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Groups with sharp non-linear irreducible characters

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ABSTRACT

A complex character χ of a finite group G is called sharp if $|G| = \prod_{a \in L_\chi} (\chi(1) - a)$ where $L_\chi := \{\chi(g) \mid 1 \neq g \in G\}$. In this paper we give a characterization of finite groups all non-linear irreducible characters of which are sharp.

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1. Introduction

Let G be a finite group and $L_\chi := \{\chi(g) \mid 1 \neq g \in G\}$ where χ is a complex character of G . It is known that the order of G divides $\prod_{a \in L_\chi} (\chi(1) - a)$ [2, 13]. The pair (G, χ) (or briefly, the character χ) is called sharp of type L or L -sharp if $|G| = \prod_{a \in L} (\chi(1) - a)$, where $L = L_\chi$. Clearly, χ is faithful whenever (G, χ) is L -sharp. The notion of sharpness was first introduced for permutation characters by Ito and Kiyota in [12], as a generalization of sharply t -transitive permutation representations. Thereafter Cameron and Kiyota extended this concept to arbitrary group characters and posed the problem of determining all the L -sharp pairs (G, χ) for a given set L [6]. After that for several cases those problems have been studied by many authors. For instance, in [1] Alvis and Nozawa gave a complete classification of sharp pairs of type L when L contains an irrational number.

Theorem 1.1 ([1, Theorem 1.3]). *Suppose (G, χ) is sharp and χ assumes an irrational value. Then one of the following holds:*

- (1) G is cyclic of order m , and either $m \geq 3$ and χ is linear, or else $m \geq 5$ and χ is the sum of two complex conjugate linear characters of G .
- (2) G is dihedral of order $2m$, where $m \geq 5$ is odd, and χ is irreducible of degree 2.
- (3) G is dihedral or generalized quaternion of order $2m$, where $m \geq 8$ is even, and $\chi = \psi$ or $\chi = \psi + \varepsilon$, where ψ is irreducible of degree 2 and ε is linear with cyclic kernel of order m .
- (4) G is isomorphic to the binary octahedral group and χ is irreducible of degree 2.
- (5) G is isomorphic to $\text{SL}(2, 5)$ and χ is irreducible of degree 2.
- (6) G is isomorphic to A_5 and χ is irreducible of degree 3.

Among the groups which appear in Theorem 1.1, dihedral groups of order $2m$, where $m \geq 5$ is an odd prime, are the only groups all non-linear irreducible characters of which are sharp (see Proposition 2.2). This observation raises the question if there are other groups with this property? Since all nonprincipal

irreducible characters of simple groups are faithful, it seems that some simple groups have this property. But we will show that the group with the mentioned property must be solvable (See [Corollary 2.16](#)). Note that if a group G has a sharp linear character λ , then the derived subgroup of G is trivial and consequently G is abelian. Since λ is faithful the center of G is cyclic and thus G is cyclic. Moreover, if all nonprincipal linear characters of a group G are sharp, then G is cyclic of prime order.

In this paper we shall study the nonabelian group all non-linear irreducible characters of which are sharp. More precisely, we prove the following.

Main Theorem: Suppose that G is a finite nonabelian group such that every non-linear irreducible character of G is sharp. Then one of the following holds:

- (1) G is dihedral of order $2m$, where $m \geq 5$ is an odd prime;
- (2) G is an extraspecial 2-group;
- (3) $G \cong E(p^n) : \mathbb{Z}_t$ is a Frobenius group with kernel $E(p^n) = G'$ such that $t \geq 2$ and it is even if p is odd.

Moreover, in case (3) every non-linear irreducible character of G contains exactly one negative value.

2. Proof of main theorem

In this section we first consider the finite nonabelian groups G such that (G, χ) is sharp for every non-linear irreducible character $\chi \in \text{Irr}(G)$ but L_χ contains an irrational value for some χ .

Lemma 2.1. *Suppose that G is a finite nonabelian group such that (G, χ) is sharp for every non-linear irreducible character $\chi \in \text{Irr}(G)$. Then G' is the unique minimal normal subgroup of G .*

Proof. It is known that each normal subgroup of G is the intersection of $\ker \chi$ for some $\chi \in \text{Irr}(G)$. By hypothesis, the kernel of every non-linear irreducible character is trivial. This implies that each normal subgroup of G is the intersection of some of $\ker \lambda$, for some linear irreducible characters λ of G . So for each normal subgroup N of G we have

$$\{1\} \neq G' = \bigcap \{\ker \lambda \mid \lambda \in \text{Irr}(G), \lambda(1) = 1\} \subseteq N.$$

Hence G' is the unique minimal normal subgroup of G . □

Proposition 2.2. *Suppose that G is a finite nonabelian group such that (G, χ) is sharp for all non-linear irreducible characters $\chi \in \text{Irr}(G)$. If χ assumes an irrational value for some χ , then G is dihedral of order $2m$, where $m \geq 5$ is an odd prime.*

Proof. Since G contains a sharp character with irrational values, it follows from [Theorem 1.1](#) that G must be one of the groups appear in cases (1) – (6).

Since G is not an abelian group, (1) does not occur. Suppose that $G \cong D_{2m}$ where $m \geq 5$ is odd. By [16, Theorem 3.2], every faithful irreducible character of G is sharp. But one can see that every non-linear irreducible character D_{2m} is faithful iff m is prime. So (2) holds iff $m \geq 5$ is an odd prime.

If $G \cong D_{2m}$ where $m \geq 8$ is even, then G has a non-linear irreducible character such that it is not faithful, and if G is a generalized quaternion group of order $2m$ where $m \geq 8$ is even, then $Z(G) < G'$, which contradicts [Lemma 2.1](#). So (3) does not hold. The cases (4), (5), and (6) do not occur, because binary octahedral group and $\text{SL}(2, 5)$ have irreducible characters of degrees 2, 5, respectively, that are not faithful and A_5 has an irreducible character of degree 5 which is not sharp. □

For the rest of the paper, we will refer to a group G as a $\mathbb{Q}_1$ -group if every non-linear irreducible character is rational valued. When every irreducible character is rational valued, G will be called a $\mathbb{Q}$ -group. It is clear that every $\mathbb{Q}$ -group is a $\mathbb{Q}_1$ -group.

Lemma 2.3 ([7, Corollary 3.5]). *Let G be a p -group. If G is a $\mathbb{Q}_1$ -group, then $p = 2$.*

Theorem 2.4 ([7, Theorem 3.6]). *Let G be a $\mathbb{Q}_1$ -group. Then $Z(G)$ is an elementary abelian 2-group.*

Theorem 2.5 ([7, Theorem 3.10]). *Let G be a nonsolvable group. Then G is a $\mathbb{Q}_1$ -group iff G is a $\mathbb{Q}$ -group.*

Theorem 2.6 ([8, Theorem B]). *Let G be a noncyclic finite simple group. Then G is a composition factor of a rational group if and only if G is isomorphic to an alternating group or one of the following groups:*

- (i) $\mathrm{PSp}_4(3)$, $\mathrm{Sp}_6(2)$, $\mathrm{O}_8^+(2)$,
- (ii) $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_4(3)$.

Lemma 2.7 ([8, Corollary B.1]). *Let G be a noncyclic simple group. Then G is a rational group if and only if $G \cong \mathrm{Sp}_6(2)$ or $\mathrm{O}_8^+(2)$.*

The next Lemmas will be needed later.

Lemma 2.8 ([3, Lemma 5]). *Let N be a minimal normal subgroup of G so that $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$, a nonabelian simple group. Let A be the automorphism group of S . If $\sigma \in \mathrm{Irr}(S)$ extends to A , then $\sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$ extends to G .*

Proposition 2.9. *Let G be a finite group and N be a normal subgroup of G . Suppose that $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$ for some $\theta \in \mathrm{Irr}(N)$. If χ is sharp, then $\prod_{a \in L_1} (\theta(1) - a)$ divides $|G|$ where $L_1 := \{\theta(x) \mid x \in N; \theta(x) \neq \theta(1)\}$.*

Proof. Put $L = \{\chi(g) \mid 1 \neq g \in G\}$. Since χ is sharp, $|G| = \prod_{b \in L} (\chi(1) - b)$. But $L_1 \subseteq L$ and $\theta(1) = \chi(1)$, thus

$$\prod_{a \in L_1} (\theta(1) - a) \mid \prod_{b \in L} (\chi(1) - b).$$

This completes the proof. □

Theorem 2.10. *Suppose that G is a $\mathbb{Q}_1$ -group such that (G, χ) is sharp for all non-linear $\chi \in \mathrm{Irr}(G)$. If $G' \cong S^t$ where S is a non-abelian simple group such that $G' \leq G \leq \mathrm{Aut}(G')$ and G/G' is an elementary abelian 2-group, then*

$$S \notin \{\mathrm{PSp}_4(3), \mathrm{Sp}_6(2), \mathrm{O}_8^+(2), \mathrm{PSL}_3(4), \mathrm{PSU}_4(3)\}.$$

Proof. First we note that since G/G' is an elementary abelian 2-group, if p is an odd prime divisor of $|G|$, then $p \mid |G'|$.

Suppose that $S \cong \mathrm{PSp}_4(3)$ and $G' \cong (\mathrm{PSp}_4(3))^t$. By character table of $\mathrm{PSp}_4(3)$ in [5], χ_{19} is the Steinberg character of S and $L_{\chi_{19}} = \{-8, -2, -1, 0, 1, 4\}$. By Lemma 2.8, $(\chi_{19})^t$ extends to G . Hence $(-1)^t \in L_{(\chi_{19})^t}$. Then by Proposition 2.9, $(\chi_{19}(1))^t - (-1)^t \mid |G|$. But $\chi_{19}(1) = 64$, therefore

$$64^t - (-1)^t = (64 + 1)\alpha \mid |G|,$$

for some integer α . Thus 13 divides $|G|$ and hence 13 divides $|G'|$, a contradiction.

If $S \cong \mathrm{PSU}_4(3)$ and $G' \cong (\mathrm{PSU}_4(3))^t$, then character table of $\mathrm{PSU}_4(3)$ in [5] shows that χ_{19} is the Steinberg character of S and $L_{\chi_{19}} = \{-3, -1, 0, 1, 9\}$. It follows from Lemma 2.8 that $(\chi_{19})^t$ extends to G and so $(-1)^t \in L_{(\chi_{19})^t}$. Thus $(\chi_{19}(1))^t - (-1)^t$ divides $|G|$ by Lemma 2.9. Since $\chi_{19}(1) = 729$, we have $729^t - (-1)^t$ divides $|G|$. Thus 73 divides $|G|$, a contradiction.

Let $S \cong \text{PSL}_3(4)$ and $G' \cong (\text{PSL}_3(4))^t$. By character table of $\text{PSL}_3(4)$ in [5], χ_{10} is the Steinberg character of S and $L_{\chi_{10}} = \{-1, 0, 1\}$. By Lemma 2.8, $(\chi_{10})^t$ extends to G . Therefore $(-1)^t \in L_{(\chi_{10})^t}$ and by Proposition 2.9, $(\chi_{10}(1))^t - (-1)^t$ divides $|G|$. Thus $64^t - (-1)^t$ divides $|G|$ and then 13 divides $|G|$. This is a contradiction.

Now suppose that $S \cong \text{Sp}_6(2)$ and $G' \cong (\text{Sp}_6(2))^t$. The character table of $\text{Sp}_6(2)$ in [5], shows that χ_{30} is the Steinberg character of S and $L_{\chi_{30}} = \{-16, -4, -1, 0, 1, 2, 8\}$. Since $(\chi_{30})^t$ extends to G we have $1^t = 1 \in L_{(\chi_{30})^t}$. So $(\chi_{30}(1))^t - 1$ divides $|G|$ and then $512^t - 1$ divides $|G|$. This implies that 73 divides $|G|$, a contradiction.

Finally, if $S \cong \text{O}_8^+(2)$ and $G' \cong (\text{O}_8^+(2))^t$ then by [5], χ_{51} is the Steinberg character of S and $L_{\chi_{51}} = \{-8, -4, -1, 0, 1, 64\}$. Since $(\chi_{51})^t$ extends to G we have $1 \in L_{(\chi_{51})^t}$. So $(\chi_{51}(1))^t - 1$ divides $|G|$ and $4096^t - 1$ must divide $|G|$. Since $13 \nmid |G|$ we have a contradiction. $\square$

Lemma 2.11 ([4, Lemma 2.2]). *Let $n \geq 5$ be an integer and let p be a prime. If the p -part of $n!$ is p^v , then $v \leq n/(p-1)$.*

Lemma 2.12. *Let G be a finite group and p be a prime. Let m and n be positive integers. If p^{mn} divides $|G^n|$, then p^m divides $|G|$.*

Proof. If the p -part of $|G|$ is p^r for some $r \in \mathbb{N}$, then since p^{mn} divides $|G^n| = |G|^n$, we have p^{mn} divides p^{rn} . Hence $p^{rn} = sp^{mn}$ for some $s \in \mathbb{N}$. Since $s = (p^r/p^m)^n$ is a positive integer, we must have p^m divides p^r , and thus divides $|G|$. $\square$

Lemma 2.13. *Let i, j and k be positive integers. Then:*

1. $(k-i)(k-i-1) \neq (k-j)(k-j-1)$ for each $i, j \leq k-1$ and $i \neq j$;
2. $(k-i)(3k-3i-1) \neq (k-j)(3k-3j-1)$ for each $i, j \leq k$ and $i \neq j$;
3. $(k-i)(3k-3i+1) \neq (k-j)(3k-3j+1)$ for each $i, j \leq k$ and $i \neq j$;
4. $(k-i)(3k-3i-1) \neq (k-4)(3k-13)$ for each $5 \leq i \leq k$;
5. $(k-i)(3k-3i+1) \neq (k-4)(3k-11)$ for each $5 \leq i \leq k$.

Proof. (1) Suppose that there exists $i, j \leq k-1$ such that $i \neq j$ and $(k-i)(k-i-1) = (k-j)(k-j-1)$. Then $(i-j)(2k-(i+j)-1) = 0$, and so $2k-(i+j)-1 = 0$. Thus $i+j = 2k-1$, a contradiction. The other cases can be proved similarly. $\square$

Lemma 2.14. *Let i, j and k be positive integers. Then:*

1. $(k-i)(3k-3i-1) \neq (k-j)(3k-3j-1) - 4$ for each $i, j \leq k$;
2. $(k-i)(3k-3i+1) \neq (k-j)(3k-3j+1) - 4$ for each $i \leq k-1$ and $j \leq k$;
3. $(k-i)(3k-3i-1) \neq (k-j)(3k-3j-1) - 2$ for each $i \leq k-1$ and $j \leq k$;
4. $(k-i)(3k-3i+1) \neq (k-j)(3k-3j+1) - 2$ for each $i, j \leq k$;
5. If $k \neq 3$, then $(k-i)(3k-3i-1) \neq (k-4)(3k-13) - 4$ for each $i \leq k$;
6. If $k \neq 3, 5$, then $(k-i)(3k-3i-1) \neq (k-4)(3k-13) - 2$ for each $i \leq k$;
7. If $k \neq 5$, then $(k-i)(3k-3i+1) \neq (k-4)(3k-11) - 4$ for each $i \leq k$;
8. If $k \neq 3$, then $(k-i)(3k-3i+1) \neq (k-4)(3k-11) - 2$ for each $i \leq k$;
9. If $k \neq 4$, then $(k-i)(3k-3i-1) - 2 \neq (k-4)(3k-13)$ for each $i \leq k$;
10. If $k \neq 3$, then $(k-i)(3k-3i+1) - 2 \neq (k-4)(3k-11)$ for each $i \leq k$.

Proof. (1) Suppose on the contrary that there exists $i, j \leq k$ that $(k-i)(3k-3i-1) = (k-j)(3k-3j-1) - 4$. Then $(i-j)(6k-3(i+j)-1) = 4$. Since $i-j$ is even iff $6k-3(i+j)-1$ is odd, we must have

$$(i-j, 6k-3(i+j)-1) \in \{(1, 4), (4, 1), (-1, -4), (-4, -1)\}.$$

Then

- (a) if $(i - j, 6k - 3(i + j) - 1) = (1, 4)$, then $6(k - j) = 8$, a contradiction,
- (b) if $(i - j, 6k - 3(i + j) - 1) = (4, 1)$, then $6(k - j) = 14$, a contradiction,
- (c) if $(i - j, 6k - 3(i + j) - 1) = (-1, -4)$, then $6(k - j) = -6$. So $j = k + 1$, a contradiction,
- (d) if $(i - j, 6k - 3(i + j) - 1) = (-4, -1)$, then $6(k - j) = -12$. So $j = k + 2$, a contradiction.

The other cases can be proved similarly. □

Theorem 2.15. *Under the hypothesis of Theorem 2.10, $S \not\cong A_n$ for every $n \in \mathbb{N}$.*

Proof. Suppose on the contrary that $S \cong A_n$ for some $n \geq 5$, and $G' \cong (A_n)^t$. If $t = 1$, then $A_n \leq G \leq S_n$ for some $n \geq 5$. Suppose that $G \cong A_n$. Then by Theorem 2.5, G is a rational group which contradicts Lemma 2.7. Let $G \cong S_n$. Then function $\chi : G \rightarrow \mathbb{C}$ defined by $\chi(\sigma) = |\text{Fix}(\sigma)| - 1$, where

$$\text{Fix}(\sigma) = \{i \in [n] \mid \sigma(i) = i\},$$

is a non-linear irreducible character of G of degree $n - 1$ and $L_\chi = \{-1, 0, 1, \dots, n - 3\}$. Let θ be the sign character of G , (that is, $\theta(\sigma) = 1$ if σ is an even permutation and $\theta(\sigma) = -1$ if σ is an odd permutation). Then the product of χ by θ is again a non-linear irreducible character of G of degree $n - 1$ (see [11]). By hypothesis χ and $\chi\theta$ must be sharp. This implies that $|G| = \prod_{l \in L_\chi} (n - 1 - l) = \prod_{l \in L_{\chi\theta}} (n - 1 - l)$, which is impossible because $\prod_{l \in L_{\chi\theta}} (n - 1 - l) = \prod_{l \in A} (n - 1 - l) \prod_{l \in B} (n - 1 + l)$ where $A = \{\chi(\sigma) \mid 1 \neq \sigma \text{ is an even permutation}\}$ and $B = \{\chi(\sigma) \mid \sigma \text{ is an odd permutation}\}$ and since $L_\chi = A \cup B$ we have

$$\prod_{l \in L_\chi} (n - 1 - l) < \prod_{l \in A} (n - 1 - l) \prod_{l \in B} (n - 1 + l).$$

Thus χ and $\chi\theta$ can not be sharp simultaneously, a contradiction. Thus we can assume that $t \geq 2$.

Suppose that $S \cong A_5$ and $G' \cong (A_5)^t$. The character table of A_5 in [5] shows that $\chi_5(1) = 5$ and χ_5 is extendible to S_5 . Moreover, $L_{\chi_5} = \{-1, 0, 1\}$. Thus by Lemma 2.8, $(\chi_5)^t$ is extendible to G . Since $-1 \in L_{(\chi_5)^t}$, Proposition 2.9 shows that $(\chi_5(1))^t - (-1) = 5^t + 1$ divides $|G|$. On the other hand, by Zsigmondy Theorem in [19], there is a prime number q such that $q \mid 5^t + 1$ and $q \nmid 5 + 1$. Clearly, $q \neq 2, 3$. Since $q \mid |G|$, we must have $q = 5$, a contradiction.

Let $S \cong A_6$ and $G' \cong (A_6)^t$. From the character table of A_6 in [5], we have $\chi_2(1) = 5$ with $L_{\chi_2} = \{-1, 0, 1, 2\}$ and χ_2 is extendible to S_6 . Thus by Lemma 2.8, $(\chi_2)^t$ is extendible to G . Also $-1 \in L_{(\chi_2)^t}$. By Proposition 2.9, $(\chi_2(1))^t - (-1) = 5^t + 1$ divides $|G|$. Similarly, Zsigmondy Theorem in [19] shows that there is a prime number q that divides $5^t + 1$ and does not divide $5 + 1$. Since $q \neq 2, 3$ and $q \mid |G|$, we have $q = 5$, a contradiction.

If $S \cong A_7$ and $G' \cong (A_7)^t$, then the character table of A_7 in [5], shows that $\chi_8(1) = 21$ with $(-1) \in L_{\chi_8}$ and χ_8 is extendible to S_7 . Thus $(-1)^t \in L_{(\chi_8)^t}$ and by Lemma 2.8, $(\chi_8)^t$ is extendible to G . By Proposition 2.9, $(\chi_8(1))^t - (-1)^t$ divides $|G|$. Therefore $21^t - (-1)^t$ divides $|G|$ and so 11 divides $|G|$, a contradiction.

Let $\pi(g) = \text{Fix}(g)$ and $\delta(g)$ be the number of 2-cycles in the cycle decomposition of g on $\{1, 2, \dots, n\}$. We put

$$\sigma := \frac{\pi(\pi - 3)}{2} + \delta, \quad \tau := \frac{(\pi - 1)(\pi - 2)}{2} - \delta.$$

If $n \geq 6$, then by ([10, Theorem 11.9]) and ([3, Theorem 3]), σ and τ are non-linear irreducible characters of A_n such that are extendible to S_n . Moreover, Lemma 2.8 shows that σ^t and τ^t are also extendible to G . If $S \cong A_8$ and $G' \cong (A_8)^t$, then $\sigma(1) = 20$ and the character table of A_8 in [5] shows that $\sigma = \chi_4$ and $1 \in L_\sigma$. Hence $1 \in L_{(\sigma)^t}$ and by Lemma 2.8, $(\sigma)^t$ is extendible to G . But by Proposition 2.9, $(\sigma(1))^t - 1$ divides $|G|$, therefore $20^t - 1$ divides $|G|$ and so 19 divides $|G|$, a contradiction. An argument similar to that given in above shows that

- (a) if $S \cong A_9$ and $G' \cong (A_9)^t$, then $\sigma = \chi_5$ and 13 divides $|G|$, a contradiction;
- (b) if $S \cong A_{10}$ and $G' \cong (A_{10})^t$, then $\sigma = \chi_3$ and 17 divides $|G|$, a contradiction;
- (c) if $S \cong A_{11}$ and $G' \cong (A_{11})^t$, then $\sigma = \chi_3$ and 43 divides $|G|$, a contradiction;
- (d) if $S \cong A_{12}$ and $G' \cong (A_{12})^t$, then $\sigma = \chi_3$ and 47 divides $|G|$, a contradiction;
- (e) if $S \cong A_{13}$ and $G' \cong (A_{13})^t$, then $\sigma = \chi_3$ and 17 divides $|G|$, a contradiction.

So we can assume that $n \geq 14$. Set:

$$\begin{aligned}\rho_i &:= (1\ 2\ 3)(4\ 5\ 6) \cdots (3i-2\ 3i-1\ 3i) \text{ for } 1 \leq i \leq k; \\ \beta_0 &:= (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12); \\ \beta_i &:= (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)(13\ 14\ 15) \cdots (3i-2\ 3i-1\ 3i) \text{ for } 5 \leq i \leq k; \\ \alpha_0 &:= (1\ 2)(3\ 4)(5\ 6)(7\ 8\ 9\ 10\ 11\ 12); \\ \alpha_i &:= (1\ 2)(3\ 4)(5\ 6)(7\ 8\ 9\ 10\ 11\ 12)(13\ 14\ 15) \cdots (3i-2\ 3i-1\ 3i) \text{ for } 5 \leq i \leq k.\end{aligned}$$

Now we consider the following cases.

- (I) $n = 3k$ for some positive integer k .

In this case $\sigma(1) = 3^2k(k-1)/2$ and $\sigma(\rho_i) = 3^2(k-i)(k-i-1)/2$ for each $1 \leq i \leq k$. It follows from Lemma 2.13(1) that $\sigma(\rho_i), i \leq k-1$ are different numbers. Then Proposition 2.9 shows that $\prod_{1 \leq i \leq k-1} (\sigma(1)^t - \sigma(\rho_i)^t)$ divides $|G|$. So $3^{2t(k-1)}$ divides $|G|$ and thus divides $|G'|$. Since $G' \cong A_n^t$, Lemma 2.12 implies that 3^{2k-2} divides $|A_n|$. Let the 3-part of $|A_n|$ is 3^v . By Lemma 2.11, $v \leq n/2$. Then $2k-2 \leq 3k/2$. So $k \leq 4$ and $n \leq 12$, a contradiction.

- (II) Let $n = 3k+1$ for some positive integer k .

In this case $\tau(1) = 3k(3k-1)/2$ and

$$\begin{aligned}\tau(\rho_i) &= \frac{3(k-i)(3k-3i-1)}{2}, \text{ for } 1 \leq i \leq k; \\ \tau(\beta_i) &= \frac{3((k-i)(3k-3i-1)-4)}{2}, \text{ for } 5 \leq i \leq k; \\ \tau(\alpha_i) &= \frac{3((k-i)(3k-3i-1)-2)}{2}, \text{ for } 5 \leq i \leq k; \\ \tau(\beta_0) &= \frac{3((k-4)(3k-13)-4)}{2}; \\ \tau(\alpha_0) &= \frac{3((k-4)(3k-13)-2)}{2}.\end{aligned}$$

Let $k \neq 5$. By Lemmas 2.13 and 2.14, the numbers

$$\tau(\rho_i), 1 \leq i \leq k-1, \tau(\beta_i), 5 \leq i \leq k, \tau(\alpha_i), 5 \leq i \leq k-1, \tau(\beta_0), \tau(\alpha_0)$$

are different numbers. Put $A = \prod_{1 \leq i \leq k-1} (\tau(1)^t - \tau(\rho_i)^t)$, $B = \prod_{5 \leq i \leq k} (\tau(1)^t - \tau(\beta_i)^t)$, and $C = \prod_{5 \leq i \leq k-1} (\tau(1)^t - \tau(\alpha_i)^t)$. Then Proposition 2.9 shows that the number

$$(\tau(1)^t - \tau(\beta_0)^t) \times (\tau(1)^t - \tau(\alpha_0)^t) \times A \times B \times C$$

must divide $|G|$. Then $3^{t(3k-8)}$ divides $|G|$ and thus divides $|G'|$. Since $G' \cong A_n^t$, Lemma 2.12 implies that 3^{3k-8} divides $|A_n|$. Let the 3-part of $|A_n|$ be 3^v . By Lemma 2.11, $v \leq n/2$. Thus $3k-8 \leq 3k+1/2$. Hence $k \leq 5$ and $n \leq 16$ which is impossible, since $n \geq 14$ and $k \neq 5$.

Now, let $k = 5$, so $n = 16$. Thus $\tau(1) = 105$. Using the character table of A_{16} (can be computed by GAP [9]) we see that $26 \in L_\tau$. Then an argument similar to that given in above shows that 79 divides $|G|$ and thus 79 divides $|G'|$, a contradiction.

(III) $n = 3k + 2$ for some positive integer k .

In this case $\tau(1) = 3k(3k + 1)/2$ and

$$\begin{aligned}\tau(\rho_i) &= \frac{3(k-i)(3k-3i+1)}{2} \quad \text{for } 1 \leq i \leq k; \\ \tau(\beta_i) &= \frac{3((k-i)(3k-3i+1)-4)}{2} \quad \text{for } 5 \leq i \leq k; \\ \tau(\alpha_i) &= \frac{3((k-i)(3k-3i+1)-2)}{2} \quad \text{for } 5 \leq i \leq k; \\ \tau(\beta_0) &= \frac{3((k-4)(3k-11)-4)}{2}; \\ \tau(\alpha_0) &= \frac{3((k-4)(3k-11)-2)}{2}.\end{aligned}$$

Let $k \neq 5$. Then using an argument analogous to that in (II), it follows that $3^{t(3k-7)}$ divides $|G|$ and therefore divides $|G'|$. Since $G' \cong A_n^t$, Lemma 2.12 shows that 3^{3k-7} divides $|A_n|$. Let the 3-part of $|A_n|$ is 3^v . By Lemma 2.11, $v \leq n/2$. Thus $3k - 7 \leq 3k + 2/2$. Hence $k \leq 5$ and $n \leq 17$. But $n \geq 14$ and $k \neq 5$. Therefore $n = 14$ and $\tau(1) = 78$. From the character table of A_{14} (using GAP [9]) we obtain $21 \in L_\tau$. An argument similar to that given in above shows that 19 divides $|G|$ and thus 19 divides $|G'|$, a contradiction.

Now, let $k = 5$, so $n = 17$. Therefore $\tau(1) = 120$. Using the character table of A_{17} (can be computed by GAP [9]) we can see that $27 \in L_\tau$. An argument similar to that given in above shows that 31 divides $|G'|$, a contradiction. □

Corollary 2.16. *Suppose that G is a finite nonabelian group such that all of its non-linear irreducible characters are sharp. If G is a $\mathbb{Q}_1$ -group, then G is solvable.*

Proof. Let G be a nonsolvable group. Then G' is non-abelian. Moreover, Lemma 2.1 shows that G' is the unique minimal normal subgroup of G . Thus $G' \cong S^t$, where S is a non-abelian simple group and $t \in \mathbb{N}$, and $G' \leq G \leq \text{Aut}(G')$. Since G is a $\mathbb{Q}_1$ -group, by Theorem 2.5, G is a $\mathbb{Q}$ -group and by Theorem 2.6, S is isomorphic to $\text{PSp}_4(3)$, $\text{Sp}_6(2)$, $\text{O}_8^+(2)$, $\text{PSL}_3(4)$, $\text{PSU}_4(3)$ or an alternating group. Also G/G' is an elementary abelian 2-group. Now by Theorems 2.10 and 2.15 we have a contradiction. So G must be solvable. □

Theorem 2.17. *Suppose that G is a finite nonabelian group such that (G, χ) is sharp for all non-linear irreducible characters $\chi \in \text{Irr}(G)$. If G is a $\mathbb{Q}_1$ -group, then one of the following occurs:*

- (i) G is an extraspecial 2-group;
- (ii) $G \cong E(p^n) : \mathbb{Z}_t$ is a Frobenius group, where $E(p^n) = G'$, p is an odd prime, $n \geq 1$, and $t \geq 2$ is even.
- (iii) $G \cong G' : \mathbb{Z}_t$ is a Frobenius group, where the derived subgroup G' of G is a rational elementary abelian 2-group and $t \geq 3$ is odd.

Also in cases (ii) and (iii), every non-linear irreducible character of G contains exactly one negative value.

Proof. By Corollary 2.16 and Lemma 2.1, G is solvable and G' is the unique minimal normal subgroup of G . Then by ([11, Lemma 12.3]), all non-linear irreducible characters of G have equal degree f and one of the following situations occurs:

- (1) G is a p -group, $Z(G)$ is cyclic and $G/Z(G)$ is elementary abelian of order f^2 .
- (2) G is a Frobenius group with an abelian Frobenius complement of order f . Also, G' is the Frobenius kernel and is an elementary abelian p -group.

Suppose that case (1) holds. Then since G is a p -group, $Z(G) \neq 1$ and by Lemma 2.3, $p = 2$. Let $|G| = 2^k$ for some $k \in \mathbb{N}$. Since G' is the unique minimal normal subgroup we have $G' \leq Z(G)$. But $Z(G)$ is cyclic and by Theorem 2.4, $Z(G)$ is also an elementary abelian 2-group, so $|Z(G)| = 2$ and thus $Z(G) = G'$. Then G is an extraspecial 2-group and $|G| = 2^{2\alpha+1}$ where $2^\alpha = f$. Now by ([14, Proposition 3.5] and [14, Lemma 3.8]), we have (i).

Now, suppose that case (2) occurs. From ([15, Main Theorem]), we have:

- (a) $G \cong E(p^n) : \mathbb{Z}_f$ is a Frobenius group, where $E(p^n) = G'$, p is an odd prime, $n \geq 1$ and $f \geq 1$ is even;
- (b) $G \cong G' : \mathbb{Z}_f$ is a Frobenius group, where the derived subgroup G' of G is a rational 2-group and $f \geq 1$ is odd.

If $f = 1$, then $G \cong G'$ and therefore G is nonsolvable, a contradiction. So $f > 1$. Now let χ be an arbitrary non-linear irreducible character of G and $L_\chi = \{l_1, l_2, \dots, l_r\}$. If $|G'| = q^n$, where q is a prime, then we have $|G| = fq^n = (\chi(1) - l_1) \cdots (\chi(1) - l_r)$. But $\chi(g) = 0$ for each $g \in G \setminus G'$, thus $0 \in L_\chi$. Let $l_1 = 0$. Then $fq^n = f(f - l_2) \cdots (f - l_r)$. Therefore we have $f - l_i = q^{m_i}$, for $2 \leq i \leq r$ and $m_i \geq 0$.

Claim A: $m_i \geq 1$ for $2 \leq i \leq r$.

Suppose that $m_j = 0$ and $x \in G'$ such that $\chi(x) = l_j$. Then $l_j = f - 1$. Since $o(x) = q$, ([18, Proposition 3]) shows that $\chi(x^q) \equiv \chi(x) \pmod{q}$. Hence $f \equiv l_j = f - 1 \pmod{q}$ and so $q \mid 1$, a contradiction. Thus $m_i \geq 1$ for $2 \leq i \leq r$.

Claim B: $-f \notin L_\chi$.

Let $-f \in L_\chi$. We may assume without loss of generality that $l_2 = -f$. Thus $|G| = fq^n = 2f^2(\chi(1) - l_3) \cdots (\chi(1) - l_r)$ and so $f \mid q^n$, a contradiction.

Suppose L_χ contains at least two negative integers. Let $l_i < l_j < 0$. Thus $f - l_i = q^{m_i}$ and $f - l_j = q^{m_j}$. Hence $m_j < m_i$ and $f < q^{m_i}$ and $f < q^{m_j}$. Also $l_i, l_j > -f$ and we have

$$\begin{aligned} -l_i < f &\Rightarrow f - l_i = q^{m_i} < 2f \Rightarrow \frac{q^{m_i}}{2} < f < q^{m_i} \\ -l_j < f &\Rightarrow f - l_j = q^{m_j} < 2f \Rightarrow \frac{q^{m_j}}{2} < f < q^{m_j} \end{aligned}$$

Hence

$$0 < l_j - l_i < f \Rightarrow q^{m_i} - q^{m_j} < f < q^{m_j} \Rightarrow q^{m_j}(q^{m_i-m_j} - 1) < q^{m_j}$$

Thus $q^{m_i-m_j} - 1 < 1$, a contradiction. □

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