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## Nilpotent probability of compact groups

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## ABSTRACT

Let  $k$  be any positive integer and  $G$  a compact (Hausdorff) group. Let  $\mathbf{np}_k(G)$  denote the probability that  $k+1$  randomly chosen elements  $x_1, \dots, x_{k+1}$  satisfy  $[x_1, x_2, \dots, x_{k+1}] = 1$ . We study the following problem: If  $\mathbf{np}_k(G) > 0$  then, does there exist an open nilpotent subgroup of class at most  $k$ ? The answer is positive for profinite groups and we give a new proof. We also prove that the connected component  $G^0$  of  $G$  is abelian and there exists a closed normal nilpotent subgroup  $N$  of class at most  $k$  such that  $G^0 N$  is open in  $G$ . In particular, a connected compact group  $G$  with  $\mathbf{np}_k(G) > 0$  is abelian.

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## 1. Introduction and results

Let  $G$  be a compact (Hausdorff) group and denote by  $\mathbf{m}_G$  the (unique) Haar measure on  $G$ . The following question is proposed in [9, Question 1.20.]:

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**Question 1.1.** Let  $G$  be a compact group, and suppose that

$$\mathcal{N}_k(G) := \{(x_1, \dots, x_{k+1}) \in G^{k+1} \mid [x_1, \dots, x_{k+1}] = 1\},$$

has positive Haar measure in  $G^{k+1}$ . Does  $G$  have an open  $k$ -step nilpotent subgroup?

Here  $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$  and  $[x_1, \dots, x_{k+1}]$  is inductively defined by  $[[x_1, \dots, x_k], x_{k+1}]$  for all  $k > 1$ . The Haar measure of  $\mathcal{N}_k(G)$  is called the degree of  $k$ -nilpotence of the group  $G$ .

Denote by  $\text{PA}_k$  the class of compact groups giving positive answer to Question 1.1. In [8] it is proved that  $\text{PA}_1$  contains profinite groups and in [6] it is proved that all compact groups are included in  $\text{PA}_1$  (see also [1]). In [2] it is shown that  $\text{PA}_2$  contains all compact groups. Martino et al. have shown that  $\text{PA}_k$  contains all profinite groups for any  $k$  (see [9, Theorem 1.19.]). Here in Theorem 2.12 below, we give another proof of the latter result without using submultiplicativity of the  $k$ -nilpotence degree of a finite group (see [9, Theorem 1.21.]).

Our main result is the following.

**Theorem 1.2.** *Let  $G$  be a compact group with  $\text{np}_k(G) > 0$ . Then the connected component  $G^0$  of  $G$  is abelian and there exists a closed normal nilpotent subgroup  $N$  of class at most  $k$  such that  $G^0N$  is open in  $G$ .*

## 2. Relative nilpotent probability and submultiplicativity

We denote the set of continuous functions on a compact group  $G$  by  $\mathcal{C}(G)$ . If  $N$  is a closed normal subgroup of  $G$ , then  $G/N$  is a compact group, equipped with quotient topology. There exists a projection-like map  $P : \mathcal{C}(G) \rightarrow \mathcal{C}(G/N)$  defined by

$$Pf(\bar{g}) := \int_N f(gy) \, dy,$$

where  $dy$  stands for the Haar measure on  $N$ . We will use the following two simple lemmas repeatedly:

**Lemma 2.1.** *Let  $G$  be a compact group and  $N$  be a closed normal subgroup of  $G$ . Then, for all  $f \in \mathcal{C}(G)$*

$$\int_G f(g) \, dg = \int_{G/N} Pf(\bar{g}) \, d\bar{g}.$$

**Proof.** The functional

$$I : \mathcal{C}(G) \rightarrow \mathbb{C}, \quad f \mapsto \int_{G/N} P f(\bar{g}) \, d\bar{g}$$

is nonzero, positive and left-invariant, also  $I(1) = 1$ , so by the uniqueness of Haar integral the result holds.  $\square$

**Remark 2.2.** We have an analog of Lemma 2.1 whenever  $H$  is a closed not necessarily normal subgroup, see [4, (2.49) Theorem].

The Uniqueness of Haar measure has another achievement; consider a continuous epimorphism  $\phi : G \rightarrow H$  between compact groups. Then

$$I : \mathcal{C}(H) \rightarrow \mathbb{C}, \quad f \mapsto \int_G f \circ \phi(g) \, dg$$

defines a nonzero, positive and left-invariant functional. Also,  $I(1) = 1$ , so we have

**Lemma 2.3.** *Let  $\phi : G \rightarrow H$  be a continuous epimorphism between compact groups. Then*

$$\int_H f(h) \, dh = \int_G f \circ \phi(g) \, dg \quad (f \in \mathcal{C}(H)).$$

**Lemma 2.4.** *Let  $H$  be a closed subgroup of a group  $G$ . Then  $yC_G(x) \cap H$  is either empty or a coset of  $C_H(x)$ , for all  $x, y \in G$ . In particular, we have  $\mathbf{m}_H(yC_G(x) \cap H) \leq \mathbf{m}_H(C_H(x))$ , for all  $x, y \in G$ .*

**Proof.** If  $z \in C_G(x) \cap y^{-1}H$ , then one can easily check that  $C_G(x) \cap y^{-1}H = zC_H(x)$ .  $\square$

Let  $H$  be a closed subgroup of a compact group  $G$ . For  $x_1, x_2, \dots, x_k$  in  $G$ , we define

$$\mathcal{N}(H; x_1, \dots, x_k) := \{(y_1, \dots, y_k) \in H^k : [x_1 y_1, \dots, x_k y_k] = 1\}$$

and  $\mathbf{np}(H; x_1, \dots, x_k) = \mathbf{m}_{H^k}(\mathcal{N}(H; x_1, \dots, x_k))$ .

Also, by  $\mathbf{cp}(G)$ , we denote the probability that a randomly chosen pair of elements of  $G$  commute, i.e.  $\mathbf{m}_{G^2}(\{(x, y) \in G^2 : xy = yx\})$ .

**Proposition 2.5.** *Let  $H$  be a closed subgroup of a compact group  $G$ . Then  $\mathbf{np}(H; x, y) \leq \mathbf{cp}(H)$  for all  $x$  and  $y$  in  $G$ .*

**Proof.** By the above lemma, we have

$$\begin{aligned}
 \mathbf{np}(H; x, y) &= \int_H \int_H 1_{\mathcal{N}(H; x, y)}(\xi, \eta) \, d\eta \, d\xi \\
 &= \int_H \mathbf{m}_H(x^{-1}C_G(y\eta) \cap N) \, d\eta \\
 &\leq \int_H \mathbf{m}_H(C_H(y\eta)) \, d\eta \\
 &= \int_H \mathbf{m}_H(y^{-1}C_G(\xi) \cap H) \, d\xi \\
 &\leq \int_H \mathbf{m}_H(C_H(\xi)) \, d\xi = \mathbf{cp}(H). \quad \square
 \end{aligned}$$

**Remark 2.6.** In [9, Lemma 6.2.], using combinatorics methods, the authors have shown that for a normal subgroup  $N$  of a *finite* group  $G$ ,

$$\mathbf{np}(N; x_1, \dots, x_n) \leq \mathbf{np}(N; \overbrace{1, \dots, 1}^{n\text{-times}})$$

for all  $x_k \in G$ ,  $1 \leq k \leq n$ . We could not implement these methods in the case of infinite compact groups to have the above inequality for a closed normal subgroup  $N$ , and of course we do not need this generalization.

Now, we are going to prove that  $\mathbf{np}(N; x_1, \dots, x_n)$  is bounded away from 1, for all closed normal subgroup  $N$  of a compact group  $G$  which is not nilpotent of class at most  $k$  and all  $x_k \in G$ ,  $1 \leq k \leq n$ .

**Proposition 2.7.** *Let  $H$  be a closed subgroup of a compact group  $G$ . Then*

$$\mathbf{np}(H; x_1, \dots, x_{n+1}) \leq \frac{1}{2} \left( 1 + \mathbf{np} \left( \frac{H}{\mathbf{Z} \cap H}; \overline{x_1}, \dots, \overline{x_n} \right) \right),$$

for all  $x_1, \dots, x_n \in G$ , where  $\mathbf{Z}$  is the center of  $G$ , and  $\overline{x}$  denotes the image of  $x$  in  $H/(\mathbf{Z} \cap H)$ .

**Proof.** Put  $K := \mathbf{Z} \cap H$ ,

$$X = \{\mathbf{y} = (y_1, \dots, y_n) \in H^n : [x_1 y_1, \dots, x_n y_n] \in K\}$$

and

$$\mathcal{N} = \mathcal{N}\left(\frac{H}{K}; \overline{x_1}, \dots, \overline{x_n}\right).$$

Then one can easily verify that

$$1_X(\mathbf{y}\mathbf{z}) = 1_X(\mathbf{y}) \quad (\mathbf{y} \in H^n, \mathbf{z} \in K^n) \quad (2.1)$$

( $1_X$  is the characteristic function of  $X$ ) and

$$1_X(\mathbf{y}) = 1 \quad \text{if and only if} \quad 1_{\mathcal{N}}(\bar{\mathbf{y}}) = 1. \quad (2.2)$$

Now, we have

$$\begin{aligned} \mathbf{np}(H; x_1, \dots, x_{n+1}) &= \int_{H^n} \mathbf{m}_H(x_{n+1}^{-1} C_G([x_1 y_1, \dots, x_n y_n]) \cap H) \, d\mathbf{y} \\ &\leq \int_{H^n} \mathbf{m}_H(C_H([x_1 y_1, \dots, x_n y_n])) \, d\mathbf{y} \\ &= \int_X \mathbf{m}_H(C_H([x_1 y_1, \dots, x_n y_n])) \, d\mathbf{y} \\ &\quad + \int_{H^n \setminus X} \mathbf{m}_H(C_H([x_1 y_1, \dots, x_n y_n])) \, d\mathbf{y} \\ &\leq \mathbf{m}_{H^n}(X) + \frac{1}{2}(1 - \mathbf{m}_{H^n}(X)) \\ &= \frac{1}{2}(1 + \mathbf{m}_{H^n}(X)). \end{aligned}$$

The latter inequality holds because on  $X^c$ ,  $C_G([x_1 y_1, \dots, x_n y_n])$  has index at least 2. We compute  $\mathbf{m}_{H^n}(X)$ :

$$\begin{aligned} \mathbf{m}_{H^n}(X) &= \int_{H^n} 1_X(\mathbf{y}) \, d\mathbf{y} \\ &= \int_{H^n/K^n} \int_{K^n} 1_X(\mathbf{y}\mathbf{z}) \, d\mathbf{z} \, d\bar{\mathbf{y}}, \quad \text{by Lemma 2.1,} \\ &= \int_{H^n/K^n} 1_X(\mathbf{y}) \, d\bar{\mathbf{y}}, \quad \text{by (2.1),} \\ &= \int_{(H/K)^n} 1_{\mathcal{N}}(\bar{\mathbf{y}}) \, d\bar{\mathbf{y}}, \quad \text{by (2.2),} \\ &= \mathbf{np}\left(\frac{H}{\mathbf{Z} \cap H}; \overline{x_1}, \dots, \overline{x_n}\right). \quad \square \end{aligned}$$

By Theorem 2.7 we obtain the following characterization of closed and nilpotent normal subgroups of a compact group:

**Proposition 2.8.** *Let  $H$  be a closed subgroup of a compact group  $G$ . Then  $H$  is nilpotent of class at most  $n$  if and only if*

$$\mathbf{np}(H; x_1, \dots, x_{n+1}) = 1$$

for some  $x_k \in G$ ,  $1 \leq k \leq n+1$ .

**Proof.** Only one direction needs proof.

Using induction on  $n$ , we prove the result. For  $n = 1$ , assume that  $\mathbf{np}(H; x, y) = 1$  for some  $x, y \in G$ , so by Proposition 2.5,  $\mathbf{cp}(H) = 1$ , whence implies that  $H$  is an abelian group. Assuming that the result is valid for  $n - 1$ , we prove its correctness for  $n$ . If

$$\mathbf{np}(H; x_1, \dots, x_{n+1}) = 1$$

for some  $x_k \in G$ ,  $1 \leq k \leq n+1$ , then by Proposition 2.7, we get  $\mathbf{np}(\frac{H}{\mathbf{Z} \cap H}; \overline{x_1}, \dots, \overline{x_n}) = 1$ . Now, by induction hypothesis,  $\frac{H}{\mathbf{Z} \cap H}$  is nilpotent of class at most  $n - 1$ , so,  $H$  is nilpotent of class at most  $n$ .  $\square$

**Definition 2.9.** Let  $H$  be a closed subgroup of a compact group  $G$ . The *relative  $k$ -nilpotence of  $H$  in  $G$* , denoted by  $\mathbf{np}_{k,G}(H)$ , defined to be

$$\sup \{ \mathbf{np}(H; x_1, \dots, x_{k+1}) : x_1, \dots, x_{k+1} \in G \}.$$

When  $H = G$ , then we remove  $G$  from the index and simply write  $\mathbf{np}_k(G)$ .

**Remark 2.10.** Since the Haar measure is left-invariant, one can easily verify that  $\mathbf{np}_k(G) = \mathbf{np}(G; x_1, \dots, x_{k+1})$  for all  $x_1, \dots, x_{k+1} \in G$ . So,  $\mathbf{np}_k(G)$  is nothing but the degree of  $k$ -step nilpotence of  $G$  defined in [9]. Also, by Proposition 2.5, we have  $\mathbf{np}_{1,G}(H) = \mathbf{cp}(H)$  for a closed subgroup  $H$  of a compact group  $G$ .

The following result follows inductively from Proposition 2.8, 2.5 and [5, Theorem].

**Theorem 2.11.** *Let  $H$  be closed subgroup of a compact group  $G$  which is not nilpotent of class at most  $n$ . Then  $\mathbf{np}_{k,G}(H) \leq 1 - \frac{3}{2^{k+2}}$ , for all positive integer  $k \leq n$ .*

By Theorem 2.11 in hand, we give another (simple) proof for the following statement.

**Theorem 2.12.** ([9, Theorem 1.19.]) *Let  $G$  be a profinite group with  $\mathbf{np}_k(G) > 0$ . Then  $G$  has an open  $k$ -step nilpotent subgroup.*

**Proof.** By [3, Lemma 2.4.], there is an open normal subgroup  $N$  along with elements  $x_1, \dots, x_{k+1} \in G$  such that  $\mathbf{np}(N; x_1, \dots, x_{k+1}) > 1 - \frac{3}{2^{k+2}}$ , so by Theorem 2.11,  $N$  should be nilpotent of class at most  $k$ .  $\square$

If  $B$  is a measurable subgroup of a compact group  $A$ , then  $\mathbf{m}_A(B) = [A : B]^{-1}$ , by convention on “ $\frac{1}{\infty} = 0$ ”. We have

**Lemma 2.13.** *Let  $G$  be a compact group,  $H$  be a closed subgroup of  $G$  and  $N$  be a normal closed subgroup of  $G$ . Then*

$$\mathbf{m}_G(H) = \mathbf{m}_N(N \cap H) \mathbf{m}_{G/N}(NH/N).$$

**Proof.** The following is well known

$$[G : H] = [G : NH][NH : H] = [G/N : NH/N][N : N \cap H].$$

This along with the explanation before the lemma implies the result.  $\square$

**Theorem 2.14.** *Let  $G$  be a compact group with closed subgroups  $H$  and closed normal subgroup  $N$  contained in  $H$ . Then  $\mathbf{np}_{k,G}(H) \leq \mathbf{np}_{k,G/N}(H/N) \mathbf{np}_{k,G}(N)$ .*

**Proof.** For  $x_1, \dots, x_{k+1} \in G$  we have

$$\mathbf{np}(H; x_1, \dots, x_{k+1}) \leq \int_{H^k} \mathbf{m}_H(C_H([x_1 y_1, \dots, x_k y_k])) \, d\mathbf{y}.$$

By Lemmas 2.1 and 2.13,

$$\begin{aligned} \int_{H^k} \mathbf{m}_H(C_H([x_1 y_1, \dots, x_k y_k])) \, d\mathbf{y} &= \int_{H^k/N^k} \int_{N^k} \mathbf{m}_H(C_H([x_1 y_1 z_1, \dots, x_k y_k z_k])) \, d\mathbf{z} \, d\bar{\mathbf{y}} \\ &= \int_{H^k/N^k} \int_{N^k} \mathbf{m}_N(C_N([x_1 y_1 z_1, \dots, x_k y_k z_k])) \mathbf{m}_{H/N}(NC_H([x_1 y_1 z_1, \dots, x_k y_k z_k])/N) \, d\mathbf{z} \, d\bar{\mathbf{y}} \\ &\stackrel{(*)}{\leq} \int_{H^k/N^k} \int_{N^k} \mathbf{m}_N(C_N([x_1 y_1 z_1, \dots, x_k y_k z_k])) \mathbf{m}_{H/N}(C_{H/N}([\overline{x_1 y_1}, \dots, \overline{x_k y_k}])) \, d\mathbf{z} \, d\bar{\mathbf{y}} \\ &= \int_{H^k/N^k} \mathbf{m}_{H/N}(C_{H/N}([\overline{x_1 y_1}, \dots, \overline{x_k y_k}])) \left( \int_{N^k} \mathbf{m}_N(C_N([x_1 y_1 z_1, \dots, x_k y_k z_k])) \, d\mathbf{z} \right) d\bar{\mathbf{y}} \\ &= \int_{H^k/N^k} \mathbf{m}_{H/N}(C_{H/N}([\overline{x_1 y_1}, \dots, \overline{x_k y_k}])) \mathbf{np}(N; x_1 y_1, \dots, x_k y_k, 1) \, d\bar{\mathbf{y}} \end{aligned}$$

$$\begin{aligned} &\leq \mathbf{np}_{k,G}(N) \int_{(H/N)^k} \mathbf{m}_{H/N} (C_{H/N}([\overline{x_1 y_1}, \dots, \overline{x_k y_k}])) \, d\bar{\mathbf{y}} \\ &\leq \mathbf{np}_{k,G}(N) \mathbf{np}_{k,G/N}(H/N). \end{aligned}$$

The inequality (\*) obtained by the inclusion

$$NC_H([x_1 y_1, \dots, x_k y_k])/N \subset C_{H/N}([\overline{x_1 y_1}, \dots, \overline{x_k y_k}]). \quad \square$$

As a corollary of the above theorem, we state:

**Corollary 2.15.** *Let  $N$  be a closed normal subgroup of a compact group  $G$ . Then  $\mathbf{np}_k(G) \leq \mathbf{np}_k(G/N) \mathbf{np}_{k,G}(N)$ . In particular,  $\mathbf{cp}(G) \leq \mathbf{cp}(G/N) \mathbf{cp}(N)$ .*

### 3. Proofs of main results

Using Corollaries 2.15 and 2.11, we prove the following.

**Theorem 3.1.** *Let  $G$  be a compact group and assume that  $\mathbf{np}_k(G) > 0$ . Then the length of all closed normal series of  $G$  which its factors are not nilpotent of class at most  $k$  should be bounded.*

**Proof.** Let

$$1 = G_{r+1} \leq G_r \leq \dots \leq G_1 \leq G_0 = G$$

be a closed normal series such that  $G_i/G_{i+1}$ ,  $0 \leq i \leq r$ , is not nilpotent of class at most  $k$ . Using the Theorem 2.14 repeatedly, we have

$$\begin{aligned} \mathbf{np}_k(G) &\leq \mathbf{np}_{k,G_0}(G_r) \mathbf{np}_k(G_0/G_r) \\ &\leq \mathbf{np}_{k,G_0}(G_r) \mathbf{np}_{k,G_0/G_r}(G_{r-1}/G_r) \mathbf{np}_k(G_0/G_{r-1}) \\ &\vdots \\ &\leq \prod_{i=2}^{r+1} \mathbf{np}_{k,G_0/G_i}(G_{i-1}/G_i). \end{aligned}$$

The Corollary 2.11 implies that the right hand side of the above inequality is at most  $(1 - \frac{3}{2^{k+2}})^r$ . All in all, we get

$$r < \frac{\ln \mathbf{np}_k(G)}{\ln(1 - \frac{3}{2^{k+2}})}. \quad \square$$

**Theorem 3.2.** *Let  $G$  be a compact group with  $\mathbf{np}_k(G) > 0$ . Assume that there exists a decreasing net  $(H_\alpha)$  of closed normal subgroups of  $G$  such that  $\bigcap_\alpha H_\alpha = 1$ . Then  $H_\alpha$  is nilpotent of class at most  $k$  for sufficiently large  $\alpha$ .*

**Proof.** If none of  $H_\alpha$ 's are nilpotent of class at most  $k$ , then we can construct an infinite closed normal series of  $G$  which its factors are not nilpotent of class at most  $k$ , violates Theorem 3.1; indeed, suppose we have constructed a closed normal series,

$$G = H_0 \geq H_{\alpha_1} \geq \cdots \geq H_{\alpha_n} \quad (3.1)$$

such that its factors are not nilpotent of class at most  $k$ . Since  $H_{\alpha_n}$  is not  $k$ -step nilpotent, there exists  $(x_1, \dots, x_{k+1})$  in  $H_{\alpha_n}^{k+1}$  such that  $[x_1, \dots, x_{k+1}] \neq 1$ . By the assumption, we can choose  $H_{\alpha_{n+1}}$  such that  $[x_1, \dots, x_{k+1}] \notin H_{\alpha_{n+1}}$ , so  $H_{\alpha_{n+1}}$  can be added to (3.1).  $\square$

Since  $\bigcap \{N : N \text{ is an open normal subgroup of } G\} = 1$ , for a profinite group  $G$ , Theorem 3.2 gives another proof for Theorem 2.12.

Theorem 3.1 has also another implication:

**Corollary 3.3.** *Let  $(G_\alpha)_\alpha$  be a collection of compact groups. If*

$$\mathbf{np}_k \left( \prod_\alpha G_\alpha \right) > 0,$$

*then  $G_\alpha$ 's are nilpotent of class at most  $k$  for all but finitely many  $\alpha$ .*

Now we're ready to show that the class  $\mathbf{PA}_k$  contains connected compact groups. A little preparation is needed before proving the later.

**Lemma 3.4.** *Let  $G$  be a compact group. Then  $G$  contains no nontrivial closed semi simple connected compact Lie subgroup  $H$  with  $\mathbf{np}_{k,G}(H) > 0$ .*

**Proof.** Suppose  $\mathcal{N} = \mathcal{N}(H; x_1, \dots, x_{k+1})$  has positive measure for some  $x_1, \dots, x_{k+1} \in G$ . Since  $\mathcal{N}$  is Zariski-closed and has positive measure, it should be equal to  $H^{k+1}$ , so  $H$  should be nilpotent by Proposition 2.8 and hence trivial by semi simplicity.  $\square$

Also compact Lie groups belong to  $\mathbf{PA}_k$ . Next we use  $G^0$  to denote the connected component of a compact group  $G$ .

**Lemma 3.5.** *Let  $G$  be a compact Lie group with  $\mathbf{np}_k(G) > 0$ . Then  $G^0$  is torus. In particular  $G$  has an open  $k$ -step nilpotent subgroup.*

**Proof.** Denote by  $\mathbf{Z}_0$  the center of  $G^0$ . By Corollary 2.15 and Theorem 2.14, we obtain

$$0 < \mathbf{np}_k(G) \leq \mathbf{np}_{k,G}(G^0) \leq \mathbf{np}_{k,G/\mathbf{Z}_0}(G^0/\mathbf{Z}_0).$$

Now the result obtained with Corollary 3.4, because  $G^0/\mathbf{Z}_0$  is a semi simple connected compact Lie subgroup of  $G/\mathbf{Z}_0$ . The last claim of Lemma follows from the fact that  $G^0$  is a closed subgroup of  $G$  of finite index ([7, Theorem 6.36.]), so it should be open.  $\square$

**Remark 3.6.** The above lemma is attributed to Yves de Cornulier ([12]), which is given in response to a question posed by the second author ([11]). Of course, the above proof is somewhat different from the proof presented in [12].

The main result of the Lemma 3.5 is true for arbitrary compact groups. Denote by  $\mathfrak{N}(G)$  the set of all closed normal subgroups  $N$  of  $G$  such that  $G/N$  is a Lie group.

**Theorem 3.7.** *Let  $G$  be a connected compact group with  $\mathbf{np}_k(G) > 0$ . Then  $G$  is an abelian group.*

**Proof.** Suppose  $G$  is non-abelian. Since  $G$  is compact and therefore residually Lie ([7, Lemma 9.1.]), there exists a closed normal subgroup  $N \leq G$  such that  $G/N$  is a connected non-abelian compact Lie group, and it follows from Corollary 2.15 that  $\mathbf{np}_k(G/N) > \mathbf{np}_k(G) > 0$ . But this contradicts Lemma 3.5.  $\square$

**Proof of Theorem 1.2.** By Corollary 2.15,  $\mathbf{np}_k(G/N) > 0$  for all  $N \in \mathfrak{N}(G)$ . Lemma 3.5, now implies that  $(G/N)^0 = G^0 N/N$  ([7, Lemma 9.18.]) is an open abelian group and so,  $G^0 N$  becomes open and normal in  $G$ . If none of  $G^0 N$ 's is  $k$ -step nilpotent,  $N \in \mathfrak{N}(G)$ , then we can construct a closed normal series of  $G$  whose factors are not nilpotent of class at most  $k$ , violates Theorem 3.1; indeed, suppose we have constructed a closed normal series,

$$G \geq G^0 N_1 \geq \cdots \geq G^0 N_s \quad (3.2)$$

such that its factors are not nilpotent of class at most  $k$  and  $N_1, \dots, N_s \in \mathfrak{N}(G)$ . Since  $G^0 N_s$  is not  $k$ -step nilpotent, there exists  $(x_1, \dots, x_{k+1})$  in  $(G^0 N_s)^{k+1}$  such that  $[x_1, \dots, x_{k+1}] \neq 1$ . But,  $[x_1, \dots, x_{k+1}]$  is equal 1 modulo  $N_s$ , for  $G^0 N_s/N_s$  is abelian. Therefore  $[x_1, \dots, x_{k+1}] \in N_s$ , and by the fact that  $\bigcap \{N : N \in \mathfrak{N}(G)\} = 1$  ([7, Lemma 9.1.]), we obtain  $N_{s+1}$  in  $\mathfrak{N}(G)$ , which does not contain  $[x_1, \dots, x_{k+1}]$ .  $\square$

The following question is proposed in [2] in order to resolve Question 1.1.

**Question 3.8** (*Question 3.2 of [2]*). Are there finitely many words  $w_{i,j}$  ( $1 \leq i \leq n$ ,  $1 \leq j \leq k+1$ ) in the free group on  $k+1$  generators such that if  $G$  is a compact group,  $(x_1, \dots, x_{k+1}), \mathbf{u} = (u_1, \dots, u_{k+1}) \in G^{k+1}$  and  $[x_1 w_{i,1}(\mathbf{u}), \dots, x_{k+1} w_{i,k+1}(\mathbf{u})] = 1$  for all  $i \in \{1, \dots, n\}$  then  $[u_1, \dots, u_{k+1}] = 1$ ?

To settle Question 1.1, we will prove that using Theorem 1.2, we may study the following slightly simpler question on finite groups.

**Question 3.9.** Are there finitely many words  $w_{i,j}$  ( $1 \leq i \leq n$ ,  $1 \leq j \leq k+1$ ) in the free group on  $k+1$  generators such that if  $G$  is a finite group,  $(x_1, \dots, x_{k+1}), \mathbf{u} = (u_1, \dots, u_{k+1}) \in G^{k+1}$ ,  $\langle u_1, \dots, u_{k+1} \rangle$  is abelian-by- $(k$ -step nilpotent) and  $[x_1 w_{i,1}(\mathbf{u}), \dots, x_{k+1} w_{i,k+1}(\mathbf{u})] = 1$  for all  $i \in \{1, \dots, n\}$  then  $[u_1, \dots, u_{k+1}] = 1$ ?

**Theorem 3.10.** *If the answer of Question 3.9 is positive, the answer of Question 1.1 is also affirmative.*

**Proof.** First note that by Theorem 1.2 together with Hall Theorem, [10, 15.4.1], all finitely generated subgroups of  $G$  are residually finite.

Let  $n$  be the positive integer in Question 3.9. By [2, Theorem 3.1] there exists an open subset  $V$  of  $G$  containing 1 such that

$$\mathbf{m} \left( \bigcap_{i=1}^n \mathcal{N}_k(G) \mathbf{y}_i \right) > 0, \quad (3.3)$$

where  $\mathbf{y}_i \in V^{k+1}$ ,  $i = 1, \dots, n$ . By continuity of word maps, we can find an open neighborhood  $U$  of 1, such that  $w_{i,j}(\mathbf{u}) \in V$  for

$$\mathbf{u} = (u_1, \dots, u_{k+1}) \in U^{k+1},$$

and  $1 \leq i \leq n$ ,  $1 \leq j \leq k+1$ . Now, for  $\mathbf{u} = (u_1, \dots, u_{k+1}) \in U^{k+1}$ , (3.3) ensures the existence of  $x_1, \dots, x_{k+1} \in G$ , such that

$$[x_1 w_{i,1}(\mathbf{u}), \dots, x_{k+1} w_{i,k+1}(\mathbf{u})] = 1, \quad (1 \leq i \leq n). \quad (3.4)$$

But  $\langle x_i, u_i : i = 1, \dots, k+1 \rangle$  is residually finite, so by passing to its finite quotients and by hypothesis on Question 3.9, we get from (3.4) that  $[u_1, \dots, u_{k+1}] = 1$ , whence the open subgroup  $\langle U \rangle$  is a  $k$ -step open subgroup of  $G$ .  $\square$

## Data availability

No data was used for the research described in the article.

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