

\[
E \left( \sum_{j=1}^{N} \left[ \int_{\Omega^*} \frac{1}{\alpha} \phi_j(x) u^*(x, y) d\Omega \right] u^T(x_j) \right) + G \left( \sum_{j=1}^{N} \left[ c(y) \phi_j(y) + \int_{L^*} \phi_j(x) \frac{\partial u^*}{\partial n} (x, y) d\Gamma + \int_{\Gamma^*} \phi_j(x) \frac{\partial u^*}{\partial n} (x, y) d\Gamma \right] \right) \tag{3.13}
\]

\[
- \int_{\Omega^*} \frac{1}{\kappa} \nabla \cdot \nabla \phi_j(x) u^*(x, y) d\Omega \right] u^T(x_j) = r(y).
\]
Imposing (3.12) at \( y = x_i \), \( 1 \leq i \leq N_1 \) and (3.13) at \( y = x_i \), \( N_1 + 1 \leq i \leq N \) and setting
\[
H_1(i,j) = \phi_j(x_i), \quad 1 \leq i \leq N_1, \quad 1 \leq j \leq N, \\
H_2(i,j) = \int_{\Omega_i} \frac{1}{\alpha(x)} \phi_j(x) u^*(x,x_i) d\Omega, \quad N_1 + 1 \leq i \leq N, \quad 1 \leq j \leq N, \\
K_2(i,j) = c(x_i) \phi_j(x_i) + \int_{L_i} \phi_j(x) \frac{\partial u^*}{\partial n}(x,x_i) d\Gamma + \int_{\Gamma_i} \phi_j(x) \frac{\partial u^*}{\partial n}(x,x_i) d\Gamma \\
- \int_{\Omega_i} \frac{1}{\kappa} \nabla \cdot \nabla \phi_j(x) u^*(x,x_i) d\Omega, \quad N_1 + 1 \leq i \leq N, \quad 1 \leq j \leq N, \\
R_1(i,:) = \mathbf{u}(x_i), \quad N_1 + 1 \leq i \leq N, \\
R_2(i,:) = \mathbf{r}(x_i), \quad N_1 + 1 \leq i \leq N, \\
U(j,:) = \mathbf{u}(x_j), \quad 1 \leq j \leq N,
\]
give
\[
\begin{align*}
E(H_1 U)^T & = R_1^T \\
E(H_2 U)^T + G(K_2 U)^T & = R_2^T 
\end{align*}
\]
Let \( H = [H_1^T \quad H_2^T]_{N \times N}, \ K = [0 \quad K_2^T]_{N \times N} \) and \( R = [R_1^T \quad R_2^T]_{F \times N} \), thus we have
\[
EU^T H + GU^T K = R, \tag{3.14}
\]
which is a generalized Sylvester equation. This equation should be solved by some linear algebra techniques that we will discuss in the following. Note that \( U(j,\ell) = u(x_j,t_\ell) \) for \( 1 \leq j \leq N \) and \( 1 \leq \ell \leq F \). When \( U \) is derived from equation (3.14), the solution at every point \((x,t) \in \Omega \times [0,t_F]\) can be approximated by
\[
u(x,t) \approx \tilde{u}(x,t) = \sum_{j=1}^{N} \phi_j(x) u(x_j,t) \\
= \sum_{j=1}^{N} \sum_{\ell=1}^{F} \phi_j(x) \varphi(\ell) u(x_j,t_\ell).
\]
If we set \( \phi(x) = [\phi_1(x), ..., \phi_N(x)] \) and \( \varphi(t) = [\varphi_1(t), ..., \varphi_F(t)] \) then
\[
u(x,t) \approx \phi(x) U \varphi^T(t). \tag{3.15}
\]
Note that, \( E \) and \( G \) are time and \( H \) and \( K \) are space matrices. The method provides the time and space matrices separately. Therefore other meshless methods in space can be easily replaced by the MLBIE, presented in this paper.

The advantages of this method over the finite difference discretization in time can be expressed as bellow. In time discretization method we should use a difference equation of order \( O(\Delta t) \) like \( [u(t + \Delta t) - u(t)]/\Delta t \) instead of \( \partial u/\partial t \). Thus a small time step \( \Delta t \) requires to get accurate results. In the method of this paper the
convergence rate in the time domain is of order $h_{\tau_{[0,t_F]}}^{m+1}$. Therefore, using not so many small time fill distance $h_{\tau_{[0,t_F]}}$, accurate results are obtained.

Laplace transform method is another technique to treat the time dependent problems. Several linear system of equations should be solved to get the solution at a sample time $t$. Unfortunately, the accuracy of the space method (here ML-BIE) will finally lose due to the poor accuracy of Laplace inversion techniques. In addition Laplace transforms of the body force and the boundary conditions should be provided.

However, the method presented in this paper needs a linear algebra technique to solve the generalized Sylvester equation (3.14). There are several strategies.

This equation can be converted to

$$TU^T + UT^S = C,$$

(3.16)

with the time matrix $T = G^{-1}E$, the space matrix $S = KH^{-1}$ and the right-hand side $C = G^{-1}RH^{-1}$, providing $G$ and $H$ are invertible matrices. Equation (3.16) is a standard Sylvester equation with unknown $U$. In MATLAB the function

$$X = \text{lyap} (T,S,-C)$$

gives the solution $U = X^T$. Numerical results show, matrix $H$ is invertible and if 0 is not included in $\{t_1, ..., t_F\}$, matrix $G$ is also invertible. Note that there is no force to include zero.

Equation (3.14) can be also solved directly using the Bartels-Stewart (BS) or Hessenberg-Schur (HS) algorithms presented in [25] based on QZ factorization, without any assumption on invertibility. Gardiner et.al [25] derived: if $N \geq F$ (if is not just change $N$ and $F$), the computational costs of BS and HS algorithms are

BS: $33N^3 + 33F^3 + 3N^2F + 3F^2N,$

HS: $8.8N^3 + 33F^3 + (5 + 5.5p)N^2F + 3F^2N,$

where $0 \leq p \leq 0.5$. In this method, $F$ is the number of meshless points in one-dimensional time domain, and in many cases can be ignored in contrast to the number of meshless points in multi-dimensional space domain, i.e. $N$.

These costs are roughly the same as the costs of solving many linear system of equations which is required in time discretization technique. Note that, solving a single generalized sylvester equation (3.14) gives the results at all time and space points, at once. This means that solving a generalized Sylvester equation is not a noticeable problem in sense of computational costs. Anyway, since in time difference method, $LU$ decomposition and backward and forward substitutions can be used in some cases, the method of this paper is slightly more expensive.
4. Numerical results

In numerical results, the basis

\[ B = \left\{ \frac{(x-z)^\beta}{h^{|eta|}} \right\}_{0 \leq |eta| \leq m} \]

is used where \( h \) can be \( q_X, h_X, \Omega \) or an average of them. And \( z \) is a fixed evaluation point such as a test point or a Gaussian point for integration in weak-form based techniques. Here \( \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2 \) is a multi-index and \( |eta| = \beta_1 + \beta_2 \). If \( x = (x_1, x_2) \) then \( x^\beta = x_1^{\beta_1} x_2^{\beta_2} \). This choice of basis function, instead of \( B = \{x^\beta\}_{0 \leq |eta| \leq m} \), leads to a well-conditioned matrix \( A = PWP^T \) in the MLS approximation (Equation (2.2)) and \( A^{-1} \) is computed accurately, especially when the mesh-size is reduced. The effect of this variation on the conditioning of the MLS algorithm has been analytically investigated in [26].

We take \( m = 2 \) to span the trial spaces by the MLS approximation in both time and space domains. The local sub-domain's size is chosen as \( \sigma = 0.002 \). Numerical results in [27] show that it is a good choice. For integration over the local sub-domains in MLBIE (space) we use the Gauss integration formulas [12]. A fifteen points Gauss formula is employed for regular integrals and a seven points special Gauss formula for integrations with singular kernel \( \ln r \) in \([0, 1]\). For the time integration (to generate the matrix \( G \)), if the time domain is large it is better to use some more accurate quadratures. Here we use a fifty points Gauss formula. If the time domain is small we can use some fewer points quadrature formulas. Note that the integration in the time domain is based on calling the one-dimensional MLS shape functions which has extremely low cost in comparison to the integration in multi-dimensional space domain. Therefore, the cost of integration in time can be really ignored.

4.1. Problem 1. Consider a finite strip with a unidirectional variation of the thermal conductivity and diffusivity. The same exponential spatial variation for both parameters is taken

\[
\begin{align*}
\kappa(x) &= \kappa_0 \exp(\gamma x_1), \\
\alpha(x) &= \alpha_0 \exp(\gamma x_1),
\end{align*}
\]

with \( \alpha_0 = 0.17 \times 10^{-4} \text{ m}^2\text{s}^{-1} \) and \( \kappa_0 = 17 \text{ Wm}^{-1}\text{C}^{-1} \). Three different exponential parameters \( \gamma = 0.2, 0.5 \) and \( 1.0 \text{ cm}^{-1} \) \((20, 50 \text{ and } 100 \text{ m}^{-1})\) are selected in numerical calculation. On both opposite sides parallel to the \( x_2 \)-axis two different temperatures are prescribed. One side is kept to zero temperature and the other has the Heaviside step time variation i.e., \( u = TH(t) \) with \( T = 1 \text{ C}^0 \). On the lateral sides of the strip the heat flux vanishes. See Fig. 1. Such problem has been considered in [5] using MLBIE method with Laplace transform in time, and in [9] using an RBF based meshless collocation method with time difference approximation.
In numerical calculations, a square with a side-length \( a = 0.04 \) m and a \( 11 \times 11 \) regular node distribution is used for the MLS approximation in the space domain (Fig. 1). Also a regular node distribution in the time domain \( t \in [0, 60] \) with fill distance (mesh-size) \( h_{T,[0,t_F]} = 2 \) sec. is used. The values of unknown function \( u \) at arbitrary point \((x, t) \in \Omega \times [0, t_F]\) is approximated using equation (3.15), after solving \( U \) from (3.14).

The special case with an exponential parameter \( \gamma = 0 \) corresponds to a homogeneous material. In such a case an analytical solution is available

\[
   u(x, t) = \frac{Tx_1}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{T \cos n\pi}{n} \sin \frac{n\pi x_1}{a} \times \exp \left( -\frac{\alpha_0 n^2 \pi^2 t}{a^2} \right),
\]

which can be used to check the accuracy of the present numerical method. Numerical results are computed at three locations along the \( x_1 \)-axis with \( x_1/a = 0.25, 0.5 \) and \( 0.75 \). Results are depicted in Fig. 2. An excellent agreement between numerical and analytical results is obtained.

It is known that the numerical results are rather inaccurate at very early time instants and at points close to the application of thermal shocks. Therefore in Fig. 3 we have compared the numerical and analytical solutions at very early time instants. Numerical results are obtained for \( x_1/a = 0.5, t_F = 2 \) and \( h_{T,[0,t_F]} = 0.02 \) and then given for time-lengths \( 4 \times 10^{-3} \) (using Equation (3.15)) by full-rectangles in this figure. The absolute error is approximately bounded by \( 2 \times 10^{-4} \). The results are more accurate for points closer to the starting time. Besides, in Fig. 4 the numerical and analytical solutions at points very close to the application of thermal shocks are given and compared for \( t = 10 \) sec. The maximum absolute error is \( 1.45 \times 10^{-3} \). Consequently, the method gives accurate results in these cases.

The discussion above concerns heat conduction in homogeneous materials only since analytical solutions can be used for verification. To illustrate the application of the proposed algorithm, consider now the cases \( \gamma = 20, \) and \( 50 \) m\(^{-1} \), respectively. The variation of temperature with time for three \( \gamma \)-values and at position \( x_1 = 0.01 \) and \( 0.02 \) m are presented in Figs. 5 and 6, respectively.

Fig. 6 is in good agreement with Figure 11 presented in [9], but some differences are evident between Fig. 5 of this paper and Figure 4 of [5]. This disagreement has also been reported in Table 1 of [9]. Note that in this case the results of this paper are in agreement with Table 1 of [9].

For final steady state an analytical solution can be obtained as

\[
   u(x, t \to \infty) = T \frac{\exp(-\gamma x_1) - 1}{\exp(-\gamma a) - 1}, \Bigg( u \to T \frac{x_1}{a}, \text{as} \ \gamma \to 0 \Bigg).
\]

Analytical and numerical results computed at time \( t = 60 \) sec. corresponding to stationary or static loading conditions are presented in Fig. 7. Numerical results are in good agreement with analytical ones for the steady state temperatures.
As expected, it is found from Figs. 5, 6 and 7 that the temperature increases along with an increase in $\gamma$-values (or equivalently in thermal conductivity), and the temperature approaches a steady state when $t > 20$ sec.

4.2. **Problem 2.** In this example, an infinitely long and functionally graded thick-walled hollow cylinder is considered, where the following radii $R_1 = 8 \times 10^{-2}$ m and $R_2 = 10 \times 10^{-2}$ m are selected. This problem has been considered in [5] using MLBIE method, where Laplace transform employed to eliminate the time variable.

Heaviside step boundary condition is prescribed on the external surface of the hollow cylinder for the time-dependent thermal loading as a thermal shock with $T = 1 C^\circ$. The inner surface is kept at zero temperature. Due to the symmetry in geometry and boundary conditions it is sufficient to analyze only a quarter of the cross section of the hollow cylinder. The boundary conditions and the total nodes lying in the domain and on the global boundary are depicted in Fig. 8.

The thermal conductivity and diffusivity functions are chosen such as (4.1) with same constants $\alpha_0$ and $\kappa_0$. For comparison purpose, a homogeneous material ($\gamma = 0$) is first considered. In this homogeneous case, an analytical solution is known as:

$$u(r, t) = T \ln(r/R_1) \frac{\ln(R_2/R_1)}{\ln(R_2/R_1)} - \pi \sum_{n=1}^{\infty} T \frac{J_0^2(R_1 \beta_n) U_0(r \beta_n)}{J_0^2(R_1 \beta_n) - J_0^2(R_2 \beta_n)} \times \exp\left(-\alpha_0 \beta_n^2 t\right),$$

where

$$U_0(r \beta_n) = J_0(r \beta_n) Y_0(R_2 \beta_n) - J_0(R_2 \beta_n) Y_0(r \beta_n),$$

and $\beta_n$ are the roots of the following transcendental equation

$$J_0(r) Y_0(r R_2/R_1) - Y_0(r) J_0(r R_2/R_1) = 0,$$

with $J_0(r)$ and $Y_0(r)$ being Bessel functions of first and second kinds and zeroth order. In numerical results we use time step $h_{T,[0, t_F]} = 0.5$ sec. in the time domain $[0, 16]$.

Fig. 9 depicts the numerical and exact solutions for various times at $r = 0.09$ m. The results show an excellent agreement with the analytical solution.

Fig. 10 shows the variation of the temperature with radial coordinate at four different time instants.

Finally, we consider the functionally graded hollow cylinder with the thermal diffusivity and conductivity being graded in the radial direction. Numerical results for the time variation of the temperature are shown in Fig. 11. Similar to the results of a finite strip in the first example, the temperature level at interior points in the steady state increases with increasing $\gamma$-value. Here, again there are some differences between the results of this paper and the results presented in Figure 9 of [5] using MLBIE method and Laplace transform. It can be seen that, before getting the steady state solution, the results of this paper are larger than those obtained by MLBIE and Laplace transform. This disagreement also observed in the previous example. For $\gamma = 50$ m$^{-1}$ and 100 m$^{-1}$ the temperature grows rapidly and gets
the steady-state in lower times. To deal with this argument we use very fine time steps $h_T,[0,t_F]\] = 0.1$ and $h_T,[0,t_F]\] = 0.005$, respectively. The results are presented in Fig. 12 and 13. As we can see, in the first case for $t > 1$ sec., and in the second case for $t > 0.05$, we get the steady state solutions.

Finally Fig. 14 illustrates the variation of the temperature with the normalized radial coordinate $r/R_1$ for different choices of $\gamma$ at $t = 2$ sec.

5. Conclusion

Heat conduction problem in functionally graded materials (FGMs) is investigated in this paper using the truly meshless local boundary integral equation (MLBIE) method. This method is based on the moving least squares (MLS) approximation in trial space. We avoid using Laplace transform and time discretization methods to eliminate the time variable. We employ again MLS as an approximation method in the time domain. Up to the best of our knowledge, it is a new technique to treat the time variable (at least) in the field of meshless methods. Numerical results which are provided for a finite strip and a hallow cylinder, show the accuracy and efficiency of this technique. Although the process has been presented for the MLBIE method, it can be extend to other MLS based meshless methods, similarly.

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References


**Figure 1.** Boundary conditions and node distribution for a finite strip.

**Figure 2.** Time variation of the temperature in a finite strip at three positions with $\gamma = 0$. 
Figure 3. Accuracy of method for early time instants at position $x_1/a = 0.5$ of the functionally graded finite strip.

Figure 4. Accuracy of method for points close to the thermal shock at time $t = 10$ sec. of the functionally graded finite strip.
Figure 5. Time variation of the temperature at position $x_1/a = 0.25$ of the functionally graded finite strip.

Figure 6. Time variation of the temperature at position $x_1/a = 0.5$ of the functionally graded finite strip.
Figure 7. Distribution of temperature along $x_1$-axis for a functionally graded finite square strip under steady-state loading conditions.

Figure 8. Boundary conditions and node distribution for a hollow cylinder.
Figure 9. Exact and numerical solution at $r = 0.09$ m in hallow cylinder with homogeneous material properties ($\gamma = 0$).

Figure 10. Temperature variation with radial coordinate $r$ in hallow cylinder at different time levels with homogeneous material properties ($\gamma = 0$).
Figure 11. Time variation of the temperature at $r = 0.09$ m in a functionally graded hollow cylinder.

Figure 12. Time variation of the temperature at $r = 0.09$ m for $\gamma = 50$ m$^{-1}$ for a functionally graded hollow cylinder.
Figure 13. Time variation of the temperature at $r = 0.09$ m for $\gamma = 100$ m$^{-1}$ for a functionally graded hollow cylinder.

Figure 14. Temperature variation with radial coordinate $r$ at $t = 2$ sec. for a functionally graded hollow cylinder.