



The sequence reconstruction of permutations under hamming metric with small errors

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Abstract

The sequence reconstruction problem asks for the recovery of a sequence from multiple noisy copies, where each copy may contain up to r errors. In the case of permutations on n letters under the Hamming metric, this problem is closely related to the parameter $N(n, r)$, the maximum intersection size of two Hamming balls of radius r . While previous work has resolved $N(n, r)$ for small radii ($r \leq 4$) and established asymptotic bounds for larger r , we present new exact formulas for $r \in \{5, 6, 7\}$ using group action techniques. In addition, we develop a formula for $N(n, r)$ based on the irreducible characters of the symmetric group S_n , along with an algorithm that enables computation of $N(n, r)$ for larger parameters, including cases such as $N(43, 8)$ and $N(24, 14)$.

Keywords Sequence reconstruction · Hamming distance · Permutation codes

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1 Introduction and results

The sequence reconstruction problem, introduced by Levenshtein in 2001 [8, 9], involves reconstructing a transmitted sequence from multiple noisy copies received over distinct channels, each introducing at most r errors. Later, this problem has attracted interest in applications such as DNA storage [3, 7], racetrack memories [2], and communication systems. In combinatorial terms, a key challenge is to determine the maximum intersection size $N(n, r)$ of two metric balls of radius r centered at distinct sequences. For permutations under the Hamming metric, this problem remains central due to the relevance of permutation codes in flash memories [6], power-line communications [10, 13], and DNA storage [1, 12].

Main contributions of [14] resolves the sequence reconstruction problem for permutations under Hamming errors for small radii ($r \leq 4$) and provides asymptotic bounds for larger r . In particular, it is proved that $N(n, 2) = 3$ ($n \geq 3$) [14, Theorem 4]; $N(n, 3) = 4n - 6$ ($n \geq 3$) [14, Theorem 5]; and $N(n, 4) = 7n^2 - 31n + 36$ ($n \geq 4$) [14, Theorem 6]. For example, the latter implies that unique reconstruction requires $N(n, 4) + 1 = 7n^2 - 31n + 37$ distinct permutations at distance ≤ 4 .

Here by using some elementary results about action of groups on sets, we find an algorithm with input r and output $N(n, r)$. Applying the algorithm we find the exact values of $N(n, r)$ for $r \in \{5, 6, 7\}$, as follows:

Theorem 1.1 *The following hold:*

- (a) $N(n, 5) = \frac{32}{3}n^3 - 89n^2 + \frac{739}{3}n - 220$, for $n \geq 5$.
- (b) $N(n, 6) = \frac{181}{12}n^4 - \frac{401}{2}n^3 + \frac{11783}{12}n^2 - \frac{4153}{2}n + 1590$, for $n \geq 6$.
- (c) $N(n, 7) = \frac{607}{30}n^5 - \frac{4675}{12}n^4 + \frac{8807}{3}n^3 - \frac{129041}{12}n^2 + \frac{190471}{10}n - 12978$, for $n \geq 7$.

Furthermore, in Theorem 4.2, we provide a formula for computing $N(n, r)$ using the irreducible characters of the symmetric group S_n and an algorithm for computing $N(n, r)$ in this way. The results obtained from implementing this algorithm in GAP [11] are presented in Table 2, which displays the computed values of $N(n, r)$.

2 Preliminaries

Let n be a positive integer and let $[n] := \{1, \dots, n\}$. For any non-empty subset A of $[n]$ we denote by S_A the set of all permutations on $[n]$ such that they induce the identity map on $[n] \setminus A$; in particular $S_{[m]}$ is denoted by S_m for any $m \in [n]$. We denote the identity element of S_n by I_n . For $\sigma \in S_n$, $M(\sigma) := \{i \in [n] \mid i^\sigma \neq i\}$, where i^σ denotes the image of i under σ . We define T_r^n as $\{\sigma \in S_n \mid |M(\sigma)| \leq r\}$. The composition $\sigma\tau$ of two permutations $\sigma, \tau \in S_n$ is defined by $i^{\sigma\tau} = (i^\sigma)^\tau$ for any $i \in [n]$. We use cycle notation for

permutations: for distinct elements $a_1, a_2, \dots, a_k$, the cycle $(a_1 a_2 \dots a_k)$ denotes the permutation that maps $a_1 \mapsto a_2, a_2 \mapsto a_3, \dots, a_{k-1} \mapsto a_k$, and $a_k \mapsto a_1$, while fixing all other elements. It is well-known that every permutation in the S_n can be uniquely expressed as a product of disjoint cycles, except for changes in the order of the factors, so that the rule of commutation is the property of being disjoint. The cycle type of a permutation $\pi \in S_n$ is a sequence $1^{t_1}, \dots, n^{t_n}$, where t_i denotes the number of disjoint cycles of length i in the cycle decomposition of π . By convention, any term with $t_i = 0$ is omitted; this applies to 1^{t_1} as well, so fixed points are not shown. Note that for a permutation π with such a cyclic type we have $|M(\pi)| = \sum_{i=2}^n i \times t_i$.

For two permutations δ and σ , we denote $\delta^{-1}\sigma\delta$ by σ^δ which is called the conjugate of σ by δ . This defines an equivalence relation on S_n , whose equivalence classes are called conjugacy classes. The conjugacy class of a permutation $\sigma \in S_n$ is the set $\{\sigma^\delta \mid \delta \in S_n\}$. It is well known that two permutations in S_n are conjugate if and only if they have the same cycle type; therefore, each cycle type corresponds to a unique conjugacy class of S_n . For any $\pi \in S_n$ and $r, s \in [n]$ such that $r \leq s$, we denote the set $\{(\sigma, \tau) \in T_r^s \times T_r^s \mid \sigma\tau = \pi\}$ by $\Omega_{s,r}(\pi)$. We denote by $C_{S_A}(\sigma)$ the centralizer of σ in S_A , that is, $\{\delta \in S_A \mid \delta\sigma = \sigma\delta\}$.

For any two permutations $\sigma, \pi \in S_n$, the Hamming distance $d(\sigma, \pi)$ between σ and π is defined by $|M(\sigma\pi^{-1})|$. For a given $\sigma \in S_n$ and a non-negative integer r , the Hamming ball of radius r centered at σ is defined as $B_r^n(\sigma) := \{\pi \in S_n \mid d(\sigma, \pi) \leq r\}$. For integers d and r , define $I(n, d, r)$ as the maximum possible size of the intersection of two metric balls of radius r in S_n , whose centers are exactly at distance d :

$$I(n, d, r) := \max\{|B_r^n(\pi) \cap B_r^n(\tau)| : \pi, \tau \in S_n, d(\pi, \tau) = d\}.$$

Then $N(n, r)$, which represents the minimum number of channels needed to guarantee correct decoding of any transmitted permutation with up to r errors per channel, can be expressed as

$$N(n, r) := \max_{\pi, \tau \in S_n, d(\pi, \tau) \geq 2} |B_r^n(\pi) \cap B_r^n(\tau)| = \max_{d \geq 2} I(n, d, r).$$

The authors in [14] determined the exact values of $I(n, d, r)$ for $d = 2, 3, 4$ and for any $r \geq 2$. Moreover, they proposed the following conjecture:

Conjecture 2.1 [14] For any $r \geq 5$ and $n \geq r$, we have $N(n, r) = I(n, 2, r)$.

In [14], it is also shown that Conjecture 2.1 is valid for $r = 3, 4$ and for any $r \geq 5$ when n is large enough [14, Theorem 9] and [14, Lemma 3]. In this paper, we will prove the validity of this conjecture for $r = 5, 6$, and 7. Most of the following lemma, which will be used in later sections, has been proved in [15, Lemma 7]. It is presented here for the reader's convenience and to clarify its relation to the new notations.

Lemma 2.2 For any $r \in [n]$ and for all $\pi, \sigma \in S_n$, we have $|\Omega_{n,r}(\pi)| = |B_r^n(\pi) \cap B_r^n(I_n)|$ and $|\Omega_{n,r}(\pi^\sigma)| = |\Omega_{n,r}(\pi)| = |T_r^n\pi \cap T_r^n|$, where $T_r^n\pi := \{\rho\pi : \rho \in T_r^n\}$. In particular, if $\pi \in T_k^n$ for some $k \in [n]$, then $|\Omega_{n,r}(\pi)| = |\Omega_{n,r}(\pi')|$ for some $\pi' \in S_k$.

Proof The map defined from $\Omega_{n,r}(\pi)$ to $\Omega_{n,r}(\pi^\sigma)$ by $(\tau_1, \tau_2) \mapsto (\tau_1^\sigma, \tau_2^\sigma)$ is a bijection. The map defined from $T_r^n \pi \cap T_r^n$ to $\Omega_{n,r}(\pi)$ by $\tau \mapsto (\pi\tau^{-1}, \tau)$ is a bijection, to see the latter we need the fact that $M(\theta) = M(\theta^{-1})$ for all $\theta \in S_n$. The proof of the first part is complete with the fact that $B_r^n(\sigma) = T_r^n \sigma$, for all $\sigma \in S_n$.

The second part follows from the first equality of the first part and the fact that one can find a $\theta \in S_n$ such that $\pi^\theta \in S_k$ whenever $|M(\pi)| \leq k$. This completes the proof. $\square$

Remark 2.3 Since for any permutation σ with $|M(\sigma)| > 2r$ we have $\Omega_{n,r}(\sigma) = \emptyset$, and in view of [14, Lemma 3] and Lemma 2.2, it follows that if $\sigma_1, \dots, \sigma_\ell$ are representatives of those conjugacy classes in S_n whose elements move at most $2r$ points, then $N(n, r) = \max\{|\Omega_{n,r}(\sigma_i)| : 1 \leq i \leq \ell\}$. Moreover, for a fixed integer d with $0 \leq d \leq 2r$, if $\sigma'_1, \dots, \sigma'_k$ are representatives of those conjugacy classes in S_n whose elements move exactly d points, then $I(n, d, r) = \max\{|\Omega_{n,r}(\sigma'_i)| : 1 \leq i \leq k\}$. In particular, $I(n, 2, r) = |\Omega_{n,r}(\tau)|$, where τ is a transposition in S_n , i.e., a permutation that interchanges two elements and fixes all others.

We will need the concept of group action in the next section, and we recall it here. A right action of a group G on a set X is a map $X \times G \rightarrow X$, $(x, g) \mapsto x \cdot g$, such that $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G$ and $x \cdot I_n = x$ for all $x \in X$. For $x \in X$, the orbit of x is $\text{Orb}(x) = \{x \cdot g : g \in G\}$, and the stabilizer of x is $\text{Stab}_G(x) = \{g \in G : x \cdot g = x\}$.

3 Exact values $N(n, r)$ for $r = 5, 6, 7$

In this section, for a given integer $r \geq 2$, we prove several lemmas showing that for any $n \geq 2r$ and any $\sigma \in S_n$ with $|M(\sigma)| \leq 2r$, the computation of $|\Omega_{n,r}(\sigma)|$ in S_n can be reduced to the corresponding computation in S_{2r} . We then present an algorithm for determining $N(n, r)$, and use this algorithm to prove Theorem 1.1.

Lemma 3.1 Let $r \in [n]$ such that $2r \leq n$ and $\sigma\tau = \pi$, where $\sigma, \tau \in T_r^n$ and $\pi \in S_{2r}$. Then there exist $\sigma', \tau' \in T_r^{2r}$ such that $\sigma'\tau' = \pi$, where $\sigma' = \sigma^\delta$, $\tau' = \tau^\delta$ for some $\delta \in C_{S_n}(\pi)$.

Proof Let $\Delta := (M(\sigma) \cup M(\tau)) \cap [2r]$ and $\Delta' := (M(\sigma) \cup M(\tau)) \setminus \Delta$. Since $\sigma, \tau \in T_r^n$, we have $|M(\sigma) \cup M(\tau)| \leq 2r$, and hence $f := |\Delta'| \leq 2r - |\Delta|$. Note that if $f = 0$, then there is nothing to prove; hence, we may assume that $f > 0$. Choose a subset $\Lambda \subseteq [2r] \setminus \Delta$ such that $|\Lambda| = f$. Choose a permutation $\delta \in S_n$ such that $x^\delta \in \Lambda$ for all $x \in \Delta'$, $x^\delta \in \Delta'$ for all $x \in \Lambda$, and $x^\delta = x$ for all $x \in [n] \setminus (\Delta' \cup \Lambda)$. Such a permutation exists since $|\Delta'| = |\Lambda|$. Conjugating the equality $\sigma\tau = \pi$ by δ , we obtain $\sigma^\delta\tau^\delta = \pi^\delta$. Since $M(\pi) \subseteq \Delta$ and δ acts trivially on Δ , it follows that $\pi^\delta = \pi$, and hence $\delta \in C_{S_n}(\pi)$. By construction, all points in $M(\sigma^\delta)$ and $M(\tau^\delta)$ lie in $[2r]$. Moreover, since $|M(\sigma^\delta)| = |M(\sigma)|$ and $|M(\tau^\delta)| = |M(\tau)|$, $\sigma^\delta, \tau^\delta \in T_r^{2r}$. Setting $\sigma' = \sigma^\delta$ and $\tau' = \tau^\delta$ completes the proof. $\square$

The following example demonstrates the construction described in Lemma 3.1.

Example 3.2 Let $n = 12$ and $r = 5$. Consider the permutation $\pi = (1\ 3\ 8\ 6) \in S_{10}$. Let $\sigma = (1\ 3\ 8\ 11\ 12)$ and $\tau = (12\ 11\ 6\ 1)$ in S_{12} . Clearly, $\sigma\tau = \pi$, $\Delta := (M(\sigma) \cup M(\tau)) \cap [2r] = \{1, 3, 6, 8\}$ and $\Delta' := (M(\sigma) \cup M(\tau)) \setminus \Delta = \{11, 12\}$. We choose $\Lambda = \{2, 4\}$ and define $\delta = (11\ 2\ 12\ 4) \in S_{12}$ so that the conditions on Λ and δ stated in the proof of Lemma 3.1 are satisfied. It is easy to see that $\delta \in C_{12}(\pi)$ and if $\sigma' = \sigma^\delta = (1\ 3\ 8\ 2\ 4)$ and $\tau' = \tau^\delta = (4\ 2\ 6\ 1)$, then σ' and τ' are two permutation in T_5^{10} such that $\sigma'\tau' = \pi$.

For any $\pi \in S_n$, the centralizer $C_{S_n}(\pi)$ acts by conjugation on the set $\Omega_{n,r}(\pi)$ as follows:

$$(\sigma, \tau) \cdot \delta := (\sigma^\delta, \tau^\delta)$$

for all $\delta \in C_{S_n}(\pi)$ and all $(\sigma, \tau) \in \Omega_{n,r}(\pi)$: for it follows from $\sigma\tau = \pi$ that $\sigma^\delta\tau^\delta = \pi^\delta = \pi$; and since $|M(\eta)| = |M(\eta^\theta)|$ for all $\eta, \theta \in S_n$, we have $(\sigma^\delta, \tau^\delta) \in \Omega_{n,r}(\pi)$.

Lemma 3.3 Let $r \in [n]$ such that $2r \leq n$ and $\pi \in S_{2r}$. Then the number of orbits of the action by conjugation of $C_{S_n}(\pi)$ on $\Omega_{n,r}(\pi)$ is equal to the one of the action of $C_{S_{2r}}(\pi)$ by conjugation on $\Omega_{2r,r}(\pi)$.

Proof It follows from Lemma 3.1 that every orbit of the conjugation action of $C_{S_n}(\pi)$ on $\Omega_{n,r}(\pi)$ contains at least one element of $\Omega_{2r,r}(\pi)$. Hence, the number of orbits of the conjugation action of $C_{S_{2r}}(\pi)$ on $\Omega_{2r,r}(\pi)$ is at least as large as the number of orbits of the conjugation action of $C_{S_n}(\pi)$ on $\Omega_{n,r}(\pi)$. Now suppose that (σ, τ) and (σ', τ') are two distinct elements of $\Omega_{2r,r}(\pi)$ that belong to the same orbit under the action of $C_{S_n}(\pi)$ on $\Omega_{n,r}(\pi)$. Then there exists $\lambda \in C_{S_n}(\pi)$ such that $(\sigma^\lambda, \tau^\lambda) = (\sigma', \tau')$. Let $\Theta = [2r] \setminus (M(\sigma) \cup M(\tau))$, and $\Theta' = [2r] \setminus (M(\sigma') \cup M(\tau'))$. Since $M(\sigma') = \{i^\lambda \mid i \in M(\sigma)\}$ and $M(\tau') = \{i^\lambda \mid i \in M(\tau)\}$, we have $|\Theta| = |\Theta'|$. Hence there exists a bijection $\rho : \Theta \rightarrow \Theta'$. Define $\delta \in S_{2r}$ by

$$i^\delta = \begin{cases} i^\rho & \text{if } i \in \Theta, \\ i^\lambda & \text{if } i \in [2r] \setminus \Theta. \end{cases}$$

It is straightforward to verify that δ is a permutation of $[2r]$. We prove that $\sigma^\delta = \sigma'$. Let $i \in [2r]$. First suppose that $i \in \Theta'$. Then $i \notin M(\sigma')$, so $i^{\sigma'} = i$. Moreover $i^{\delta^{-1}} = i^{\rho^{-1}} \in \Theta$, and hence $i^{\rho^{-1}} \notin M(\sigma)$. Therefore $i^{\delta^{-1}\sigma\delta} = i^{\rho^{-1}\sigma\delta} = i^{\rho^{-1}\delta} = i^{\rho^{-1}\rho} = i$.

Thus $i^{\sigma'} = i^{\delta^{-1}\sigma\delta}$.

Now suppose that $i \notin \Theta'$. Since $i \in M(\sigma') \cup M(\tau')$, we distinguish two cases.

If $i \in M(\sigma')$, then $i^{\lambda^{-1}} \in M(\sigma)$. Hence $i^{\delta^{-1}\sigma\delta} = i^{\lambda^{-1}\sigma\lambda} = i^{\sigma'}$.

If $i \notin M(\sigma')$, then $i^{\lambda^{-1}} \notin M(\sigma)$. Since $i^{\lambda^{-1}} \notin \Theta$, we have $i^{\delta^{-1}\sigma\delta} = i^{\lambda^{-1}\sigma\delta} = i^{\lambda^{-1}\delta} = i^{\lambda^{-1}\lambda} = i = i^{\sigma'}$.

Thus $i^{\sigma^\delta} = i^{\sigma'}$ for all $i \in [2r]$, and hence $\sigma^\delta = \sigma'$. By a similar argument we obtain $\tau^\delta = \tau'$. Consequently, $\pi^\delta = (\sigma\tau)^\delta = \sigma^\delta\tau^\delta = \sigma'\tau' = \pi$, so $\delta \in C_{S_{2r}}(\pi)$, which completes the proof. $\square$

Lemma 3.4 Let $r \in [n]$ be such that $2r \leq n$, and let $\pi \in S_{2r}$. Suppose that $\{(\sigma_1, \tau_1), \dots, (\sigma_s, \tau_s)\}$ is a complete set of representatives of orbits of the action by conjugation of $C_{S_{2r}}(\pi)$ on $\Omega_{2r, \tau}(\pi)$. Then

$$|\Omega_{n, r}(\pi)| = \sum_{i=1}^s \frac{|C_{S_n}(\pi)|}{|C_{S_n}(\pi) \cap C_{S_n}(\sigma_i)|}.$$

Proof It follows from the fact that the underlying set of an action is partitioned by the orbits and the size of the orbit of $(\sigma, \tau) \in \Omega_{n, r}(\pi)$ is equal to $|C_{S_n}(\pi) : \text{Stab}_{C_{S_n}(\pi)}((\sigma, \tau))|$. It is easy to see that $\text{Stab}_{C_{S_n}(\pi)}((\sigma, \tau)) = C_{S_n}(\sigma) \cap C_{S_n}(\tau)$. Since $\sigma\tau = \pi$, it follows that $C_{S_n}(\sigma) \cap C_{S_n}(\tau) = C_{S_n}(\sigma) \cap C_{S_n}(\pi)$. Now Lemma 3.3 completes the proof. $\square$

We denote by $C_{S_A}(\sigma, \pi)$ the intersection of two centralizers $C_{S_A}(\sigma)$ and $C_{S_A}(\pi)$ that is $C_{S_A}(\sigma) \cap C_{S_A}(\pi)$.

Lemma 3.5 For any two permutations σ and π in S_n ,

$$C_{S_n}(\sigma, \pi) = C_{S_{M(\sigma) \cup M(\pi)}}(\sigma, \pi) \times S_{[n] \setminus (M(\sigma) \cup M(\pi))}.$$

Proof It suffices to prove that for each $\tau \in C_{S_n}(\sigma, \pi)$ and each $i \in M(\sigma) \cup M(\pi)$, we have $i^\tau \in M(\sigma) \cup M(\pi)$. Suppose that $i^\tau = j \in [n] \setminus (M(\sigma) \cup M(\pi))$. Without loss of generality, we may assume that $i \in M(\sigma)$. Since $\tau \in C_{S_n}(\sigma)$, we have $i^{\sigma\tau} = i^{\tau\sigma}$. Suppose that $i^\sigma = k \in M(\sigma) \setminus \{i\}$. Then we would have $i^\tau = k^\tau = j$, which is a contradiction. This completes the proof. $\square$

Lemma 3.6 Let $r \in [n]$ such that $2r \leq n$ and $\pi \in S_{2r}$. Suppose that $\{(\sigma_1, \tau_1), \dots, (\sigma_s, \tau_s)\}$ is a complete set of representatives of orbits of the action by conjugation of $C_{S_{2r}}(\pi)$ on $\Omega_{2r, \tau}(\pi)$. Then

$$|\Omega_{n, r}(\pi)| = \sum_{i=1}^s \frac{|C_{S_{M(\pi)}}(\pi)|}{|C_{S_{M(\pi) \cup M(\sigma_i)}}(\pi, \sigma_i)|} \cdot \frac{(n - |M(\pi)|)!}{(n - |M(\pi) \cup M(\sigma_i)|)!}. \quad (3.1)$$

Proof Since $M(I_n) = \emptyset$ and by Lemma 3.5,

$$C_{S_n}(\pi) = C_{S_n}(\pi, I_n) = C_{S_{M(\sigma) \cup \emptyset}}(\pi) \times S_{[n] \setminus (M(\sigma) \cup \emptyset)} = C_{S_{M(\sigma)}}(\pi) \times S_{[n] \setminus M(\sigma)}.$$

Hence the proof follows from Lemmas 3.4 and 3.5. $\square$

In view of the above results, if $n \geq 2r$, the value of $N(n, r)$ can be obtained from Remark 2.3 together with the relation (3.1). If $r \leq n \leq 2r - 1$, then $N(n, r)$ is directly obtained from $N(n, r) = \max\{|\Omega_{n, r}(\sigma_i)| : 1 \leq i \leq \ell\}$, where $\sigma_1, \dots, \sigma_\ell$ denote a complete set of representatives of the conjugacy classes of S_n . Based on these observations, we present Algorithm 1 for computing $N(n, r)$ for a given $r \leq n$.

Algorithm 1 Computation of $N(n, r)$ for a given $r \leq n$

-
- 1: **Input:** $r \geq 2$
 - 2: **Output:** $N(n, r)$
 - 3: compute T_r^m for each $m \in [2r] \setminus [r - 1]$
 - 4: compute a complete set R_m of representatives of non-trivial conjugacy classes of elements of S_m for each $m \in [2r] \setminus [r - 1]$
 - 5: compute $\Omega_{m,r}(\pi)$ for each $\pi \in R_m$ and all $m \in [2r] \setminus [r - 1]$
 - 6: compute the set $\mathcal{O}_r$ consisting of the orbits $O_{\pi,r}$ of the action of $C_{S_{2r}}(\pi)$ on $\Omega_{2r,r}(\pi)$ for each $\pi \in R_{2r}$
 - 7: compute a set $\mathcal{RO}_r$ consisting of sets $RO_{\pi,r}$ of representatives of $O_{\pi,r}$ for each $\pi \in R_{2r}$
 - 8: compute the set of polynomials (in n) $|\Omega_{n,r}(\pi)| = \sum_{(\sigma,\tau) \in RO_{\pi,r}} \frac{|C_{S_M(\pi)}(\pi)|}{|C_{S_M(\pi) \cup M(\sigma)}(\pi, \sigma)|} \cdot \frac{(n - |M(\pi)|)!}{(n - |M(\pi) \cup M(\sigma)|)!}$ for each $RO_{\pi,r} \in \mathcal{RO}_r$.
 - 9: Now $N(n, r) = \begin{cases} \max\{|\Omega_{n,r}(\pi)| : \pi \in R_{2r}\} & \text{if } n \geq 2r \\ \max\{|\Omega_{n,r}(\pi)| : \pi \in R_n\} & \text{if } r \leq n \leq 2r - 1. \end{cases}$
-

The GAP [11] program corresponding to Algorithm 1 is given in Appendix A. Also, in Table 1, we present the results obtained from Algorithm 1 for $r = 5, 6, 7$ and $r \leq n \leq 2r - 1$, where each row corresponds to a conjugacy class of S_n represented by its cycle type, together with the associated value of $|\Omega_{n,r}(\sigma)|$ for a representative σ of that class. It is worth noting that $N(n, n)$ is clearly equal to $n!$, and this value is also included in the table for completeness. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 We consider $r = 5, 6$, or 7 . Based on the values of $N(n, r)$ obtained from Algorithm 1, which are listed in Table 1, it can be verified that for $n = r, r + 1, \dots, 2r - 1$, the corresponding values of $N(n, r)$ satisfy the relations in parts (i), (ii), and (iii) for $r = 5, 6, 7$, respectively.

Now assume that $n \geq 2r$. In this case, the number of conjugacy classes in S_n whose elements move at most $2r$ points equals the number of conjugacy classes in S_{2r} , which is exactly 41, 77, and 135 for $r = 5, 6$, and 7 , respectively. By running Algorithm 1 in GAP [11], we obtain a list of 41, 77, and 135 polynomial functions in n for $r = 5, 6, 7$, respectively, where each function represents the value of $|\Omega_{n,r}(\sigma)|$ for some conjugacy class representative σ . The GAP code and the resulting functions are provided in Appendix A.

To identify which of these functions yields the maximum value, we import the list into MATLAB [5] and use the “isAlways” command to perform symbolic comparisons. These comparisons show that, for $n \geq 2r$, the function corresponding to transpositions provides the largest value. Therefore, the formulas in parts (i), (ii), and (iii) hold for all $n \geq r$, completing the proof. $\square$

Remark 3.7 Note that, according to the proof of Theorem 1.1, for $r \in \{5, 6, 7\}$ we have $N(n, r) = |\Omega_{n,r}(\tau)|$, where τ is a transposition. Therefore, in view of Remark 2.3, Conjecture 2.1 holds for these values of r . It is worth noting that Algorithm 1 for computing $N(n, r)$

Table 1 Computation of $N(n, r)$ using Algorithm 1 for $r \in \{5, 6, 7\}$ and $n \in [2r] - [r - 1]$

Cycle Type	$r = 5$					$r = 6$					$r = 7$					(13,7)		
	(5,5)	(6,5)	(7,5)	(8,5)	(9,5)	(6,6)	(7,6)	(8,6)	(9,6)	(10,6)	(11,6)	(7,7)	(8,7)	(9,7)	(10,7)		(11,7)	(12,7)
2	120	358	802	1516	2564	720	2612	6946	15234	29350	51530	5040	21514	66222	165318	357458	696228	1252572
3	120	327	678	1206	1944	720	2409	5931	12189	22245	37320	5040	20013	57216	133797	273402	507102	874320
4	120	304	576	936	1384	720	2248	5128	9784	16640	26120	5040	18768	50064	109560	210360	368040	600648
5	120	285	465	660	870	720	2115	4360	7510	11620	16745	5040	17715	43645	87995	156195	253940	387190
6	120	270	354	438	522	720	2004	3594	5490	7692	10200	5040	16818	37518	68604	111540	167790	238818
7		238	245	252	252	720	1911	2821	3752	4704	5677	5040	16051	31472	51380	75852	104965	138796
8			98	98	98	720	2032	2240	2448	2448	2656	5040	15392	25376	36112	47600	59840	72832
9				2	2	720	909	909	918	918	927	5040	19179	22608	26064	29547	33057	33057
10						720	170	170	170	170	170	5040	10570	10990	11410	11830	11830	11830
11						720					11	5040		2706	2706	2717	2728	2728
12																312	312	312
13																		13
2 ²	120	306	586	964	1444	720	2252	5162	9914	16990	26890	5040	18786	50242	110358	212838	374228	614004
2 ³		270	342	414	486	720	2004	3582	5454	7620	10080	5040	16830	37518	68532	111300	167250	237810
2 ⁴				102	102	720		2062	2302	2548	2800	5040	15406	25390	36340	48268	61186	75106
2 ⁵										260	260	5040			10660	11260	11860	12460
3 ²	120	272	356	442	530	720	2004	3602	5518	7756	10320	5040	16820	37540	68698	111810	168410	240050
3 ³					18			948	966	966	984	5040		19200	22656	26166	29730	33348
4 ²				98	98	720		2050	2258	2468	2680	5040	15394	25378	36132	47660	59966	73054
5 ²																		
3, 2	120	286	466	660	868	720	2116	4366	7528	11660	16820	5040	17722	43690	88150	156590	254780	388772
4, 2	120		350	430	510	720		2026	2226	2426	2626	5040	16822	37518	68580	111460	167610	238482
5, 2			242	252	262	720	1912	2822	3762	4732	5732	5040	16054	31470	51398	75948	105230	139354
6, 2				72	72	720		2026	2242	2458	2674	5040	15394	25378	36154	47722	60082	73234
7, 2					14	720			924	938	952	5040		19182	22626	26112	29640	33210
8, 2										176	176	5040			10576	11056	11536	12016
9, 2											18	5040				2790	2808	2826
4, 3	120		230	238	246	720	1910	2824	3756	4706	5674	5040	16052	31474	51382	75852	104960	138782

Table 1 (continued)

Cycle Type	$r = 5$					$r = 6$					$r = 7$							
	(5,5)	(6,5)	(7,5)	(8,5)	(9,5)	(6,6)	(7,6)	(8,6)	(9,6)	(10,6)	(11,6)	(7,7)	(8,7)	(9,7)	(10,7)		(11,7)	(12,7)
5, 3				82	82	720		2032	2232	2432	2632	5040	15392	25368	36086	47546	59748	72692
6, 3					12	720			922	934	946	5040		19174	22636	26144	29698	33298
7, 3										154	154	5040			10544	10964	11384	11804
8, 3											16	5040				2736	2752	2768
5, 4					2				882	892	902	5040		19164	22602	26062	29544	33048
6, 4										176	176	5040			10576	10984	11392	11800
7, 4											2	5040				10964	10964	10964
3, 2, 2		270	238	252	266	720	1912	2826	3776	4762	5784	5040	16058	31470	51418	76044	105490	139898
4, 2, 2				82	82	720	2038		2262	2488	2716	5040	15398	25382	36216	47904	60450	73858
5, 2, 2					12	720		2032		932	952	5040		19170	22638	26158	29730	33354
6, 2, 2										188	188	5040			10588	11116	11644	12172
7, 2, 2											28	5040				10932	10932	10932
4, 3					10	720			910	928	946	5040		19150	22660	26236	29878	33586
5, 3										192	192	5040			10622	11102	11584	12068
6, 3											14	5040			10576	10984	11384	11804
3 ² , 2	120			76	76	720	15394			11044		5040	15394	25362	36102	47614	59698	72954
4, 3 ²										220	220	5040			10680	11208	11740	12276
5, 4, 2										202	202	5040			10642	11062	11484	11908
3, 2 ³					6	720			886	916	946	5040		19150	22600	26144	29698	33298
4, 2 ³										212	212	5040			10612	11164	11716	12268
5, 2 ³											32	5040				2972	3014	3056
3 ² , 3									948			5040		19186	22624	26098	29608	33154
4, 3, 2 ²										144	144	5040			10524	10524	11340	11748
5, 3, 2 ²											12	5040				2702	2724	2746
3 ² , 2 ²										202	202	5040			10642	11116	11584	12068
4 ² , 2										188	188	5040			10588	11044	11500	11956

for $r \geq 8$ was not feasible in GAP [11], since computing the values of T_r^n and the orbits of the action specified in the algorithm became infeasible for such r . In the next section, we present a method that enables the computation of $N(n, r)$ for $r \geq 8$, even up to $r = 14$, for small values of n .

4 Computation of $N(n, r)$ based on irreducible characters of S_n

In this section, we present a formula for computing $N(n, r)$, which is based on the irreducible characters of the symmetric group S_n . We then provide an algorithm implementing this formula. One of the main advantages of this approach is that it allows the computation of $N(n, r)$ for larger values of the parameters. For instance, quantities such as $N(43, 8)$ or $N(24, 14)$ can be determined using this method. At the end of this section, we present these computations in Table 2.

We begin by reviewing the necessary notation. Let $\text{Irr}(S_n)$ denote the set of all complex irreducible characters of S_n , and let $C_1, \dots, C_m$ denote the conjugacy classes of S_n . For each $i \in [m]$ and each character $\chi \in \text{Irr}(S_n)$, we define $\chi_i := \chi(\sigma)$, where $\sigma \in C_i$. Since characters are constant on conjugacy classes, the value χ_i is well-defined.

Lemma 4.1 *Let $n \geq 3$ and $2 \leq r \leq n$ be integers and let $\sigma \in S_n$. Let $C_1, \dots, C_m$ denote the conjugacy classes of S_n , where C_1 is the class containing the identity element. If $\sigma \in C_s$, then*

$$|\Omega_{n,r}(\sigma)| = \sum_{i,j \in \Delta_r^n} \frac{|C_i||C_j|}{n!} \sum_{\chi \in \text{Irr}(S_n)} \frac{\chi_i \chi_j \chi_s}{\chi_1},$$

Table 2 $N(n, r)$ obtained from Algorithm 2 for $8 \leq r \leq 14$

$n \setminus r$	8	9	10	11	12	13	14
8	40320						
9	197864	362880					
10	691886	2012014	3628800				
11	1937162	7877738	22428812	39916800			
12	4645488	24447408	97202778	272082658	479001600		
13	9940944	64396224	331165914	1293005254	3569113616	6227020800	
14	19493964	150186636	950495706	4797853066	18454503238	50349389446	87178291200
15	35674212	318841668	2399645250	14903556670	74082374082	281399134434	760174857236
16	61722264	628057176	5483265534	40494217510	247620078452	1215098293428	4566528353042
17	101940096	1163818056	11567835966	99095215906	720472798732	4348516104892	21105666402962
18	161900378	2049683418	22857719238	222920301958	1879927189494	13489665769206	80518266961486
19	248674574	3457905742	42761973402	467894961682	4492054545698	37386313854882	265283557908634
20	371079848	5622549032	76369870820	926635847380	9980994911416	94566233135032	778257965296288
21	539944776	8854770984	131054690436	1746560045900	20861318069976	221751258851064	2077485960431616
22	768393864	13560434184	217226966604	3154509431084	41384025479236	487806116103876	5127142870055256
23	1072150872	20260211352	349259994492	5489504307332	78473125853804	1015959516165548	11841078608718768
24	1469860944	29612349648	546612008868	9245486999828	143048793168360	2018021573791848	25833283577408932
25	1983431544	42438259056	835171069860	15126179983580	251855106281752	3845552935810104	—
26	2638392198	59751089862	1248850306068	24114464568020	429935481079172	—	—
27	3464273042	82787464242	1831462782192	37558985066492	713927345918412	—	—
28	4495002176	113042526976	2638906875126	57280999780526	1156379586164896	—	—
29	5769321824	152308480304	3741694659254	85704834017354	—	—	—
30	7331223300	202716767940	5227857418470	126015641735670	—	—	—
31	9230400780	266784073260	7206264019230	182348551279170	—	—	—
32	11522723880	347462296680	9810389495730	260013657009930	—	—	—
33	14270729040	448192677240	13202572815090	365761722494190	—	—	—
34	17544129714	572964223410	17578804407210	508095882221610	—	—	—
35	21420345366	726376618134	23174085660750	697635067655550	—	—	—
36	25985049272	913707763128	30268404203472	947535339716400	—	—	—
37	3132735128	1140986127448	39193370401968	1273975783592448	—	—	—
38	37567302464	1415068065344	50339562132584	—	—	—	—
39	44802660864	1743720268416	64164626492136	—	—	—	—
40	53163352992	2135707517088	8120218873800	—	—	—	—
41	62785196424	—	—	—	—	—	—
42	73815944286	—	—	—	—	—	—
43	86415964698	—	—	—	—	—	—

where Δ_r^n are the sets of indices corresponding to classes whose elements move at most r points.

Proof According to the definition of $\Omega_{n,r}(\sigma)$, we have

$$|\Omega_{n,r}(\sigma)| = \left| \bigcup_{i,j \in \Delta_r^n} \{(\tau, \rho) \in \mathcal{C}_i \times \mathcal{C}_j \mid \tau\rho = \sigma\} \right|.$$

Let $i, j, s \in [m]$. For each $\sigma \in \mathcal{C}_s$, the number of pairs $(x, y) \in \mathcal{C}_i \times \mathcal{C}_j$ such that $xy = \sigma$ is clearly constant; we denote this number by $\mathcal{C}_{i,j}^s$. Hence, $|\Omega_{n,r}(\sigma)| = \sum_{i,j \in \Delta_r^n} \mathcal{C}_{i,j}^s$. Moreover, due to the fact that for every $\tau \in S_n$, τ and τ^{-1} are into the same conjugate class and in view of [4, Lemma 2.15 and problem 3.9], for all $i, j, s \in [m]$, we have

$$\mathcal{C}_{i,j}^s = \frac{|\mathcal{C}_i||\mathcal{C}_j|}{n!} \sum_{\chi \in \text{Irr}(S_n)} \frac{\chi_i \chi_j \chi_s}{\chi_1}.$$

This completes the proof. $\square$

Theorem 4.2 Let $n \geq 3$ and $2 \leq r \leq n$ be integers and let $\mathcal{C}_1, \dots, \mathcal{C}_m$ denote the conjugacy classes of S_n , where $\mathcal{C}_1$ is the class containing the identity element. Then

$$N(n, r) = \max_{s \in \Delta_{2r}^n} \sum_{i,j \in \Delta_r^n} \frac{|\mathcal{C}_i||\mathcal{C}_j|}{n!} \sum_{\chi \in \text{Irr}(S_n)} \frac{\chi_i \chi_j \chi_s}{\chi_1}, \quad (4.1)$$

where Δ_r^n and Δ_{2r}^n are the sets of indices corresponding to classes whose elements move at most r and $2r$ points, respectively.

Proof The result follows from Lemma 4.1 and Remark 2.3. $\square$

Algorithm 2 is presented for computing $N(n, r)$ for given values of n and r as expressed in relation (4.1). As specified in the algorithm, the output of `ComputeClassOmegaList` is a list of ordered pairs, where the first element of each pair is a representative of a conjugacy class that moves at most $2r$ elements, and the second element is the value of $\Omega_{n,r}$ for that permutation. The function `computeN` returns a list of pairs from the list produced by `computelistomega` whose second components are equal to the maximum value among all second components in the list. In fact, the second component of each element in the output corresponds to $N(n, r)$, while the first component identifies the conjugacy class whose representative achieves this value of $\Omega_{n,r}$.

By running Algorithm 2 in GAP [11], the values of $N(n, r)$ were computed as summarized in Table 2. In most cases the computations were performed quickly, and even in the most time-consuming instances the running time did not exceed three days; entries in Table 2 marked with “—” indicate cases where the computation required more time. Because of the large number of conjugacy classes whose representatives move at most $2r$ points, we did not report the values of $\Omega_{n,r}(\sigma)$ for each σ . It is worth noting that all results in Table 2 confirm Conjecture 2.1; in other words, in every case, $N(n, r)$ coincides with $I(n, 2, r) = |\Omega_{n,r}(\tau)|$, where τ is a transposition in S_n .

Algorithm 2 Compute $N(n, r)$ as obtained from Equation (4.1)

```

1: Input: Integer numbers  $n$  and  $2 \leq r \leq n$ 
2: Output: A list of pairs  $(\sigma, N(n, r))$ , where each  $\sigma$  is a representative of a
   conjugacy class whose  $\Omega_{n,r}(\sigma)$  attains the value  $N(n, r)$ .
3: function SUMCHAR( $G, i, j, \ell$ )
4:    $total \leftarrow 0$ 
5:   for all  $\chi \in \text{Irr}(G)$  do
6:      $total \leftarrow total + \frac{\chi_i \cdot \chi_j \cdot \chi_\ell}{\chi_1}$ 
7:   end for
8:   return  $total$ 
9: end function
10: function COMPUTECLASSOMEGALIST( $n, r$ )
11:    $G \leftarrow$  the symmetric group  $S_n$ 
12:    $C \leftarrow$  a list of conjugacy classes of  $G$ 
13:    $\Delta_r \leftarrow \{\ell \in [|C|] : C[\ell] \subseteq T_r^n\}$   $\triangleright$  Indices of classes moving at most  $r$  points
14:    $\Delta_{2r} \leftarrow$  a list of the indices of the nontrivial conjugacy classes of  $G$  that
     move at most  $2r$  points.
15:    $resultList \leftarrow []$   $\triangleright$  Stores computed pairs  $(\sigma_i, |\Omega_{n,r}(\sigma_i)|)$ 
16:   for all  $\ell \in \Delta_{2r}$  do
17:      $N_\ell \leftarrow 0$ 
18:     for all  $i \in \Delta_r$  do
19:       for all  $j \in \Delta_r$  do
20:          $w \leftarrow \frac{|C[i]| \cdot |C[j]|}{|G|}$ 
21:          $s \leftarrow \text{SumChar}(G, i, j, \ell)$ 
22:          $N_\ell \leftarrow N_\ell + w \cdot s$ 
23:       end for
24:     end for
25:     Append (an representative of  $C[\ell], N_\ell$ ) to  $resultList$ 
26:   end for
27:   return  $resultList$ 
28: end function
29: function COMPUTEN( $n, r$ )
30:    $data \leftarrow \text{ComputeClassOmegaList } n, r)$ 
31:   return The list of pairs in  $data$  whose second components attain the max-
     imum value among all second components of pairs in  $data$ .
32: end function

```

Appendix A

The following GAP [11] code computes $N(n, 5)$ using Algorithm 1 for $n \geq 10$:

```

u:=function(sigma,tau)
return Union(MovedPoints(sigma),MovedPoints(tau));
end;;
su:=function(sigma,tau)
return Size( u(sigma,tau));
end;;
csu:=function(sigma,tau)
return Size(Centralizer(SymmetricGroup(u(sigma,tau)),Group(sigma,tau)));
end;;
n:=Indeterminate(Rationals,"n");
p:=function(sigma,tau)
local b,l,ll,i;
l:=su(sigma,()); ll:= su(sigma,tau);
if l=ll then b:=1; fi;
if l<ll then b:=Product([l..ll-1],i->n-i); fi;
return b;
end;;
CU:=function(sigma,tau)
return csu(sigma,())/csu(sigma,tau)*p(sigma,tau);
end;;
S10:=SymmetricGroup(10);
C10:=ConjugacyClasses(S10);
T5:=Union(Filtered(C10,x->Size(MovedPoints(Representative(x)))<=5));;
c:=List(C10,Representative);;
F:=List(c,a->[a,Filtered(T5,x->x^-1*a in T5)]);;
F2:=List(F,X->[X[1],Orbits(Centralizer(S10,X[1]),X[2])]);;
F3:=List(F2,X->[X[1],List(X[2],Representative)]);;
F4:=Difference(F3,[F3[1]]);;
List(F4,X->Sum(X[2],x->CU(X[1],x)));

```

In the following, we present the result of the above program in GAP:

```

[32/3*n^3-89*n^2+739/3*n-220, 2/3*n^3+35*n^2-779/3*n+460, 72*n-162,
102,
0, 11/2*n^3-27*n^2+7/2*n+90, 7*n^2+89*n-500, 14*n+140, 6, n^2+71*n
-190,
76, 4, 18, 44*n^2-300*n+520, 80*n-210, 82, 0, 8*n+174, 10, 0, 98, 0,
15/2*n^2+165/2*n-480, 10*n+172, 12, 82, 2, 2, 2, 84*n-234, 72, 0,
12,
0, 7*n+189, 14, 0, 80, 0, 9, 0 ]

```

By running a similar program in GAP to compute $N(n, 6)$ and $N(n, 7)$, we obtained the following results, respectively.

```
[ 181/12*n^4-401/2*n^3+11783/12*n^2-4153/2*n+1590,
  3/4*n^4+329/6*n^3-2735/4*n^2+15769/6*n-3250,
  147*n^2-627*n-810, 3*n^2+189*n+358, 260, 20,
  53/8*n^4-193/4*n^3-197/8*n^2+3265/4*n-1455,
  29/3*n^3+224*n^2-8231/3*n+7030, 18*n^2+644*n-3478,
  30*n+616, 26, 2/3*n^3+143*n^2-1979/3*n-614,
  200*n+426, 220, 2, 18*n+786, 6, 6,
  212/3*n^3-808*n^2+9172/3*n-3800, 151*n^2-679*n-642,
  n^2+207*n+318, 212, 8, 9*n^2+779*n-3984, 18*n+748,
  18, 144, 2, n^2+191*n+458, 188, 4, 22, 0,
  55/6*n^3+465/2*n^2-8375/3*n+7120, 15*n^2+685*n-3618,
  20*n+732, 32, 200*n+432, 192, 0, 12, 10*n+792,
  12, 0, 202, 0, 153*n^2-705*n-558, 216*n+298, 188, 2,
  12*n+814, 14, 2, 176, 2, 2, 2, 21/2*n^2+1505/2*n-3871,
  14*n+798, 28, 154, 0, 14, 0, 208*n+368, 176, 0, 16, 0,
  9*n+828, 18, 0, 170, 0, 11, 0 ]
```

```
[ 607/30*n^5-4675/12*n^4+8807/3*n^3-129041/12*n^2+190471/10*n-12978,
  11/15*n^5+327/4*n^4-8917/6*n^3+37061/4*n^2-730247/30*n+22582, 238*
  n^3-1263*n^2-9487*n+51702,
  2*n^3+429*n^2+2257*n-31130, 600*n+4660, 740, 0, 309/40*n^5-615/8*n
  ^4-1031/8*n^3+31551/8*n^2-308017/20*n+18543,
  47/4*n^4+2809/6*n^3-36467/4*n^2+302381/6*n-89810, 71/3*n^3+1629*n
  ^2-52250/3*n+39018, 33*n^2+2883*n-9470, 66*n+2246,
  20, 3/4*n^4+1345/6*n^3-4795/4*n^2-56365/6*n+50848, 386*n^2+3406*n
  -36558, 2*n^2+486*n+5620, 494, 12,
  27*n^2+2943*n-9474, 36*n+2292, 42, 438, 0, 103*n^4-1730*n^3+10649*
  n^2-28222*n+26880, 242*n^3-1351*n^2-8851*n+50190,
  2/3*n^3+407*n^2+8761/3*n-34354, 552*n+5092, 536, 0, 38/3*n^3+1901*
  n^2-58931/3*n+45052, 23*n^2+3025*n-9914,
  38*n+2410, 32, 408*n+6444, 398, 4, 6, 2/3*n^3+367*n^2+10801/3*n
  -37238, 456*n+6028, 436, 0, 22*n+2546, 20, 4, 240,
  0, 265/24*n^4+5825/12*n^3-222565/24*n^2+612325/12*n-90755, 55/3*n
  ^3+1761*n^2-55498/3*n+41958, 26*n^2+2974*n-9702,
  42*n+2510, 38, 371*n^2+3669*n-37704, n^2+459*n+5932, 414, 6, 22*n
  +2460, 34, 0, 11*n^2+3229*n-10788, 22*n+2510, 22,
  294, 2, 26, n^2+399*n+6552, 384, 4, 26, 0, 244*n^3-1395*n^2-8533*n
  +49434, 396*n^2+3252*n-35966, 528*n+5308, 434, 0,
  18*n^2+3096*n-10136, 24*n+2492, 38, 350, 0, 408*n+6496, 386, 0,
  14, 0, 12*n+2510, 14, 0, 362, 0,
  77/6*n^3+1897*n^2-117677/6*n+44975, 21*n^2+3045*n-9924, 28*n+2592,
  44, 420*n+6344, 366, 2, 16, 14*n+2578, 16, 2,
  324, 2, 2, 2, 376*n^2+3592*n-37408, 480*n+5776, 384, 0, 16*n+2560,
  32, 0, 288, 0, 16, 0, 27/2*n^2+6345/2*n-10467,
  18*n+2592, 36, 306, 0, 18, 0, 420*n+6370, 360, 0, 20, 0, 11*n
  +2585, 22, 0, 312, 0, 13, 0 ]
```

Here is a code written in GAP [11] to compute $N(n, r)$ from Algorithm 2:

```

sumchar:= function(G,i,j,s,NCju)
local A,irr,x;
irr:=Irr(G);
A:=[];
for x in [1..NCju] do
Add(A, (irr[x][i]*irr[x][j]*irr[x][s])/irr[x][1]);
od;
return (Sum(A));
end;
N:=function(n,r)
local G,Cju,NCju,Ir,I2r,H,k,list,c,l,i,j,cijl;
G:=SymmetricGroup(n);
Cju:=ConjugacyClasses(G);
NCju:=NrConjugacyClasses(G);
Ir:=Filtered([1..NCju],t->NrMovedPoints(Representative(Cju[t]))<=r);
I2r:=Filtered([2..NCju],t->NrMovedPoints(Representative(Cju[t]))<=2*r);
resultlist:=[];
for l in I2r do
k:=0;
for i in Ir do
for j in Ir do
cijl:=((Size(Cju[i])*Size(Cju[j]))/Size(G))*sumchar(G,i,j,l,NCju);
k:=k+cijl;
od;
od;
Add(resultlist,[Representative(Cju[l]),k]);
od;
return(resultlist);
end;
ComputN:=functiona(n,r)
local data,max, maxdata;
data:=nr(n,r);
max:=Maximum(List(data,i->i[2]));
maxdata:=Filtered(data,i->i[2]=max);
return(maxdata);
end;

```

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval and Consent to participate Not applicable. The current manuscript does not report on or involve the use of any animal or human data or tissue.

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