

On 2-integral Cayley graphs

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Abstract

In this paper, we introduce the concept of k -integral graphs. A graph Γ is called k -integral if the extension degree of the splitting field of the characteristic polynomial of Γ over rational field $\mathbb{Q}$ is equal to k . We prove that the set of all finite connected graphs with given algebraic degree and maximum degree is finite. 1-integral graphs are just integral ones, graphs all of whose eigenvalues are integer. We study 2-integral Cayley graphs over finite groups G with respect to Cayley sets which are a union of conjugacy classes of G . Among other general results, we completely characterize all finite abelian groups having a connected 2-integral Cayley graph with valency 2, 3, 4 and 5. Furthermore, we classify finite groups G for which all Cayley graphs over G with bounded valency are 2-integral.

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1 Introduction and results

A graph is called *integral* if the eigenvalues of its adjacency matrix are all integers. This concept was first introduced by Harary and Schwenk [13] and they raised the question: which graphs are integral? After this, classification and construction of the integral graph have become an active topic. A general approach to studying integral graphs is to focus on some special classes of graphs, such as Cayley graphs. A *Cayley graph* over a group G with respect to a subset S of G , denoted by $\text{Cay}(G, S)$, is a (di)graph with vertex set G such that (g, h) is an arc if and only if $hg^{-1} \in S$. If $S = \emptyset$ then $\text{Cay}(G, S)$ is the empty graph, and if $1_G \in S$ then the corresponding Cayley graph has a loop at each vertex. Although in this paper the connection set S is not necessarily inverse-closed, if S is inverse-closed, i.e., $S = S^{-1} = \{s^{-1} \mid s \in S\}$, then $\text{Cay}(G, S)$ is an undirected graph. Furthermore, $\text{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$. The Cayley graph over a cyclic group is also called *circulant*. A subset S of a group G is called *normal* if for any $g \in G$, we have $g^{-1}Sg = S$. Clearly, a subset of any abelian group is a normal set. The Cayley graph over a group G with respect to a normal subset S is called a *normal Cayley graph*. A large number of results on the eigenvalues of Cayley graphs have been produced over the past more than four decades; for a survey on this topic see [18]. All graphs in this paper are loop-free, non-empty, and are not necessarily undirected, but whenever a graph is, we will mention it.

Mönius, Steuding and Stumpf [22] introduced the concept of the splitting field and the algebraic degrees of graphs to investigate which graph properties prevent integral eigenvalues. For a graph Γ , its *splitting field* $\mathbb{SF}(\Gamma)$ is the smallest field extension of the rational number field $\mathbb{Q}$ which contains all the eigenvalues of the adjacency matrix of Γ . The extension degree $[\mathbb{SF}(\Gamma) : \mathbb{Q}]$ is called the *algebraic degree* of Γ , denoted by $\deg(\Gamma)$. Since eigenvalues of graphs are all algebraic integers, a graph Γ is integral if and only if $\deg(\Gamma) = 1$. Some works have been done, very recently, on determining algebraic degrees of Cayley graphs and their generalization [14, 19, 25, 26, 28, 29]. Note that the question raised by Harary and Schwenk can be translated as: which graphs have algebraic degree 1? It is natural to ask which graphs have algebraic degree k for a given positive integer k ? To study this question, we define k -integral graphs as follows.

Definition. For a given positive integer k , a graph Γ is called *k -integral* if $\deg(\Gamma) = k$.

To study k -integral graphs, it is crucial to know that the number of such connected graphs are finite. In Section 3, we will give a positive answer to this question. We prove that for any positive integers k and Δ , the set of all finite connected graphs with algebraic degree at most k and maximum degree at most Δ is finite (see Theorem 3.2). Our result further improves [9, Theorem 2] which states that all regular, connected, integral graphs of a fixed degree are finite.

Let $G = \langle a \rangle \cong \mathbb{Z}_n$ be a cyclic group of order n and S be an inverse-closed subset of G . In 2005, W. So proved that a loop-free graph $\text{Cay}(G, S)$ is integral if and only if

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S is a union of $G_n(d)$ s, where $d \neq n$ is a divisor of n and $G_n(d) = \{a^k \mid (k, n) = d\}$ [24, Theorem 7.1]. Since then some authors tried to generalize So's result to finite abelian groups and non-abelian groups [4, 5, 17]. An interesting question is how to construct k -integral Cayley graphs over cyclic groups. For $G = \langle a \rangle \cong \mathbb{Z}_n$ and any divisor $d \neq n$ of n , if we put $[a^d] = \{a^k \mid (k, n) = d\}$, then by So's result, $\text{Cay}(G, [a^d])$ is integral. In Section 4, we show that using some special subsets of $[a^d]$, one can construct k -integral Cayley graphs over cyclic groups (see Proposition 4.5). As a corollary, we also give a method to construct a p -integral Cayley graph over cyclic group $\mathbb{Z}_n$, where $n \geq 5$ and p is a prime divisor of n (see Corollary 4.6).

The problem of classifying all finite groups having a connected integral undirected Cayley graph of given valency is started by Abdollahi and Vatanadost [2]. They classified all such groups for valency 2 and 3 in [2] and partially for valency 4 in [3]. Then, the problem completely solved, using computer, for valency 4 in [21] by Minchenko and Wanless. We are also interested in classifying all finite abelian groups having a connected 2-integral undirected Cayley graph with small valency. To this purpose, we define the set $\mathcal{G}_k$ to be the set of all finite groups having a connected undirected 2-integral Cayley graph with valency k , and using Corollary 4.9 and some other general results, we completely characterize all finite abelian groups belong to $\mathcal{G}_k$ for $2 \leq k \leq 5$ (see Theorems 5.1, 5.3, 5.5, and 5.10). As a corollary, we also give the classification of 2-integral abelian Cayley graphs with valency 2, 3, 4 and 5.

In [10], Estélyi and Kovács determined all finite groups G for which all undirected graphs $\text{Cay}(G, S)$ are integral when $4 \leq |S| \leq k$ for each integer $k \geq 4$. Ma and Wang [20] characterized the finite groups each of whose cubic undirected Cayley graphs are integral. Inspired by the above works, it is interesting to classify the finite groups G for which all undirected graphs $\text{Cay}(G, S)$ are 2-integral when $2 \leq |S| \leq k$ for each integer $k \geq 2$. In Section 6, we focus on this problem. We show that there is no group G for which all undirected graphs $\text{Cay}(G, S)$ are 2-integral when $4 \leq |S|$ (see Theorem 6.3). Moreover, we completely classify the finite groups G that all undirected graphs $\text{Cay}(G, S)$ are 2-integral when $2 \leq |S| \leq k$ where $k = 2$ and 3 (see Theorems 6.5 and 6.11).

2 Preliminaries and notations

In this paper, (di)graphs are finite, loop-free and without multiple edges. Also the groups are finite. Our notations are standard and mainly taken from [12] and [16], but for the reader's convenience we recall some of them as follows:

- $\langle g \rangle$: the cyclic group generated by g .
- $\mathbb{Z}_n$: the additive group of integers modulo n .
- $\mathbb{Z}_n^*$: the multiplicative group of $\mathbb{Z}_n$.
- $\text{Aut}(G)$: the automorphism group of the group G .
- φ : the Euler function.
- τ : the automorphism of an abelian group which maps every element to its inverse.
- $G_1 \times G_2$: the direct product of groups G_1 and G_2 .
- $[g]$: the set $\{h \in G \mid \langle h \rangle = \langle g \rangle\}$, where $g \in G$.

- $Z(G)$: the center of G .
- $g^H : \{g^\sigma \mid \sigma \in H\}$, where $g \in G$ and $H \leq \text{Aut}(G)$.
- $\text{Irr}(G)$: the set of all inequivalent $\mathbb{C}$ -irreducible characters of group G .
- K_n : the complete graph with n vertices.
- $K_{n,m}$: the complete bipartite graph with parts having m and n vertices.
- $\Gamma_1 \vee \Gamma_2$: the join of graphs Γ_1 and Γ_2 .
- $\Gamma_1 \square \Gamma_2$: the Cartesian product of graphs Γ_1 and Γ_2 .
- $\Gamma_1 \otimes \Gamma_2$: the direct product of Γ_1 and Γ_2 .
- $\Gamma_1 \boxtimes \Gamma_2$: the strong product of Γ_1 and Γ_2 .
- $\Gamma_1[\Gamma_2]$: the lexicographic product of Γ_1 and Γ_2 .
- $\mathbb{SF}(\Gamma)$: the splitting field of a graph Γ .

Let Γ be a graph and $\mathbb{SF}(\Gamma)$ be the splitting field of Γ . By the definition of a k -integral graph, Γ being 2-integral means that $[\mathbb{SF}(\Gamma) : \mathbb{Q}] = 2$. It is known that $[\mathbb{SF}(\Gamma) : \mathbb{Q}] = 2$ implies that $\mathbb{SF}(\Gamma) = \mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{SF}(\Gamma) \setminus \mathbb{Q}$ and $\alpha^2 \in \mathbb{Z}$. Next we will show that there exists a 2-integral graph with n vertices for any integer $n \geq 3$.

Before the proof, we recall that the eigenvalues of the complete bipartite graph $K_{m,n}$ are 0 with multiplicity $m + n - 2$ and $\pm\sqrt{mn}$ with multiplicity 1. Hence if mn is not a square integer, $\mathbb{SF}(K_{m,n}) = \mathbb{Q}(\sqrt{mn})$ and so $K_{m,n}$ is 2-integral.

Lemma 2.1. *For any integer $n \geq 3$, there exists an undirected 2-integral graph with n vertices.*

Proof. The eigenvalues of P_3 are 0 and $\pm\sqrt{2}$, and the eigenvalues of P_4 are $\frac{\pm 1 \pm \sqrt{5}}{2}$. Thus P_3 and P_4 are both 2-integral. So we may assume that $n \geq 5$. If n is odd, then there exist distinct integers n_1 and n_2 such that $n = n_1 + n_2$ and $\sqrt{n_1 n_2}$ is not an integer. Now K_{n_1, n_2} is 2-integral with n vertices. Let $n = 2m$ be even and $m \geq 3$. In this case, all distinct eigenvalues of $K_{m-2, m+2}$ are 0 and $\pm\sqrt{m^2 - 4}$. Clearly, $\pm\sqrt{m^2 - 4} \notin \mathbb{Z}$ and $K_{m-2, m+2}$ is 2-integral. This completes the proof. $\square$

Next we introduce a special subset of a group, which will be useful to study k -integral Cayley graphs. Let G be a finite group, and for any $g \in G$, define $[g] := \{h \in G \mid \langle g \rangle = \langle h \rangle\}$. If g and h are two elements of G , then $[g] = [h]$ or $[g] \cap [h] = \emptyset$. Hence the set $\Omega(G) := \{[g] \mid g \in G\}$ is a partition of G . In the case that $G = \langle a \rangle \cong \mathbb{Z}_n$ is a cyclic group of order n generated by a , for a divisor $d \neq n$ of n , we have $[a^d] = \{a^k \mid 1 \leq k \leq n-1, (k, n) = d\}$ which is denoted by $G_n(d)$ in the literature [24].

3 The number of k -integral undirected graphs

One of the main questions in studying k -integral graphs is whether the number of such connected graphs with fixed maximum degree are finite. In this section, we give a positive answer to this question.

Proposition 3.1. *There exists a function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that the number of distinct eigenvalues of any finite undirected graph with maximum degree Δ and algebraic degree k is at most $f(k, \Delta)$.*

Proof. Let Γ be an undirected graph with algebraic degree k and $p(x)$ be the characteristic polynomial of the adjacency matrix of Γ . Then we have $p(x) = p_1(x)^{t_1} \cdots p_l(x)^{t_l}$, where $p_i(x)$ s are monic polynomials on $\mathbb{Q}[x]$, irreducible and pairwise relatively prime and t_i 's are positive integers. Since $p_i(x)$ s are irreducible and pairwise relatively prime in $\mathbb{Q}$, the roots of $p_i(x)$ s are pairwise distinct and $p_i(x)$ s has no repeated roots.

Let $p_i(x) = x^{k_i} + a_1x^{k_i-1} + \cdots + a_{k_i-1}x + a_{k_i}$, where a_i s are integers. Since the roots of $p_i(x)$ are also the roots of $p(x)$, k_i divides k . Furthermore, since the roots of $p_i(x)$ are eigenvalues of Γ , the absolute value of any root of $p_i(x)$ is less than or equal to $\Delta(\Gamma)$, the maximum degree of vertices of Γ . Now by Vieta's formula and triangle inequality, for each j , we have

$$|a_j| \leq \binom{k_i}{j} \Delta(\Gamma)^{k_i-j}.$$

This means that the number of such polynomials and so the number of distinct eigenvalues of Γ is less than or equal to a function of k and $\Delta(\Gamma)$, as desired. $\square$

Theorem 3.2. *For any positive integers k and Δ , the set of all finite connected undirected graphs with algebraic degree at most k and maximum degree at most Δ is finite.*

Proof. Since for any connected undirected graph with s distinct eigenvalues and diameter D , we have $D \leq s - 1$, the result directly follows from Proposition 3.1 and the fact that the order of a connected graph with given diameter and maximum degree is bounded. $\square$

4 k -integral Cayley graphs

In this section, we are going to study the algebraic degrees of normal Cayley graphs. First let us recall two known results about the algebraic degrees of abelian Cayley graphs and normal Cayley (di)graphs. Note that in these two results, S is not necessarily inverse-closed and so Γ can be a Cayley digraph. Furthermore, by S^k we mean the multiset $\{s^k \mid s \in S\}$.

Lemma 4.1 ([19, Theorem 1]). *Let G be an abelian group of order n and $\Gamma = \text{Cay}(G, S)$ for some subset S of G . Then the algebraic degree of Γ is $\deg(\Gamma) = \frac{\varphi(n)}{|H|}$, where $H = \{k \in \mathbb{Z}_n^* \mid S^k = S\}$.*

Lemma 4.2 ([26, Corollary 3.10]). *Let G be a finite group with exponent m , that is the smallest positive integer such that $g^m = 1$ for all $g \in G$, and $\Gamma = \text{Cay}(G, S)$ for some normal subset S of G . Then the algebraic degree of Γ is $\deg(\Gamma) = \frac{\varphi(m)}{|H'|}$, where $H' = \{k \in \mathbb{Z}_m^* \mid S^k = S\}$.*

By the following two lemmas, we reduce the study of k -integral normal Cayley graphs to the study of k -integral $\text{Cay}(\langle g \rangle, S)$, where $\emptyset \neq S \subseteq [g]$.

Lemma 4.3. *Let G be a finite group, $g \in G$ and $\emptyset \neq S \subseteq [g]$. Then $\langle S \rangle = \langle g \rangle$ and the splitting field of $\text{Cay}(G, S)$ and $\text{Cay}(\langle g \rangle, S)$ are the same.*

Proof. Since $x \in [g]$ if and only if $\langle x \rangle = \langle g \rangle$, the first part is clear. On the other hand, $\text{Cay}(G, S)$ is isomorphic to the disjoint union of $|G : \langle S \rangle|$ copies of $\text{Cay}(\langle S \rangle, S)$, where $\text{Cay}(\langle S \rangle, S)$ is a connected component of $\text{Cay}(G, S)$. Hence the sets of all distinct eigenvalues of $\text{Cay}(G, S)$ and $\text{Cay}(\langle S \rangle, S)$ are the same and so their splitting fields are, as desired. $\square$

Lemma 4.4. *Let G be a finite group, $\Omega(G) = \{[g_1], \dots, [g_t]\}$, $\emptyset \neq S$ be a normal subset of G , and $\Gamma = \text{Cay}(G, S)$. Then $S = \bigcup_{i=1}^t S_i$, where $S_i = [g_i] \cap S$ for each $1 \leq i \leq t$, and for $S_i \neq \emptyset$, $\deg(\Gamma_i)$ is a divisor of $\deg(\Gamma)$, where $\Gamma_i = \text{Cay}(G, S_i)$.*

Proof. Since $\Omega(G)$ is a partition of G , $X = \{S_i \mid 1 \leq i \leq t, S_i \neq \emptyset\}$ is a partition of S . Let m be the exponent of G and m_i be the order of g_i . Then $m_i | m$. By Lemma 4.2, $\deg(\Gamma) = \frac{\varphi(m)}{|H|}$, where $H = \{k \in \mathbb{Z}_m^* \mid S^k = S\}$. If $S_i \neq \emptyset$, then by Lemmas 4.3 and 4.1, $\deg(\Gamma_i) = \deg(\text{Cay}(\langle g_i \rangle, S_i)) = \frac{\varphi(m_i)}{|H_i|}$, where $H_i = \{k \in \mathbb{Z}_{m_i}^* \mid S_i^k = S_i\}$.

Suppose $k \in H$. Then $(k, m) = 1$ and $S^k = S$. Since $m_i | m$, clearly $(k, m_i) = 1$. So $\langle g_i^k \rangle = \langle g_i \rangle$ and therefore $[g_i]^k = [g_i^k] = [g_i]$. Thus

$$S_i^k = ([g_i] \cap S)^k \subseteq [g_i]^k \cap S^k = [g_i] \cap S = S_i.$$

Since X is a partition of S , we have $S_i^k = S_i$.

Since $m_i | m$, the map $\pi : \mathbb{Z}_m^* \rightarrow \mathbb{Z}_{m_i}^*$ which maps any $k \pmod{m}$ to $k \pmod{m_i}$ is an onto group homomorphism. Let K be the kernel of π . Then $|K| = \frac{\varphi(m)}{\varphi(m_i)}$, by the First Isomorphism Theorem. Let π' be the restriction of π to H . Then, by the above argument, $\pi'(H) \leq H_i$. Hence $|H|$ divides $|K'| |H_i|$, where K' is the kernel of π' , again by the First Isomorphism Theorem. On the other hand, $K' = K \cap H$ and so $|K'|$ divides $|K|$. This implies that $|H|$ divides $\frac{\varphi(m)}{\varphi(m_i)} |H_i|$ and so $\frac{\varphi(m_i)}{|H_i|}$ divides $\frac{\varphi(m)}{|H|}$ as desired. $\square$

Let $G = \langle a \rangle \cong \mathbb{Z}_n$ and $\Gamma = \text{Cay}(G, S)$. By Lemma 4.1, we know that $\deg(\Gamma)$ is a divisor of $\varphi(n)$. In what follows, we are going to give a way to find possible inverse-closed generating sets of G such that $\deg(\text{Cay}(G, S)) = 1$ or p for some prime $p \mid \varphi(n)$.

Proposition 4.5. *Let $n \geq 3$, $d \neq n$, $d \mid n$, $G = \langle a \rangle \cong \mathbb{Z}_n$ and $A = \text{Aut}(G)$. Let $T_d = \{\sigma \in A \mid g^\sigma = g, \forall g \in [a^d]\}$ and K_d be a subgroup of A containing T_d . For a fixed integer $1 \leq k \leq n - 1$ with $(k, n) = d$, put $S_{k,d} = \{(a^k)^\sigma \mid \sigma \in K_d\}$ and $\Gamma_{k,d} = \text{Cay}(G, S_{k,d})$. Then*

- (1) $S_{k,d}$ is inverse-closed if K_d contains the element τ of A which maps a to a^{-1} .
- (2) $|S_{k,d}| = |K_d : T_d| \leq \varphi(\frac{n}{d})$.
- (3) $S_{k,d}$ is a generating set of G if and only if $d = 1$. In this case, $T_1 = \{1_A\}$.
- (4) $\deg(\Gamma_{k,d})$ is a divisor of $|A : K_d|$.
- (5) $\Gamma_{k,d}$ is integral if and only if $A = K_d$. In this case, $S_{k,d} = [a^d]$.

In particular, if $|A : K_d| = p$ for some prime p , then $\Gamma_{k,d}$ is p -integral.

Proof. Part (1) is obvious by the definition of τ . Note that $[a^d]$ is the set of all elements of G with order $\frac{n}{d}$. Since the order of a^k is $\frac{n}{(k,n)} = \frac{n}{d}$ and each automorphism of G preserves the order of elements of G , $S_{k,d} \subseteq [a^d]$. Moreover, A can act on $[a^d]$ and T_d is the kernel of this action. Hence A/T_d is a permutation group on $[a^d]$. Note that $S_{k,d}$ is the orbit of a^k under the action of K_d on G . Since A is abelian, K_d/T_d is abelian. This implies that K_d/T_d is regular on $S_{k,d}$, which means $|S_{k,d}| = |K_d : T_d| \leq |[a^d]| = \varphi(\frac{n}{d})$. This proves (2). Since $S_{k,d} \subseteq [a^d]$, $\langle S_{k,d} \rangle \subseteq \langle a^d \rangle$. Hence $S_{k,d}$ generates G if and only if $d = 1$. Clearly if $d = 1$ then T_1 fixes a and so $T_1 = \{1_A\}$. This proves (3).

By Lemma 4.1, $\deg(\Gamma_{k,d}) = \frac{|A|}{|H|}$, where $H = \{\sigma \in A \mid (S_{k,d})^\sigma = S_{k,d}\}$. Clearly, we have $T_d \leq H$. On the other hand $K_d \leq H$. Hence we have $K_d/T_d \leq H/T_d \leq A/T_d$ and so $|A : K_d| = |A/T_d : K_d/T_d| = |A/T_d : H/T_d| |H/T_d : K_d/T_d| = |A : H| |H/T_d : K_d/T_d|$ which implies that $\deg(\Gamma_{k,d})$ divides $|A : K_d|$, which proves (4).

If $K_d = A$ then $\deg(\Gamma_{k,d}) = 1$, by (4), which means $\Gamma_{k,d}$ is integral. Conversely, suppose that $\Gamma_{k,d}$ is integral. Since $S_{k,d}$ is a subset of $[a^d]$, [24, Theorem 7.1] implies $S_{k,d} = [a^d]$. Now A acts transitively on $[a^d]$ and so A/T_d is a regular permutation group on $[a^d]$. Thus $|A/T_d| = |K_d/T_d|$, which means $A = K_d$. This completes the proof. $\square$

As an interesting application of Proposition 4.5, one can construct Cayley graphs over cyclic groups with prime algebraic degrees. More precisely, for any integer $n \geq 5$ and $n \neq 6$, one can construct a p -integral Cayley graph over $\mathbb{Z}_n$, where p is a prime divisor of $\varphi(n)$.

Corollary 4.6. *Let $n \geq 5$ be an integer.*

- (1) *If $\varphi(n)$ is not a power of 2, then for every odd prime divisor of $\varphi(n)$, there exists an undirected p -integral Cayley graph over $\mathbb{Z}_n$ with valency $\frac{\varphi(n)}{p}$;*
- (2) *If $\varphi(n)$ is a power of 2, then there exists an undirected 2-integral Cayley graph over $\mathbb{Z}_n$ with valency $\frac{\varphi(n)}{2}$.*

In particular, for every prime p and integer $k \geq 3$, there exists a connected p -integral undirected circulant graph of order p^k and valency $p^{k-2}(p-1)$.

Proof. Let $A = \text{Aut}(\mathbb{Z}_n)$ and τ be the element of A which maps each element to its inverse. Following the notation in Proposition 4.5 and letting $d = k = 1$, we have $T_d = T_1 = \{1_A\}$.

First suppose that $\varphi(n)$ is not a power of 2 and p is an odd prime divisor of $\varphi(n)$. Since $p \mid \varphi(n)$ and A is an abelian group of order $\varphi(n)$, there exists a subgroup K of A such that $|A : K| = p$. Let $K_1 := K$, $S_{1,1} := \{a^\sigma \mid \sigma \in K\}$ and $\Gamma_{1,1} = \text{Cay}(G, S_{1,1})$. Next we will show that $\tau \in K$ and so $S_{1,1}$ is an inverse-closed generating set of G by (1) and (3) of Proposition 4.5. Put $H := \langle K, \tau \rangle$. If $\tau \notin K$ then K is a proper subgroup of H and $|H : K| = 2$. Since $p = |A : K| = |A : H| |H : K| = 2|A : H|$, we have $p = 2$ and $A = H$, a contradiction. Hence $\tau \in K$. Moreover, by Proposition 4.5(2), $|S_{1,1}| = |K_1 : T_1| = |K| = \frac{\varphi(n)}{p}$ and $\deg(\Gamma_{1,1}) = p$. This proves (1).

Next suppose $\varphi(n)$ is a power of 2, that is, $\varphi(n) = 2^k$ for some $k \geq 3$. Hence A is an abelian 2-group of order 2^k . We know that there exists a maximal subgroup $M \neq 1$ of A containing $\langle \tau \rangle$. Since A is abelian, $|A : M|$ must be a prime, which means $|A : M| = 2$ and the second part follows from Proposition 4.5 as above.

For every prime p and integer $k \geq 3$, by (1), (2) and the fact $\varphi(p^k) = p^{k-1}(p-1)$, there exists a connected p -integral circulant graph of order p^k and valency $p^{k-2}(p-1)$. This completes the proof. $\square$

Suppose $G = \langle a \rangle \cong \mathbb{Z}_n$ and keep the notations in Proposition 4.5. For a prime divisor p of $\varphi(n)$, to construct a p -integral Cayley graph over G , it suffices to construct a subgroup K_d of A for any given $d \mid n$ such that $|A : K_d| = p$ and $\langle \tau, T_d \rangle \leq K_d$. Next we are going to give the element τ of A which maps a to a^{-1} first. Then we give a concrete example to illustrate how to construct the subgroup K_d of A containing $\langle \tau, T_d \rangle$ and p -integral Cayley graph over G .

Example 4.7. Let $G = \langle a \rangle \cong \mathbb{Z}_{20}$. Then $G = P_1 \times P_2$, where $P_1 = \langle a^5 \rangle \cong \mathbb{Z}_4$ and $P_2 = \langle a^4 \rangle \cong \mathbb{Z}_5$. Let $A = \text{Aut}(G)$, $A_1 = \text{Aut}(P_1)$ and $A_2 = \text{Aut}(P_2)$. Then $A_1 = \langle \pi_1 \rangle \cong \mathbb{Z}_2$, $A_2 = \langle \pi_2 \rangle \cong \mathbb{Z}_4$ and $A = A_1 \times A_2 = \langle \pi_1 \rangle \times \langle \pi_2 \rangle = \{ \pi_2^j, \pi_1 \pi_2^j \mid 0 \leq j \leq 3 \}$, where

$$\begin{aligned} \pi_1 : P_1 &\mapsto P_1, & a^5 &\mapsto a^{15}, \\ \pi_2 : P_2 &\mapsto P_2, & a^4 &\mapsto a^{12}. \end{aligned}$$

We have $\tau = \pi_1 \pi_2^2$. For convenience, let $\sigma_j = \pi_2^j$ and $\theta_j = \pi_1 \pi_2^j$ for each $0 \leq j \leq 3$. Then $\tau = \theta_2$. Moreover, by $a = (a^5)^{-3}(a^4)^4$, we have the following for each $0 \leq j \leq 3$,

$$\begin{aligned} \sigma_j : G &\rightarrow G, & a &\mapsto a^{-15+16 \times 3^j}, \\ \theta_j : G &\rightarrow G, & a &\mapsto a^{-5+16 \times 3^j}. \end{aligned}$$

In order to construct an undirected 2-integral Cayley graph over G , it is enough to construct a subgroup K_d of A with index 2 containing $\langle \tau, T_d \rangle$ for any given $d \mid n$ by Proposition 4.5. Clearly, $d \mid n$ if and only if $d = 1, 2, 4, 5, 10, 20$.

First let $d = 1$. Then $T_d = \{1_A\}$. Put $K_1 := \langle \tau, \sigma_2 \rangle$. Since $\tau \notin \langle \sigma_2 \rangle$, $K_1 = \langle \sigma_2 \rangle \times \langle \tau \rangle$. Note that $|\langle \sigma_2 \rangle| = o(\pi_2^2) = 2$. Then $K_1 = \{1, \tau, \sigma_2, \sigma_2 \tau\}$, and so $|A : K_1| = 2$. Now let $1 \leq k \leq 19$ be an integer coprime to 20. Then, by the notations of Proposition 4.5,

$$S_{k,1} = \{a^k, (a^k)^{\sigma_2}, a^{-k}, (a^{-k})^{\sigma_2}\} = \{a^k, a^{9k}, a^{-k}, a^{-9k}\}.$$

Furthermore, $\text{Cay}(G, S_{k,1})$ is a connected 4-regular 2-integral Cayley graph, where $S_{1,1} = S_{9,1} = S_{11,1} = S_{19,1} = \{a, a^{19}, a^9, a^{11}\}$, $S_{3,1} = S_{7,1} = S_{13,1} = S_{17,1} = \{a^3, a^{17}, a^7, a^{13}\}$.

Now let $d = 2$. Then for $1 \leq l \leq 20$, we have $(l, 20) = 2$ if and only if $l = 2, 6, 14, 18$. By the above discussion and an easy calculation, we have $T_2 = \{1_A, \theta_0\}$. Put $K_2 := \langle \tau, \theta_0 \rangle$. Then $|K_2| = 4$, $|A : K_2| = 2$, $|K_2 : T_2| = 2$, and $\text{Cay}(G, S_{k,2})$, where $k = 2, 6, 14, 18$, is a 2-regular and 2-integral Cayley graph, where $S_{2,2} = S_{18,2} = \{a^2, a^{18}\}$, $S_{6,2} = S_{14,2} = \{a^6, a^{14}\}$.

Now let $d = 4$. Then, since $a^{20} = 1$ and no power of π_2 maps a^{16} to itself, similar to the previous paragraph, we see that $T_4 = \{\sigma_0, \theta_0\}$. Hence, similar to the case $d = 2$, we conclude that $\text{Cay}(G, S_{k,4})$, where $k = 4, 8, 12, 16$ is 2-integral. More precisely, $S_{4,4} = S_{16,4} = \{a^4, a^{16}\}$ and $S_{8,4} = S_{12,4} = \{a^8, a^{12}\}$.

Now let $d = 5$. Since $a^{25} = a^5$ and $a^{20} = 1$, similar to the above, we see that $T_5 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, which means that $\langle \tau, T_5 \rangle = A$ and so, by Proposition 4.5, the corresponding Cayley graphs are integral. So in this case we can not construct any 2-integral Cayley graph over G .

In the case, $d = 10, 20$, clearly $T_d = A$, which again we can not construct any 2-integral Cayley graph over G . Hence we have shown that if S is one of the following sets, then $\text{Cay}(G, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_{20}$ is 2-integral:

$$\{a, a^{19}, a^9, a^{11}\}, \{a^3, a^{17}, a^7, a^{13}\}, \{a^2, a^{18}\}, \{a^6, a^{14}\}, \{a^4, a^{16}\}, \{a^8, a^{12}\}.$$

Another important application of Proposition 4.5 is that it enables us to study the 2-integral Cayley graphs. First we give a characterization of the 2-integrality of $\text{Cay}(G, S)$, where $\emptyset \neq S \subseteq [g]$ for some $g \in G$.

Lemma 4.8. *Let G be a finite group, $g \in G$ be an element of order n , $\emptyset \neq S \subseteq [g]$, maybe not inverse-closed, and $\Gamma = \text{Cay}(G, S)$. Then Γ is 2-integral if and only if there exists a subgroup H of $\text{Aut}(\langle g \rangle)$ such that $|\text{Aut}(\langle g \rangle) : H| = 2$, $S = s^H$ for all $s \in S$ and $|S| = \frac{\varphi(n)}{2}$.*

Proof. Suppose Γ is 2-integral. Then $\text{Cay}(\langle g \rangle, S)$ is 2-integral by Lemma 4.3 and $S \neq [g]$ by [24, Corollary 7.2]. Furthermore, by Lemma 4.1, $|A : H| = 2$, where $A = \text{Aut}(\langle g \rangle) \cong \mathbb{Z}_n^*$ and $H = \{\sigma \in A \mid S^\sigma = S\}$. Since for any $\sigma \in A$ there exists an integer $1 \leq i \leq n$ with $(i, n) = 1$ such that $g^\sigma = g^i$, A acts transitively on $[g]$. Note that $|A| = |[g]|$, which means that A acts regularly on $[g]$. Hence the action of H on each orbit of H is regular. Let $s \in S$. Then $|s^H| = |H| = \frac{|A|}{2} = \frac{\varphi(n)}{2}$. Since H acts semiregularly on S , we have $|H|$ divides $|S|$. On the other hand, $s^H \subseteq S \subset [g]$ implies $\frac{\varphi(n)}{2} = |H| = |s^H| \leq |S| < |[g]| = \varphi(n)$ and so $S = s^H$.

Conversely, suppose that $S = s^H$ for some $s \in S$, where H is a subgroup of $\text{Aut}(\langle g \rangle)$ such that $|\text{Aut}(\langle g \rangle) : H| = 2$ and $|S| = \frac{\varphi(n)}{2}$. Let $\Sigma = \text{Cay}(\langle g \rangle, S)$. Then, by Proposition 4.5 (putting $d = 1$), $\deg(\Sigma) = 2$. Now Lemma 4.3 implies that Γ is 2-integral as desired. $\square$

By Lemma 4.8, we can characterize the generating sets of 2-integral normal Cayley graphs.

Corollary 4.9. *Let G be a finite group, $\Omega(G) = \{[g_1], \dots, [g_n]\}$, S is a normal subset of G and $\Gamma = \text{Cay}(G, S)$. Then $S = \bigcup_{i=1}^n S_i$, where $S_i = [g_i] \cap S$ for each $1 \leq i \leq n$. If Γ is 2-integral, then for $S_i \neq \emptyset$,*

- (1) *there exists $s_i \in S$ such that $S_i = [s_i]$ or $S_i = s_i^H \subset [s_i]$ for some subgroup H of $\text{Aut}(\langle s_i \rangle)$ of index 2, and $|S_i| = \frac{\varphi(n_i)}{2}$, where $n_i = o(s_i)$;*
- (2) *there exists $1 \leq i_0 \leq n$ such that S_{i_0} is of the latter form.*

Proof. Suppose Γ is 2-integral. By Lemma 4.4, $S = \bigcup_{i=1}^n S_i$ is a disjoint union of sets $S_1, \dots, S_n$, where $S_i = [g_i] \cap S$ for each $1 \leq i \leq n$, and if $S_i \neq \emptyset$, $\deg(\Gamma_i) = 1$ or 2, where $\Gamma_i = \text{Cay}(G, S_i)$. For $S_i \neq \emptyset$, $\deg(\Gamma_i) = 1$ if and only if $S_i = [g_i]$ by [5, Corollary 7.2], and $\deg(\Gamma_i) = 2$ if and only if there exists a subgroup H of $\text{Aut}(\langle g_i \rangle)$ such that $|\text{Aut}(\langle g_i \rangle) : H| = 2$, $S_i = s^H$ for all $s \in S_i$ and $|S_i| = \frac{\varphi(o(g_i))}{2}$ by Lemma 4.8. If for each i with $S_i \neq \emptyset$, we have $\deg(\Gamma_i) = 1$, then S is a union of $[g_i]$ s and so $\deg(\Gamma) = 1$, by [5, Proposition 4.1], which contradicts the 2-integrality of Γ . Hence there exists i_0 such that $\deg(\Gamma_{i_0}) = 2$. $\square$

5 Finite abelian groups admitting a connected 2-integral undirected Cayley graph with small valency

Recall that a cyclic group $G = \langle a \rangle$ of order n admits a connected integral undirected Cayley graph of valency 2 if and only if $n = 3, 4, 6$ [2, Lemma 2.7]. Also recall that an undirected Cayley graph $\text{Cay}(G, S)$ over an abelian group G is integral if and only if S is a union of some $[g]_S$ s, where $g \in S$ [4, 5]. If X be a non-empty subset of a group G and χ be a character of G , we set $\chi(X) = \sum_{x \in X} \chi(x)$. It is well-known that if S be a conjugate-closed subset of a group G and $\Gamma = \text{Cay}(G, S)$, then eigenvalues of Γ are $\frac{\chi(S)}{\chi(1)}$ with multiplicity $\chi(1)^2$, where χ runs over $\text{Irr}(G)$, see [30, Theorem 1] or [15]. In this section, we use this fact frequently.

Let $\mathcal{G}_k$ be the set of all finite groups admitting a connected 2-integral undirected Cayley graph with valency k . In this section, we completely characterize all finite abelian groups belong to $\mathcal{G}_k$ for $2 \leq k \leq 5$.

5.1 $\mathcal{G}_2$

Theorem 5.1. *Let G be a finite abelian group. Then $G \in \mathcal{G}_2$ if and only if $G \cong \mathbb{Z}_n$, where $n = 5, 8, 10, 12$. Furthermore, a cycle with order n is 2-integral if and only if $n = 5, 8, 10, 12$.*

Proof. Let $S = \{x, y\}$ be an inverse-closed generating set of G and $\Gamma = \text{Cay}(G, S)$ is 2-integral. Then either $x^2 = y^2 = 1$ or $y = x^{-1}$. In the first case, $S = S_1 \cup S_2$, where $S_1 = \{x\} = [x]$, $S_2 = \{y\} = [y]$, which means $\text{Cay}(G, S_i)$, $i = 1, 2$, are both integral which contradicts Corollary 4.9. In the later case, we have $S \subset [x]$ and $G = \langle x \rangle$. Let $n = o(x)$. Then, Lemma 4.8 implies that $\varphi(n) = 4$ and so $n = 5, 8, 10, 12$.

Conversely, suppose that $G = \langle x \rangle \cong \mathbb{Z}_n$, where $n = 5, 8, 10, 12$. Let $A = \text{Aut}(G)$ and put $H = \langle \tau \rangle$. Then in each case $|A : H| = 2$ and $|S| = \frac{\varphi(n)}{2}$, where $S = x^H$. Hence $\text{Cay}(G, S)$ is 2-integral, by Lemma 4.8. $\square$

5.2 $\mathcal{G}_3$

To classify all finite abelian groups in $\mathcal{G}_3$, we need the following general result.

Lemma 5.2. *Let G be a finite group and S be a normal subset of G . Then for any $1 \neq x \in Z(G)$ in which $S \cap [x] = \emptyset$, $\text{Cay}(G, S)$ and $\text{Cay}(G, S \cup [x])$ have the same splitting field. In particular, if $x \in Z(G) \setminus S$ is an involution, then $\text{Cay}(G, S)$ and $\text{Cay}(G, S \cup \{x\})$ have the same splitting field.*

Proof. Let $\chi \in \text{Irr}(G)$ and $1 \neq x \in Z(G)$ such that $S \cap [x] = \emptyset$. Since $x \in Z(G)$, [16, Exercise 5 of Chapter 13] implies that for each i we have $\chi(x^i) = \lambda^i \chi(1)$, where λ is an n th root of unity and $n = o(x)$. Hence $\frac{\chi([x])}{\chi(1)} = \sum_{1 \leq i \leq n, (i, n)=1} \lambda^i$ is an integer, say t , by [16, Lemma 22.15]. Since $\frac{\chi(S \cup [x])}{\chi(1)} = \frac{\chi(S) + \chi([x])}{\chi(1)} = \frac{\chi(S)}{\chi(1)} + t$, clearly the splitting field of $\text{Cay}(G, S)$ and $\text{Cay}(G, S \cup [x])$ are the same, as desired. $\square$

Theorem 5.3. *Let G be a finite abelian group. Then $G \in \mathcal{G}_3$ if and only if G is isomorphic to $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_2$, where $n = 8, 10, 12$. Furthermore, a connected cubic abelian undirected Cayley graph Γ is 2-integral if and only if it is isomorphic to one of the following 6 Cayley graphs $\text{Cay}(G, S)$, where:*

- (i) $G = \langle x \rangle \cong \mathbb{Z}_n$ where $n = 8, 10, 12$ and $S = \{x, x^{-1}, x^{\frac{n}{2}}\}$;
- (ii) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 8, 10, 12$ and $S = \{x, x^{-1}, y\}$.

Proof. Let $S = \{x, y, z\}$ be an inverse-closed generating set of G and $\Gamma = \text{Cay}(G, S)$ is 2-integral. We deal with the following cases:

Case I. $x^2 = y^2 = z^2$. In this case, $S = S_1 \cup S_2 \cup S_3$, where $S_1 = [x]$, $S_2 = [y]$ and $S_3 = [z]$, which contradicts Corollary 4.9.

Case II. $y = x^{-1}$ and $z = x^{\frac{n}{2}}$, where $o(x) = n$ is even. Then $G = \langle x \rangle$ and $S = S_1 \cup S_2$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$ and $S_2 = [x^{\frac{n}{2}}]$. By Corollary 4.9, $4 = \varphi(n)$ which means $n = 8, 10, 12$. Hence in this case $G = \langle x \rangle \cong \mathbb{Z}_n$, where $n = 8, 10, 12$.

Case III. $y = x^{-1}$, $z^2 = 1$ and $z \notin \langle x \rangle$. In this case, $G = \langle x \rangle \times \langle z \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$, where $n = o(x)$. Furthermore, $S = S_1 \cup S_2$, where $S_1 = \{x, x^{-1}\} \subset [x]$ and $S_2 = [z]$. Similar to the Case II, we get $\varphi(n) = 4$ and so $n = 5, 8, 10, 12$.

Hence we have showed that if $\Gamma = \text{Cay}(G, S)$ is 2-integral, then $G = \langle x \rangle \cong \mathbb{Z}_n$ and $S = \{x, x^{-1}, x^{\frac{n}{2}}\}$ or $G = \langle x \rangle \times \langle z \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ and $S = \{x, x^{-1}, z\}$, where $n = 8, 10, 12$. This proves one direction.

For the converse direction, first we suppose that $G = \langle x \rangle \cong \mathbb{Z}_n$ and $S = S_1 \cup S_2$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$, $S_2 = \{x^{\frac{n}{2}}\} = [x^{\frac{n}{2}}]$ and $n = 8, 10, 12$. Note that $S_1 \cap [x^{\frac{n}{2}}] = \emptyset$, then $\text{SF}(\text{Cay}(G, S)) = \text{SF}(\text{Cay}(G, S_1))$ by Lemma 5.2. Use the same argument as in the proof of Theorem 5.1, we have $\text{Cay}(G, S_1)$ is 2-integral and so $\text{Cay}(G, S)$ is 2-integral. Hence $G \cong \mathbb{Z}_n \in \mathcal{G}_3$, where $n = 8, 10, 12$. Similar arguments show the desired results for the case $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$, where $n = 8, 10, 12$. $\square$

5.3 $\mathcal{G}_4$

To characterize all finite abelian groups belong to $\mathcal{G}_4$, we need the following general result:

Lemma 5.4. *Let G be a finite group, $S_1, \dots, S_k$ be normal subsets of G , $\langle S_i \rangle \cap \langle S_j \rangle = \{1\}$ for all distinct i, j , and $S = S_1 \cup \dots \cup S_k$. Let $\Gamma = \text{Cay}(G, S)$ and $\Gamma_i = \text{Cay}(G, S_i)$, $i = 1, \dots, k$. Then*

- (1) $\text{SF}(\Gamma_i) \subseteq \text{SF}(\Gamma)$ for all i ,
- (2) $\deg(\Gamma_i)$ divides $\deg(\Gamma)$ for all i ,
- (3) if $\deg(\Gamma_i) = \deg(\Gamma) = 2$ for some i , then $\text{SF}(\Gamma) = \text{SF}(\Gamma_i)$,
- (4) if $\text{SF}(\Gamma_i) = \mathbb{F}$ for all i , then $\text{SF}(\Gamma) = \mathbb{F}$.

Proof. Put $G_i := \langle S_i \rangle$, $i = 1, \dots, k$. Since S_i is a normal subset of G , G_i is a normal subgroup of G . Let λ be an eigenvalue of Γ_i and $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. Then $\lambda = \frac{\psi(S_i)}{\psi(1)}$ for some $\psi \in \text{Irr}(G_i)$. Let $\psi \uparrow G$ be the induced character from ψ . Then $\psi \uparrow G = d_1\chi_1 + \dots + d_k\chi_k$ for some integers $d_1, \dots, d_k$. Since S_i s are normal subsets of G and $\langle S_l \rangle \cap \langle S_{l'} \rangle = \{1\}$ for distinct l, l' , we have $(\psi \uparrow G)(S_i) = \frac{1}{|G_i|}\psi(S_i)$ and $(\psi \uparrow G)(S \setminus S_i) = 0$. Hence $\frac{1}{|G_i|}\psi(S_i) = (\psi \uparrow G)(S) = d_1\chi_1(S) + \dots + d_k\chi_k(S)$. Since the splitting fields contain the rational field and the character degrees are positive integers, we conclude that $\lambda \in \text{SF}(\Gamma)$. Hence $\text{SF}(\Gamma_i) \subseteq \text{SF}(\Gamma)$. This proves (1). Parts (2) and (3) are direct consequences of (1).

Let λ be an eigenvalue of Γ . Since S_i 's are normal subsets of G , S is also a normal subset of G . Hence there exists $\chi \in \text{Irr}(G)$ such that $\lambda = \frac{\chi(S)}{\chi(1)}$. On the other hand, S_i 's are pairwise disjoint and so $\chi(S) = \chi(S_1) + \cdots + \chi(S_k)$. Since for each i , $\chi(S_i) \in \mathbb{F}$, we conclude that $\lambda \in \mathbb{F}$. Thus $\mathbb{SF}(\Gamma) \subseteq \mathbb{F}$. Now (1) implies that $\mathbb{SF}(\Gamma) = \mathbb{F}$ as desired. $\square$

Theorem 5.5. *Let G be a finite abelian group. Then $G \in \mathcal{G}_4$ if and only if G is isomorphic to one the following groups:*

- (1) $\mathbb{Z}_n$, where $n = 8, 10, 12, 15, 16, 20, 24, 30$.
- (2) $\mathbb{Z}_n \times \mathbb{Z}_2^2$, where $n = 5, 8, 10, 12$.
- (3) $\mathbb{Z}_n \times \mathbb{Z}_2$, where $n = 8, 10, 12$.
- (4) $\mathbb{Z}_n \times \mathbb{Z}_m$, where (n, m) is one of the pairs $(3, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10), (6, 12), (5, 5), (5, 10), (8, 8), (10, 10), (12, 12)$.

Furthermore, a connected 4-regular abelian undirected Cayley graph Γ is 2-integral if and only if it is isomorphic to one of the following 39 Cayley graphs $\text{Cay}(G, S)$, where

- (i) $G = \langle x \rangle \cong \mathbb{Z}_n$ where $n = 15, 16, 20, 24, 30$, and $S = \{x, x^{-1}, x^k, x^{-k}\}$ where $1 \leq k \leq n$ such that $(k, n) = 1$,
- (ii) $G = \langle x \rangle \cong \mathbb{Z}_8$ and $S = \{x, x^2, x^6, x^7\}$,
- (iii) $G = \langle x \rangle \cong \mathbb{Z}_{10}$ and $S = \{x, x^2, x^8, x^9\}$,
- (iv) $G = \langle x \rangle \cong \mathbb{Z}_{12}$ and $S = \{x, x^2, x^{10}, x^{11}\}$ or $\{x, x^3, x^9, x^{11}\}$ or $\{x, x^4, x^8, x^{11}\}$,
- (v) $G = \langle x \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 8, 10, 12$ and $S = \{x, x^{-1}, x^{\frac{n}{2}}, w\}$,
- (vi) $G = \langle x \rangle \times \langle z \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$ where $n = 5, 8, 10, 12$ and $S = \{x, x^{-1}, z, w\}$,
- (vii) $G = \langle x \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$ where (n, m) is one of the pairs $(3, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10), (6, 12), (5, 5), (5, 10), (8, 8), (10, 10), (12, 12)$, and $S = \{x, x^{-1}, y, y^{-1}\}$.

Proof. Let $S = \{x, y, z, w\}$ be an inverse closed generating set of G and $\text{Cay}(G, S)$ is 2-integral. Then we deal with the following cases:

Case I. $x^2 = y^2 = z^2 = w^2 = 1$. In this case, G is an elementary abelian 2-group isomorphic to $\mathbb{Z}_2^n$, where $n = 3$ or 4 . Furthermore, $S = S_1 \cup S_2 \cup S_3 \cup S_4$, where $S_1 = \{x\} = [x]$, $S_2 = \{y\} = [y]$, $S_3 = \{z\} = [z]$ and $S_4 = \{w\} = [w]$, which contradicts Corollary 4.9.

Case II. $z^2 = w^2 = 1, z, w \notin \langle x \rangle$ and $y = x^{-1}$. In this case, $G = \langle x \rangle \times \langle z \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$, where $n = o(x)$. Furthermore, $S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$, $S_2 = \{z\} = [z]$ and $S_3 = \{w\} = [w]$. Now Corollary 4.9 implies that $S_1 \neq [x]$ and $\varphi(n) = 4$ which means $n = 5, 8, 10, 12$.

Case III. $y = x^{-1}, z = x^{\frac{n}{2}}$, where $n = o(x)$ is even and $w^2 = 1$. In this case, $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$. Also $S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$, $S_2 = \{x^{\frac{n}{2}}\} = [x^{\frac{n}{2}}]$ and $S_3 = \{w\} = [w]$. Then by Corollary 4.9, $S_1 \neq [x]$ and $\varphi(n) = 4$, which means $n = 8, 10, 12$.

Case IV. $y = x^k$ for some $k \neq 1, -1$, $z = x^{-1}$ and $w = y^{-1}$. In this case, $y^2 \neq 1$ and $G = \langle x \rangle \cong \mathbb{Z}_n$, where $n = o(x)$.

First assume that $(k, n) = 1$. Then $S \subset [x]$ and Lemma 4.8 implies that $\varphi(n) = 8$, which means $n = 15, 16, 20, 24, 30$.

Now let $(k, n) = d \neq 1$. Then $S = S_1 \cup S_2$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$ and $S_2 = \{x^k, x^{-k}\} \subseteq [x^k]$. If $S_1 = [x]$ and $S_2 \neq [x^k]$ then $n = 3, 4, 6$ and $\frac{n}{d} = 5, 8, 10, 12$, respectively, which is impossible. If $S_1 \neq [x]$ and $S_2 = [x^k]$, then $n = 5, 8, 10, 12$ and $\frac{n}{d} = 3, 4, 6$, respectively, which implies $(n, k) = (8, 2), (12, 2), (12, 3)$ or $(12, 4)$. If $S_1 \neq [x]$ and $S_2 \neq [x^k]$ then $n, \frac{n}{d} = 5, 8, 10, 12$, which implies $(n, k) = (10, 2)$.

Case V. $\langle x \rangle \cap \langle y \rangle = 1$. Then $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$, where $n = o(x)$ and $m = o(y)$. We may assume that $n \leq m$. In this case, $S = S_1 \cup S_2$, where $S_1 = \{x, x^{-1}\} \subseteq [x]$ and $S_2 = \{y, y^{-1}\} \subseteq [y]$. By Corollary 4.9, $S_1 = [x], S_2 \neq [y]$ or $S_1 \neq [x], S_2 = [y]$ or $S_1 \neq [x], S_2 \neq [y]$. By a similar discussion to the above cases, in the first case $n = 3, 4, 6$, $m = 5, 8, 10, 12$, in the second case $n = 5, 8, 10, 12$, $m = 3, 4, 6$ and in the later case $n, m = 5, 8, 10, 12$. The second case is impossible because $n \leq m$. Next we will show that in the last case (n, m) must be $(5, 5), (5, 10), (8, 8), (10, 10)$ or $(12, 12)$. Let $g \in G$ and $o(g) = k \geq 2$ and $C_k = \text{Cay}(\langle g \rangle, \{g, g^{-1}\})$. By an easy computation, we have $\text{SF}(C_5) = \text{SF}(C_{10}) = \mathbb{Q}[\sqrt{5}]$, $\text{SF}(C_8) = \mathbb{Q}[\sqrt{2}]$ and $\text{SF}(C_{12}) = \mathbb{Q}(\sqrt{3})$. Hence, by Corollary 4.9 and Lemma 5.4, we have the result as desired. Moreover, Cases I-V prove one direction.

The proof of converse direction of $G \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$ where $n = 5, 8, 10, 12$, and $G \cong \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 8, 10, 12$ are similar to the proof of converse direction of Theorem 5.3.

Consider the converse direction of case $G = \langle x \rangle \cong \mathbb{Z}_n$, where $o(x) = n$ is shown in (1). Let $\Sigma_k = \text{Cay}(\langle x \rangle, \{x, x^k, x^{-1}, x^{-k}\})$. First suppose that $n = 15, 16, 20, 24, 30$ and $(n, k) = 1$. Let $\sigma_k: x \mapsto x^k$ and $\tau: x \mapsto x^{-1}$. Then $H = \langle \tau, \sigma_k \rangle$ is a subgroup of index 2 in $\text{Aut}(G)$ and $x^H = \{x, x^k, x^{-1}, x^{-k}\}$. By Lemmas 4.8, we have $\text{SF}(\Sigma_k) = 2$. Now suppose that $n = 8, 10, 12$. By a tedious computation, one can see that $\text{SF}(\Sigma_k) = \mathbb{Q}(\sqrt{2})$ if $(n, k) = (8, 2), (12, 2)$ or $(12, 4)$, $\text{SF}(\Sigma_k) = \mathbb{Q}(\sqrt{3})$ or $\mathbb{Q}(\sqrt{5})$ if $(n, k) = (12, 3)$ or $(10, 2)$ respectively.

Finally, consider the converse direction of case $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$, where (n, m) shown in (3). Let $S = S_1 \cup S_2$ and $S_1 = \{x, x^{-1}\} \subseteq [x]$ and $S_2 = \{y, y^{-1}\} \subseteq [y]$. Note that C_3, C_4, C_6 are integral with valency 2 and $\mathbb{Z}_8, \mathbb{Z}_{10}, \mathbb{Z}_{12}$ are all in $\mathcal{G}_2$ by Theorem 5.1. Hence by Lemma 5.2, $\mathbb{Z}_n \times \mathbb{Z}_m \in \mathcal{G}_4$, where $(n, m) = (3, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10)$ and $(6, 12)$. On the other hand, since $\text{SF}(C_5) = \text{SF}(C_{10}) = \mathbb{Q}[\sqrt{5}]$, $\text{SF}(C_8) = \mathbb{Q}[\sqrt{2}]$ and $\text{SF}(C_{12}) = \mathbb{Q}(\sqrt{3})$, by Lemma 5.4, $\mathbb{Z}_n \times \mathbb{Z}_m \in \mathcal{G}_4$, where $(n, m) = (5, 5), (5, 10), (8, 8)$. This completes the proof. $\square$

5.4 $\mathcal{G}_5$

To characterize all finite abelian groups in $\mathcal{G}_5$, we need the following general results.

Lemma 5.6. *Let G be a finite group of even order, S be a normal subset of G and there exists an involution $x \in Z(G) \cap S$. If $\text{Cay}(G, S)$ is connected with the splitting field $\mathbb{F}$, then*

- (1) $G = \langle S \setminus \{x\} \rangle$ or $G = \langle S \setminus \{x\} \rangle \times \langle x \rangle \cong \langle S \setminus \{x\} \rangle \times \mathbb{Z}_2$;
- (2) $\text{Cay}(\langle S \setminus \{x\} \rangle, S \setminus \{x\})$ is a connected graph with the splitting field $\mathbb{F}$.

Proof. Since $\text{Cay}(G, S)$ is connected, $G = \langle S \rangle = \langle (S \setminus \{x\}) \cup \{x\} \rangle$. Since $x \in Z(G)$, $G = \langle S \setminus \{x\} \rangle \langle x \rangle$. Moreover, $S \setminus \{x\}$ is also conjugate-closed, and since x has order 2, $G = \langle S \setminus \{x\} \rangle$ or $G = \langle S \setminus \{x\} \rangle \times \langle x \rangle \cong \langle S \setminus \{x\} \rangle \times \mathbb{Z}_2$. This proves (1).

Let $\Gamma_1 = \text{Cay}(G, S \setminus \{x\})$. We claim that the splitting field of Γ_1 is $\mathbb{F}$. First we suppose that $G = \langle S \setminus \{x\} \rangle$. Then Lemma 5.2 implies that $\text{Cay}(G, S \setminus \{x\})$ and $\text{Cay}(G, S)$ have the same splitting field and so the splitting field of Γ_1 is $\mathbb{F}$. Next we suppose that $G = \langle S \setminus \{x\} \rangle \times \langle x \rangle$. Let λ be an eigenvalue of $\text{Cay}(G, S)$. Then $\lambda = \frac{\chi(S)}{\chi(1)}$, for some $\chi \in \text{Irr}(G)$. Since $x \in Z(G)$ we have $\chi(x) = \pm\chi(1)$, see [16, Exercise 5 of Chapter 13], which implies $\lambda = \frac{\chi(S \setminus \{x\})}{\chi(1)} \pm 1$. Let $H = \langle S \setminus \{x\} \rangle$ and $\chi|_H$ be the restriction of χ to H . Since $|G : H| = 2$ and $\chi(x) \neq 0$, [16, Proposition 20.5] implies that $\chi|_H$ is an irreducible character of H . Furthermore, $\chi(1) = \chi|_H(1)$ and $\chi|_H(S \setminus \{x\}) = \chi(S \setminus \{x\})$ which imply $\lambda = \theta \pm 1$, for some eigenvalue θ of Γ_1 . Now assume that μ be an eigenvalue of Γ_1 . Then $\mu = \frac{\eta(S \setminus \{x\})}{\eta(1)}$ for some $\eta \in \text{Irr}(H)$. Let ρ_0 be the principal character of $\langle x \rangle$. Then $\zeta := \eta \times \rho_0$ is an irreducible character of G and $\zeta(S) = \eta(S \setminus \{x\}) + \rho_0(x) = \eta(S \setminus \{x\}) + 1$. Hence $\mu = \xi - \frac{1}{\zeta(1)}$, where $\xi = \frac{\zeta(S)}{\zeta(1)}$ is an eigenvalue of $\text{Cay}(G, S)$. Since $\frac{1}{\zeta(1)} \in \mathbb{Q}$, we conclude that the splitting field of $\text{Cay}(G, S)$ and Γ_1 are the same. This proves (2). $\square$

Lemma 5.7. *Let Γ_1 be an undirected graph with the splitting field $\mathbb{F}$ and Γ_2 be an integral undirected graph. Then $\Gamma_1 \square \Gamma_2$, $\Gamma_1 \otimes \Gamma_2$, and $\Gamma_1 \boxtimes \Gamma_2$ have the same splitting field $\mathbb{F}$. Furthermore, if Γ_2 is a connected regular graph then $\Gamma_1[\Gamma_2]$ has also the same splitting field $\mathbb{F}$.*

Proof. Let the eigenvalues of Γ_1 and Γ_2 be $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n_1}$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n_2}$, respectively, where n_i is the number of vertices of Γ_i . By [8, Table 4], the eigenvalues of $\Gamma_1 \square \Gamma_2$, $\Gamma_1 \otimes \Gamma_2$ and $\Gamma_1 \boxtimes \Gamma_2$ are $\lambda_i + \mu_j$, $\lambda_i \mu_j$, and $\lambda_i + \mu_j + \lambda_i \mu_j$, respectively, where $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$.

Let Γ_2 be a connected k -regular graph. Then eigenvalues of $\Gamma_1[\Gamma_2]$ are $k + n_2 \lambda_i$, $1 \leq i \leq n_1$ and μ_j with multiplicity n_1 , where $2 \leq j \leq n_2$. Since eigenvalues of Γ_2 are integers, the result follows immediately. $\square$

Corollary 5.8. *Let $\Gamma_1 = \text{Cay}(G_1, S_1)$ and $\Gamma_2 = \text{Cay}(G_2, S_2)$ be two undirected connected Cayley graphs such that Γ_1 is k -integral and Γ_2 is integral. Then $\text{Cay}(G_1 \times G_2, S)$ is an undirected connected k -integral Cayley graph over $G_1 \times G_2$, where S is one of the following sets*

- (1) $S = \{(s_1, 1), (1, s_2) \mid s_1 \in S_1, s_2 \in S_2\}$,
- (2) $S = \{(s_1, s_2) \mid s_1 \in S_1, s_2 \in S_2\}$,
- (3) $S = \{(s_1, 1), (1, s_2), (s_1, s_2) \mid s_1 \in S_1, s_2 \in S_2\}$,
- (4) $S = \{(s_1, g_2), (1, s_2) \mid s_1 \in S_1, s_2 \in S_2, g_2 \in G_2\}$, where at most one of the graphs Γ_1 or Γ_2 is bipartite.

In particular, if $|S_i| = d_i$, $i = 1, 2$, then in the above cases, $|S|$ is $d_1 + d_2$, $d_1 d_2$, $d_1 + d_2 + d_1 d_2$ and $d_1 |G_2| + d_2$, respectively.

Proof. It is a direct consequence of Lemma 5.7, [12, Corollaries 5.3, 5.6, 5.10, 5.14] and [1, Theorems 3.1, 4.1, 5.3, 6.1]. $\square$

Corollary 5.9. *Let G be a finite abelian group of even order. There exists an undirected connected $(2d + 1)$ -regular Cayley graph over G with algebraic degree k if and only if*

- (1) *there exists an undirected connected $2d$ -regular k -integral Cayley graph over G , or*
- (2) *there exists a $H \leq G$ such that $G = H \times \mathbb{Z}_2$ and H admits an undirected connected $2d$ -regular k -integral Cayley graph.*

Proof. Note that the number of involutions of any finite abelian group with even order is odd. If $\text{Cay}(G, S)$ is connected with algebraic degree k and $|S| = 2d$, for some inverse-closed subset S of G , then there exists an involution $x \in G \setminus S$. Hence $\text{Cay}(G, S \cup \{x\})$ is connected $(2d + 1)$ -regular and with algebraic degree k by Lemma 5.2. If $G = H \times \mathbb{Z}_2$ and H admits a connected $2d$ -regular Cayley graph with algebraic degree k , since K_2 as a Cayley graph over $\mathbb{Z}_2$ is integral, then Corollary 5.8 implies that G admits a connected $(2d + 1)$ -regular Cayley graph with algebraic degree k .

The converse is clear by Lemma 5.6, because any inverse-closed subset of G with odd number of elements has an involution. $\square$

Theorem 5.10. *Let G be a finite abelian group. Then $G \in \mathcal{G}_5$ if and only if $G \cong H$ or $H \times \mathbb{Z}_2$, where H is one of the groups given in Theorem 5.5 and in the first case $H \not\cong \mathbb{Z}_{15}, \mathbb{Z}_5^2$. Furthermore, a connected 5-regular abelian undirected Cayley graph Γ is 2-integral if and only if it is isomorphic to one of the following 108 Cayley graphs $\text{Cay}(G, S)$, where*

- (i) $G = \langle x \rangle \cong \mathbb{Z}_n$ where $n = 16, 20, 24, 30$, and $S = \{x, x^{-1}, x^{\frac{n}{2}}, x^k, x^{-k}\}$ where $1 \leq k \leq n$ such that $(k, n) = 1$;
- (ii) $G = \langle x \rangle \cong \mathbb{Z}_8$ and $S = \{x, x^2, x^4, x^6, x^7\}$;
- (iii) $G = \langle x \rangle \cong \mathbb{Z}_{10}$ and $S = \{x, x^2, x^5, x^8, x^9\}$;
- (iv) $G = \langle x \rangle \cong \mathbb{Z}_{12}$ and $S = \{x, x^2, x^6, x^{10}, x^{11}\}$ or $\{x, x^3, x^6, x^9, x^{11}\}$ or $\{x, x^4, x^6, x^8, x^{11}\}$;
- (v) $G = \langle x \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 8, 10, 12$ and $S = \{x, x^{-1}, x^{\frac{n}{2}}, w, x^{\frac{n}{2}}w\}$;
- (vi) $G = \langle x \rangle \times \langle z \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$ where $n = 5, 8, 10, 12$ and $S = \{x, x^{-1}, z, w, s\}$ where s is any involution of G except for z and w ;
- (vii) $G = \langle x \rangle \times \langle w \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$ where (n, m) is one of the pairs $(3, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10), (6, 12), (5, 10), (8, 8), (10, 10), (12, 12)$, and $S = \{x, x^{-1}, y, y^{-1}, s\}$ where s is any involution of G .
- (viii) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$ where $n = 16, 20, 24, 30$, and $S = \{x, x^{-1}, x^k, x^{-k}, y\}$ where $1 \leq k \leq n$ such that $(k, n) = 1$;
- (ix) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ and $S = \{x, x^2, x^6, x^7, y\}$;
- (x) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$ and $S = \{x, x^2, x^8, x^9, y\}$;
- (xi) $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_{12} \times \mathbb{Z}_2$ and $S = \{x, x^2, x^{10}, x^{11}, y\}$ or $\{x, x^3, x^9, x^{11}, y\}$ or $\{x, x^4, x^8, x^{11}, y\}$;

- (xii) $G = \langle x \rangle \times \langle w \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2^2$ where $n = 8, 10, 12$ and $S = \{x, x^{-1}, x^{\frac{n}{2}}, w, y\}$;
- (xiii) $G = \langle x \rangle \times \langle z \rangle \times \langle w \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2^3$ where $n = 5, 8, 10, 12$ and $S = \{x, x^{-1}, z, w, y\}$;
- (xiv) $G = \langle x \rangle \times \langle w \rangle \times \langle y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_2$ where (n, m) is one of the pairs $(3, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10), (6, 12), (5, 10), (8, 8), (10, 10), (12, 12)$, and $S = \{x, x^{-1}, y, y^{-1}, y\}$.

Proof. Note that in any undirected graph, the number of vertices of odd degree is even. Thus G must be a group of even order. And so this result is a direct consequence of Theorem 5.5 and Corollary 5.9. $\square$

Similar to Theorem 5.5, one can find all finite abelian groups $G \in \mathcal{G}_6$ and then it is possible to characterize all finite abelian groups $G \in \mathcal{G}_7$. Recursively, one can find all finite abelian groups in $\mathcal{G}_k$ for any given integer k . For instance, if the degree is odd, then Corollary 5.9 can be used.

6 On groups all of whose undirected Cayley graphs of bounded valency are 2-integral

In this section, we are going to classify the finite groups G that all undirected Cayley graphs $\text{Cay}(G, S)$ are 2-integral when $2 \leq |S| \leq k$ for each integer $k \geq 2$. This leads us to give the following definition.

Definition 6.1. For an integer $k \geq 2$, set

$$\mathcal{B}_k :=$$

$$\{G \mid \text{Cay}(G, S) \text{ is 2-integral whenever } S \subset G \text{ such that } 1_G \notin S = S^{-1} \text{ and } 2 \leq |S| \leq k\}.$$

Clearly, $\mathcal{B}_{k+1} \subseteq \mathcal{B}_k$, and if $|G|$ is odd, then $G \in \mathcal{B}_{2k-1}$ if and only if $G \in \mathcal{B}_{2k-2}$.

To determine the sets $\mathcal{B}_k$, we start with some basic and useful results about $\mathcal{B}_k$.

Lemma 6.2. Let $k \geq 2$ be an integer. Then the following holds for every $G \in \mathcal{B}_k$.

- (1) Every proper subgroup $H < G$ with $H \not\cong \mathbb{Z}_2$ is also in $\mathcal{B}_k$.
- (2) For every $g \in G$, the order of g is in $\{1, 2, 5, 10\}$.
- (3) A Sylow 5-subgroup of G is a group of exponent 5.
- (4) If $|G|$ is even, then Sylow 2-subgroups of G are isomorphic to $\mathbb{Z}_2$.

Moreover, $|G| = 2^i 5^j$, where $i = 0, 1$ and $j \geq 1$, and the Sylow 5-subgroup of G is normal.

Proof. Let $G \in \mathcal{B}_k$. Suppose H is a proper subgroup of G and $H \not\cong \mathbb{Z}_2$. Since for a subset $S \subset H \leq G$, the Cayley graph $\text{Cay}(G, S)$ consists of disjoint union $|G : H|$ copies of $\text{Cay}(H, S)$, we have (1).

Since $G \cong \mathbb{Z}_2$ is Cayley integral, we may assume that $|G| \geq 3$. If all non-identity elements of G are involutions, then we have (2). Next suppose that G has at least one element with order great than 2, say g . Then $\text{Cay}(G, \{g, g^{-1}\})$ is 2-integral and so

$\text{Cay}(\langle g \rangle, \{g, g^{-1}\})$ is 2-integral by (1), which implies $o(g)$, the order of g , is in $\{5, 8, 10, 12\}$ by Theorem 5.1. If $o(g) = 8$ or 12 , then $h := g^2$ or $h := g^3$ has order 4, respectively. Now $\text{Cay}(G, \{h, h^{-1}\})$ must be 2-integral, which means $o(h) \in \{5, 8, 10, 12\}$, a contradiction. This proves (2).

By (2) we may assume that $|G| = 2^i 5^j$, $i, j \geq 0$. If $j = 0$ then, by 2), G is 2-elementary abelian and so it is Cayley integral, a contradiction. Hence we may assume that $j \geq 1$ and again by (2), a Sylow 5-subgroup of G is a group of exponent 5, which proves (3). Let $i \neq 0$ and P_2 be a Sylow 2-subgroup of G . Clearly, $P_2 \cong \mathbb{Z}_2^i$. If $i \geq 2$, then $P_2 \cong \mathbb{Z}_2^i \in \mathcal{B}_k$ by 1). However $\mathbb{Z}_2^i$ is Cayley integral, a contradiction. Thus $i = 1$, which proves (4).

Let P_5 be a Sylow 5-subgroup of G . Then $G = P_5$ if $i = 0$ and $|G : P_5| = 2$ if $i = 1$, which implies that P_5 is normal in G . This completes the proof. $\square$

Theorem 6.3. $\mathcal{B}_k$ is an empty set for any $k \geq 4$.

Proof. Suppose $G \in \mathcal{B}_k$ for a $k \geq 4$. Then $G \in \mathcal{B}_4$. By Lemma 6.2, G has at least one subgroup $H = \langle h \rangle \cong \mathbb{Z}_5$. Note that $\text{Cay}(H, H \setminus \{1\})$ is integral, contradicting to Lemma 6.2(1). Thus $G \notin \mathcal{B}_k$ for any $k \geq 4$. This completes the proof. $\square$

Proposition 6.4. $D_{2n} \in \mathcal{B}_k$ for $k = 2, 3$ if and only if $n = 5$.

Proof. If $D_{2n} \in \mathcal{B}_k$, then by Lemma 6.2 we have $n = 5$. For the converse direction, it can be directly calculate that $D_{10} \in \mathcal{B}_k$ for $k = 2, 3$. $\square$

By Theorem 6.3, it is enough to determine the sets $\mathcal{B}_2$ and $\mathcal{B}_3$. In what follows, we will do this.

6.1 $\mathcal{B}_2$

In this part, we will classify the groups in $\mathcal{B}_2$.

Theorem 6.5. $G \in \mathcal{B}_2$ if and only if G is a 5-group of exponent 5 or the Sylow 5-subgroup of G is a group of exponent 5 and has index 2.

Proof. One direction is clear by Lemma 6.2.

Next suppose P_5 is a 5-group of exponent 5. If $G = P_5$, then for any inverse-closed subset S of G , $S = \{g, g^{-1}\}$, where the order of g is 5, and so $\text{Cay}(\langle g \rangle, \{g, g^{-1}\})$ is 2-integral by Lemma 5.1. Hence $\text{Cay}(G, S)$ is 2-integral.

Suppose $P_5 < G$ and $|G : P_5| = 2$. Then there exist $a \in G$ such that $G = P_5 \cup aP_5 = P_5 \cup P_5a$. Furthermore, by Lemma 6.2, we may assume that $a^2 = 1$. Next we consider the form of inverse-closed subset S of G with $|S| = 2$.

Case I: $S = \{x, x^{-1}\}$, where $x \in G$ is not an involution.

If $x \in P_5$, then the order of x is 5 and so $\text{Cay}(\langle x \rangle, \{x, x^{-1}\})$ is 2-integral by Lemma 5.1. Hence $\text{Cay}(G, S)$ is 2-integral.

Suppose that $x \notin P_5$, that is to say $x = ap = p'a$ for some $p, p' \in P_5$. Then $x^2 = p'ap = p'p \in P_5$, which means that the order of x is 5 or 10. In both cases $\text{Cay}(\langle x \rangle, \{x, x^{-1}\})$ is 2-integral and so $\text{Cay}(G, S)$ is 2-integral.

Case II: $S = \{x, y\}$, where $x, y \in G$ are both involution.

In this case, $x = ap = p^{-1}a$ and $y = ap'^{-1} = p'a$ for some $p, p' \in P_5$. Note that $xy \neq yx$, otherwise $\langle x, y \rangle \cong \mathbb{Z}_2^2 \leq G$ which is impossible by Lemma 6.2(3). Furthermore, $xy \in P_5$. Hence either $xy = 1$ or xy has order 5. The first case is impossible, otherwise $x = y^{-1}$ which is a contradiction. Hence xy has order 5 and $\langle x, y \rangle \cong D_{10}$. By Proposition 6.4, $D_{10} \in \mathcal{B}_2$. Hence $\text{Cay}(G, S)$ is 2-integral, which completes the proof. $\square$

6.2 $\mathcal{B}_3$

In this part, we will classify the groups in $\mathcal{B}_3$. First we consider the nilpotent groups.

Proposition 6.6. *If G is a nilpotent group, then $G \in \mathcal{B}_3$ if and only if $G \cong P_5$ or $P_5 \times \mathbb{Z}_2$, where P_5 is a 5-group of exponent 5.*

Proof. Suppose G is a nilpotent group. If $G \in \mathcal{B}_3$, then by Lemma 6.2, $G \cong P_5$ or $P_5 \times \mathbb{Z}_2$, where P_5 is a 5-group of exponent 5.

Next suppose $G \cong P_5$ or $G \cong \mathbb{Z}_2 \times P_5$, where P_5 is a 5-group of exponent 5. First assume that $G \cong P_5$. Then by Theorem 6.5, $P_5 \in \mathcal{B}_2$. Since $|P_5|$ is odd, we have $G \in \mathcal{B}_3$. Assume that $G \cong \mathbb{Z}_2 \times P_5$, and S is an inverse-closed subset of G such that $1_G \notin S$ where $|S| = 2$ or 3 . Note that the order of any non-identity element of G is 2, 5 or 10. If $|S| = 2$, then $S = \{g, g^{-1}\}$, where the order of g is 5 or 10. Furthermore, $\langle S \rangle = \langle g \rangle \cong \mathbb{Z}_5$ or $\mathbb{Z}_{10}$. By Lemma 5.1, $\text{Cay}(\langle S \rangle, S)$ is 2-integral. Hence $\text{Cay}(G, S)$ is 2-integral. If $|S| = 3$, then $S = \{a, g, g^{-1}\}$, where $a^2 = 1_G$ and the order of g is 5 or 10. Furthermore, $\langle S \rangle = \langle a \rangle \times \langle g \rangle \cong \mathbb{Z}_5 \times \mathbb{Z}_2 = \mathbb{Z}_{10}$. By Theorem 5.3(i), $\text{Cay}(\langle S \rangle, S)$ is 2-integral. Hence $\text{Cay}(G, S)$ is 2-integral. This completes the proof. $\square$

In what follows, we consider the non-nilpotent groups in $\mathcal{B}_3$.

Proposition 6.7. *Suppose $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$ is not nilpotent. Then $G \notin \mathcal{B}_3$. Moreover, $G = \mathbb{Z}_5^n \rtimes \mathbb{Z}_2 \notin \mathcal{B}_3$ for any $n \geq 2$.*

Proof. Suppose $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2 \cong (\langle a \rangle \times \langle b \rangle) \rtimes \langle x \rangle$, where $a^5 = b^5 = x^2 = 1_G$. Write $P = \langle a \rangle \times \langle b \rangle$. Actually, G is the generalized dihedral group over P , that is, $p^x = p^{-1}$ for all $p \in P$.

Let $S = \{bx, b^2x, ax\}$. Clearly S is inverse-closed. Next we will show that $\text{Cay}(G, S)$ is not 2-integral. Note that $\text{Cay}(G, S)$ is isomorphic to $\text{BiCay}(P, \emptyset, \emptyset, T)$, where $T = \{b, b^2, a\}$, according [7, Lemma 8]. By [27, Theorem 3.2], the eigenvalues of $\text{BiCay}(P, \emptyset, \emptyset, T)$ are $\pm|\chi_p(b+b^2+a)|$, where $p \in P$ and χ_p 's are irreducible characters of P , see also [6, Section 3.]. However, one can calculate that $\pm|\chi_{b^4}(b+b^2+a)| = \sqrt{4 + \sqrt{5}}$. Thus $\text{BiCay}(P, \emptyset, \emptyset, T)$ is not 2-integral. Hence $G \notin \mathcal{B}_3$.

On the other hand, $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_2 \leq \mathbb{Z}_5^n \rtimes \mathbb{Z}_2$ for $n \geq 2$. By Lemma 6.2(1), if $\mathbb{Z}_5^n \rtimes \mathbb{Z}_2 \in \mathcal{B}_3$, then $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_2 \in \mathcal{B}_3$, a contradiction. Thus $\mathbb{Z}_5^n \rtimes \mathbb{Z}_2 \notin \mathcal{B}_3$ for any $n \geq 2$. $\square$

The structure of minimal non-abelian group of exponent 5 is given in the following lemma. To have a full characterization of non-nilpotent groups in $\mathcal{B}_3$, we need this lemma.

Lemma 6.8 ([23]). *The minimal non-abelian group of exponent 5 is $G_5 = \langle a, b, c \mid a^5 = b^5 = c^5 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$.*

Keeping the notation of Lemma 6.8, we get the following result.

Lemma 6.9. *Suppose $G = G_5 \rtimes \mathbb{Z}_2$ is not nilpotent. Then G is isomorphic to one of the following groups:*

- (1) $G_{250,1} \cong ((\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle) \rtimes \langle d \rangle$, where $a^5 = b^5 = c^5 = d^2 = 1_G$, $a^d = a^4$, $b^d = b^4$, $c^d = c$ and $b^c = ab$;
- (2) $G_{250,2} \cong ((\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle) \rtimes \langle d \rangle$, where $a^5 = b^5 = c^5 = d^2 = 1_G$, $a^d = a$, $b^d = b^4$, $c^d = c^4$ and $b^c = ab$.

Moreover, $G_{250,1}$ and $G_{250,2}$ are not contained in $\mathcal{B}_3$.

Proof. By GAP [11], G is isomorphic to $G_{250,1}$ or $G_{250,2}$. Furthermore, $G_{250,1}$ has a subgroup isomorphic to $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$ and $G_{250,2}$ has a subgroup isomorphic to $\mathbb{Z}_5 \times D_{10}$. By Lemma 6.2(1), $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_2 \notin \mathcal{B}_3$. Hence $G_{250,1} \notin \mathcal{B}_3$. Suppose $\mathbb{Z}_5 \times D_{10} \cong \langle x \rangle \times (\langle y \rangle \rtimes \langle z \rangle) \leq G_{250,2}$, where $x^5 = y^5 = z^2 = 1$. Let $S = \{y^4 z, xy^3 z, x^4 y^3 z\}$. Clearly, $S = S^{-1}$ and $\langle S \rangle \cong \mathbb{Z}_5 \times D_{10}$. By a calculation, $\sqrt{4 + \sqrt{5}}$ is an eigenvalue of $\text{Cay}(\mathbb{Z}_5 \times D_{10}, S)$. Thus $\mathbb{Z}_5 \times D_{10} \notin \mathcal{B}_3$. This implies that $G_{250,2} \notin \mathcal{B}_3$. $\square$

Now we are ready to characterize non-nilpotent groups in $\mathcal{B}_3$.

Proposition 6.10. *If G is not nilpotent, then $G \in \mathcal{B}_3$ if and only if $G \cong D_{10}$.*

Proof. Suppose G is not nilpotent and $G \in \mathcal{B}_3$. By Lemma 6.2, $G \cong P_5 \rtimes \mathbb{Z}_2$, where P_5 is a 5-group of exponent 5. If P_5 is abelian, then $P_5 \cong \mathbb{Z}_5^n$ for some $n \geq 1$. By Proposition 6.4 and 6.7, $P_5 = \mathbb{Z}_5$ and so $G \cong D_{10}$. If P_5 is not abelian, then P_5 has a minimal non-abelian subgroup isomorphic to G_5 by Lemma 6.8. Moreover, G has a subgroup isomorphic to $G_5 \rtimes \mathbb{Z}_2$. By Lemma 6.9, $G_5 \rtimes \mathbb{Z}_2 \notin \mathcal{B}_3$, which contradicts Lemma 6.2(1). Hence P_5 is abelian, and $G \cong D_{10}$. The converse direction is clear by Proposition 6.4. This completes the proof. $\square$

By Propositions 6.6 and 6.10, we have the following result.

Theorem 6.11. *$G \in \mathcal{B}_3$ if and only if $G \cong P_5$ or $P_5 \times \mathbb{Z}_2$ or D_{10} , where P_5 is a 5-group of exponent 5.*

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