

Integral Cayley graphs of symmetric groups on transpositions

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ABSTRACT

We study subsets T consisting of some transpositions (i, j) of the symmetric group S_n on the set $\{1, \dots, n\}$ such that the Cayley graph $\Gamma_T := \text{Cay}(S_n, T)$ is an integral graph, that is, all eigenvalues of its adjacency matrix are integers. Graph properties of Γ_T are determined in terms of ones of the graph G_T whose vertex set is $\{1, \dots, n\}$, and $\{i, j\}$ is an edge if and only if $(i, j) \in T$. Here, we prove that if G_T is a tree, then Γ_T is integral if and only if T is isomorphic to the star graph $K_{1,n-1}$, answering Problem 5 of Liu and Zhou (2022) [16]. Problem 6 of that article asks necessary and sufficient conditions on T for $\text{Cay}(S_n, T)$ to be integral. We show that if G_T is a graph that we call a “generalized complete multipartite graph”, then $\text{Cay}(S_n, T)$ is integral. We conjecture that $\text{Cay}(S_n, T)$ is integral only if G_T is a generalized complete multipartite graph. To support conjecture, we show its validity whenever G_T belongs to some classes of graphs, including threshold graphs, cycles and cubic graphs.

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1. Introduction

Let G be a group and S be a subset of G such that S is inverse closed (that is, $s \in S \Leftrightarrow s^{-1} \in S$) and the identity element 1 of G does not belong to S . Then, the Cayley graph $\text{Cay}(G, S)$ is the graph whose vertex set is the set of elements of G and two vertices $g_1, g_2 \in G$ are adjacent if and only if $g_1 g_2^{-1} \in S$. It is well known that $\text{Cay}(G, S)$ is connected if and only if G can be generated by S .

Let S_n be the symmetric group on $[n] := \{1, \dots, n\}$. For a set T of transpositions in S_n , G_T is the graph with vertex set $[n]$ such that $i, j \in [n]$ are adjacent if and only if $(i, j) \in T$. It is well known that $\text{Cay}(S_n, T)$ is connected if and only if G_T is connected. The (adjacency) spectrum of a finite simple graph Γ is the multiset of eigenvalues of the adjacency matrix of Γ . A graph is called integral if and only if the spectrum consists of only integers.

Here, we study the following problems proposed by Liu and Zhou in [16].

Problem 1.1. [16, Problem 5] Let T be a set of transpositions in S_n such that G_T is a tree, where $n \geq 2$. Give necessary and sufficient conditions for $\text{Cay}(S_n, T)$ to be integral.

Problem 1.2. [16, Problem 6] Let T be a set of transpositions in S_n such that G_T is connected, where $n \geq 2$. Give necessary and sufficient conditions for $\text{Cay}(S_n, T)$ to be integral.

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In Problem 1.1, a sufficient condition on T is known to be $G_T \cong K_{1,n-1}$, the star graph with n vertices (see [1], [5, Theorem 5.3], [6], [14]). We complete the answer to Problem 1.1 by showing that this condition is also necessary (see part (b) of Corollary 3.9).

About Problem 1.2, we prove that if G_T is a “generalized complete multipartite graph” (see Definition 3.10), then $\text{Cay}(S_n, T)$ is integral. We propose this as a conjecture (see Conjecture 3.14) that the converse of this condition holds. To support Conjecture 3.14, we prove its validity for some classes of graphs, including cycles and cubic graphs (see Propositions 3.16, 3.17, and 3.18).

2. Preliminaries

Throughout this paper, we use the following notations:

C_n : The cycle graph with n vertices.

K_n : The complete graph with n vertices.

$K_{n_1, n_2, \dots, n_k}$: The complete multipartite graph with k parts of orders $n_1, n_2, \dots, n_k$.

For a given graph Γ :

$V(\Gamma)$: The set of vertices of Γ .

$E(\Gamma)$: The set of edges of Γ .

$\deg_\Gamma(u)$: The degree of vertex u of Γ .

$n\Gamma$: The graph of disjoint union of n copies of the graph Γ .

$\Gamma_1 \cup \Gamma_2$: The disjoint union of graphs Γ_1 and Γ_2 .

$\Gamma_1 * \Gamma_2$: The join graph of disjoint graphs Γ_1 and Γ_2 .

$\mathcal{L}(\Gamma)$: The Laplacian matrix of Γ , which is $D - A$, where D is the diagonal matrix of the vertex-degrees and A is the adjacency matrix of Γ with the same row vertex-labeling as D .

I_Γ : The set of all isolated vertices of a graph Γ .

$\Gamma^\#$: The induced subgraph of Γ on the set $V(\Gamma) \setminus I_\Gamma$.

$\Gamma + v$: The graph obtained by adding the isolated vertex v to the graph Γ .

2.1. Eigenvalues of Cayley graphs

For the basic definitions and facts from the representation theory of finite groups, not defined here, we refer the reader to [18]. Let G be a finite group, $\text{Irr}(G) = \{\rho_1, \dots, \rho_m\}$ the set of all inequivalent irreducible complex representations of G , and $\mathbb{C}G$ the group algebra of elements of the form $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{C}$. It is clear that $\mathbb{C}G$ is a vector space over $\mathbb{C}$ of dimension $|G|$. The left regular representation $\rho_{\text{reg}} : G \rightarrow \text{GL}(\mathbb{C}G)$, where $\text{GL}(\mathbb{C}G)$ denotes the set of all invertible linear transformations on the $\mathbb{C}$ -vector space $\mathbb{C}G$, is the right translation on $\mathbb{C}G$ by the following rule

$$\rho_{\text{reg}}(g) \left(\sum_{h \in G} a_h h \right) = \sum_{h \in G} a_h (gh).$$

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley (di)graph over G with respect to a subset S of G , and A the adjacency matrix of Γ . If we view A as a linear map $A : \mathbb{C}G \rightarrow \mathbb{C}G$, then $A = \sum_{s \in S} \rho_{\text{reg}}(s)$. Since each linear transformation on a vector space with given basis is uniquely determined by a matrix, in some cases, we state our results based on matrix theory language. The following result is found in [2] and [9].

Proposition 2.1. ([2, Corollary 7] and [9]) Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph over a finite group G with irreducible matrix representations $\rho_1, \dots, \rho_m$. Let d_l be the degree of ρ_l . For each $l \in \{1, \dots, m\}$, define a $d_l \times d_l$ matrix $A_l := \rho_l(S)$, where $\rho_l(S) = \sum_{s \in S} \rho_l(s)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomials of A_l and A , respectively.

- (1) There exists a basis $\mathcal{B}$ such that $[A]_{\mathcal{B}} = \text{Diag}(A_1 \otimes I_{d_1}, \dots, A_m \otimes I_{d_m})$, where I_{d_l} is the $d_l \times d_l$ identity matrix and $\otimes$ is the tensor product of matrices.
- (2) $\chi_A(\lambda) = \prod_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.
- (3) If S is a union of conjugacy classes of G , then eigenvalues of Γ are given by $\frac{\sum_{s \in S} \text{Tr}(\rho_l(s))}{d_l}$, where Tr denotes the trace function.

A partition is a weakly decreasing finite sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_l)$. We call $|\lambda| = \lambda_1 + \dots + \lambda_l$ the size of λ . The notation $\lambda \vdash n$ is used for a partition λ of a positive integer n . It is well known that both the conjugacy classes of S_n and the irreducible characters of S_n are indexed by partitions λ of $[n]$ (see [12]).

Lemma 2.2. Let T be the set of all transpositions of S_n or, equivalently, G_T is the complete graph. Then, the eigenvalues of $\text{Cay}(S_n, T)$ are given by

$$q_\lambda := \frac{1}{2} \sum_{j=1}^m ((\lambda_j - j)(\lambda_j - j + 1) - j(j + 1)),$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ runs over all partitions of n .

Proof. It is a direct consequence of part (3) of Proposition 2.1 and [9, Lemma 7]. $\square$

Proposition 2.3. Let T be a set of transpositions in S_n such that G_T is a (disjoint) union of complete multipartite graphs. Then, $\text{Cay}(S_n, T)$ is integral.

Proof. Since, by Lemma 2.2, for every partition $\lambda \vdash n$, q_λ is an integer, it follows from [5, Theorem 5.3 and Proposition 5.2], that $\text{Cay}(S_n, T)$ is integral. $\square$

3. Main results

Given a permutation $\pi \in S_n$, we denote by $P_\pi = (p_{ij})$ the $n \times n$ permutation matrix with entries $p_{ij} = 1$ if $i^\pi = j$, and 0 otherwise.

Recall that a graph Γ is called Laplacian integral if the spectrum of $\mathcal{L}(\Gamma)$ consists of only integers.

Theorem 3.1. Let T be a set of transpositions in S_n , for $n \geq 2$. If λ is an eigenvalue of $\mathcal{L}(G_T)$ then $|T| - \lambda$ is an eigenvalue of $\text{Cay}(S_n, T)$. In particular, if $\text{Cay}(S_n, T)$ is integral then G_T is Laplacian integral.

Proof. Given $\tau = (i, j) \in T$, we have $P_\tau = I_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, where e_{rs} is the $n \times n$ matrix that its rs -th entry is 1 and the remaining entries are 0. Moreover, $\sum_{(i,j) \in T} [e_{ii} + e_{jj}]$ and $\sum_{(i,j) \in T} [e_{ij} + e_{ji}]$ are the degree matrix and the adjacency matrix of G_T , respectively. Therefore, $\sum_{\tau \in T} P_\tau = |T|I_n - \mathcal{L}(G_T)$. Besides, each eigenvalue of $\sum_{\tau \in T} P_\tau$ is an eigenvalue of $\text{Cay}(S_n, T)$, by [8, Lemma 1.1 and Proposition 2.1]. Therefore, the integrality of $\text{Cay}(S_n, T)$ implies the integrality of $\mathcal{L}(G_T)$, as desired. $\square$

Remark 3.2. Let $\sigma = (1, 2, 3, 4, 5, 6)$ and $\pi = (1, 2)$. One can find that the map

$$\begin{aligned} \varphi : S_6 &\rightarrow GL_5(\mathbb{C}) \\ \sigma &\mapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ \pi &\mapsto \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

is an irreducible representation of S_6 . Let $T = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1)\}$. Then, the characteristic polynomial of $\sum_{t \in T} \varphi(t)$ is $x(x+2)^2(x^2+2x-12)$ has non-integer roots, which means that $\text{Cay}(S_n, T)$ is a non-integral Cayley graph by Proposition 2.1. Since $G_T \cong C_6$ is Laplacian integral, this means that the converse of Theorem 3.1 does not hold in general.

Definition 3.3. Suppose $\Delta_1, \dots, \Delta_k$, for $k \geq 2$, are graphs with the same vertex set. A graph Δ is called a *weakly join* of $\Delta_1, \dots, \Delta_k$ if there exist subgraphs Γ_i of Δ_i , for $1 \leq i \leq k$, such that

- (1) for every $1 \leq i \leq k$, $E(\Gamma_i) = E(\Delta_i)$,
- (2) for every $i \neq j$, $V(\Gamma_i) \cap V(\Gamma_j) = \emptyset$, and
- (3) $\Delta^\# = \Gamma_1 * \dots * \Gamma_k$, where $*$ is the join graph of disjoint graphs.

We use the notation $\mathcal{WJ}(\Delta_1, \dots, \Delta_k)$ for the set of all weakly join graphs defined with respect to $\Delta_1, \dots, \Delta_k$.

Example 3.4. Let $n \geq 2$ be a positive integer. One can easily check that $\mathcal{WJ}(nK_1, nK_1)$ includes all complete bipartite graphs with some isolated vertices which have totally n vertices. Now suppose $\Delta \in \mathcal{WJ}(nK_1, nK_1)$. Then $\Delta^\# = \Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are both subgraphs of nK_1 with disjoint vertex sets. Thus $\Delta^\#$ is isomorphic to K_{n_1, n_2} where $n_i = |V(\Gamma_i)|$, for every $i = 1, 2$. Therefore Δ is a complete bipartite graph with some isolated vertices. This shows that $\mathcal{WJ}(nK_1, nK_1)$ is precisely the set of all complete bipartite graphs with some isolated vertices.

Remark 3.5. Let $\Gamma_1, \dots, \Gamma_k$ be graphs with disjoint vertex sets. For every $1 \leq i \leq k$, we define the graph Δ_i as follows: the vertex set of Δ_i is $V(\Delta_i) := \bigcup_{j=1}^k V(\Gamma_j)$, and two vertices u and v of $V(\Delta_i)$ are adjacent whenever they are adjacent in Γ_i . Set $\Delta = \Gamma_1 \cdots \Gamma_k$. Then $\Delta = \Delta^\#$ and so $\Delta \in \mathcal{WJ}(\Delta_1, \dots, \Delta_k)$.

Theorem 3.6. Let T_1, T_2 , and T be (possibly empty) sets of transpositions in S_n such that $G_T \in \mathcal{WJ}(G_{T_1}, G_{T_2})$. Suppose that $\text{Cay}(S_n, T_1)$ is integral. Then, $\text{Cay}(S_n, T)$ is integral if and only if $\text{Cay}(S_n, T_2)$ is integral.

Proof. Since $G_T \in \mathcal{WJ}(G_{T_1}, G_{T_2})$, we can present $G_T^\# = \Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are subgraphs with disjoint vertices of G_{T_1} and G_{T_2} , respectively, such that $E(\Gamma_i) = E(G_{T_i})$ for $i \in \{1, 2\}$. Then $E(G_{T_1}) \cap E(G_{T_2}) = \emptyset$.

Let $T_3 := \{(i, i') \in S_n \mid (i, i') \in E(G_T) \setminus (E(\Gamma_1) \cup E(\Gamma_2))\}$. In the other words, $T_3 = \{(i_l, j_k) \mid 1 \leq l \leq r, 1 \leq k \leq s\}$, where $V(\Gamma_1) = \{i_1, \dots, i_r\}$ and $V(\Gamma_2) = \{j_1, \dots, j_s\}$. Consider the elements $a_i := \sum_{t \in T_i} t$ (for $i = 1, 2, 3$) in the group algebra $\mathbb{C}S_n$. Note that if for some $i \in \{1, 2\}$, $E(\Gamma_i) = \emptyset$ then $a_i = 0$. It is easy to check that $\sum_{t \in T} t = a_1 + a_2 + a_3$. Since $E(G_{T_1}) \cap E(G_{T_2}) = \emptyset$, clearly a_1 and a_2 commute in the group algebra $\mathbb{C}S_n$.

Suppose that $t_0 = (j_p, j_q) \in T_2$. Put $A_{t_0} := \{(i_l, j_k) \mid 1 \leq l \leq r, j_k \in \{j_p, j_q\}\}$ and $B_{t_0} = T_3 \setminus A_{t_0}$. It is clear that in the group algebra $\mathbb{C}S_n$, the element $\sum_{t \in B_{t_0}} t$ commutes with t_0 . In addition, $\sum_{t \in A_{t_0}} t = \sum_{l=1}^r ((i_l, j_p) + (i_l, j_q))$ and for every $1 \leq l \leq r$, the element $(i_l, j_p) + (i_l, j_q)$ commutes with t_0 . Thus $\sum_{t \in A_{t_0}} t$ commutes with t_0 . Therefore $a_3 = \sum_{t \in T_3} t = \sum_{t \in A_{t_0}} t + \sum_{t \in B_{t_0}} t$ commutes with t_0 . Since t_0 is an arbitrary element of T_2 , we conclude that a_2 and a_3 commute. Similarly, one can see that a_1 and a_3 commute. Consequently, a_1, a_2 and a_3 commute pairwise.

Let $\rho_1, \dots, \rho_m$ be all inequivalent irreducible representations of S_n , and d_i the degree of ρ_i . Then, by part (2) of Proposition 2.1, the spectrum of $\text{Cay}(S_n, S)$ is the union of the spectra of $\rho_1(S), \dots, \rho_m(S)$, where $S \in \{T, T_1, T_2, T_3\}$. Since a_1, a_2 , and a_3 commute pairwise and ρ_i is an algebra homomorphism, $\rho_i(a_j)$, $j = 1, 2, 3$, commute pairwise. Hence, there exists a basis $\mathcal{B}$ of the vector space $\mathbb{C}^{d_i}$ such that the matrices $\rho_i(a_j)$, for $j = 1, 2, 3$ are diagonalizable with respect to $\mathcal{B}$. We can assume that $\rho_i(a_j) = \text{diag}(\lambda_{1j}^i, \lambda_{2j}^i, \dots, \lambda_{d_i j}^i)$ with respect to the basis $\mathcal{B}$. Moreover, since ρ_i is an algebra homomorphism, we have $\rho_i(T) = \rho_i(a_1) + \rho_i(a_2) + \rho_i(a_3)$. This implies that the eigenvalues of $\rho_i(T)$ are $\lambda_{k1}^i + \lambda_{k2}^i + \lambda_{k3}^i$, where $k = 1, \dots, d_i$. Since G_{T_3} is a complete bipartite graph, Proposition 2.3 implies that $\text{Cay}(S_n, T_3)$ is integral. Thus, $\text{Cay}(S_n, T)$ is integral if and only if $\text{Cay}(S_n, T_2)$ is integral. $\square$

The following result follows directly from Theorem 3.6 and induction on the number of graphs.

Corollary 3.7. Let $T_1, \dots, T_k$ be some sets of transpositions of S_n . If, for every $i = 1, \dots, k$, the graph $\text{Cay}(S_n, T_i)$ is integral, then $\text{Cay}(S_n, T)$ is integral, where $G_T \in \mathcal{WJ}(G_{T_1}, \dots, G_{T_k})$.

Remark 3.8. Let $S = \{(1, 2), (1, 3), \dots, (1, n)\}$ for some positive integer n . It is conjectured that $\text{Cay}(S_n, S)$ is integral in [1]. Using certain properties of the Jucys-Murphy elements, the validity of this conjecture was confirmed in [6]. Since the star graph $K_{1,n-1} \in \mathcal{WJ}(nK_1, nK_1)$, the integrality of $\text{Cay}(S_n, S)$ is a direct consequence of Theorem 3.6. Furthermore, it follows from Example 3.4 that $K_{1,n-1}$ is the only tree with n vertices which is in $\mathcal{WJ}(nK_1, nK_1)$.

A vertex v of a graph Γ is called a *complete vertex* if it is adjacent to each vertex $u \neq v$ of Γ . For a graph Γ , a *cut vertex* of Γ is a vertex v of Γ such that the number of connected components of $\Gamma - v$ is greater than the number of connected components of Γ . As a direct consequence of the following result, we give a complete solution to Problem 1.1.

Corollary 3.9. Let T be a set of transpositions in S_n .

- (a) If G_T is a connected graph with a cut vertex v , then $\text{Cay}(S_n, T)$ is integral if and only if v is a complete vertex and $\text{Cay}(S_n, T')$ is integral, where T' is a set of transpositions in S_n such that $G_{T'} = (G_T - v) + v$.
- (b) If G_T is a tree, then $\text{Cay}(S_n, T)$ is integral if and only if G_T is the star graph $K_{1,n-1}$.

Proof. (a) Suppose that $\text{Cay}(S_n, T)$ is integral. Then, by Theorem 3.1 and [13, Corollary 2.1], v is a complete vertex. Let T'' be the empty set. Then $G_{T''} = nK_1$ and so $\text{Cay}(S_n, T'')$ is integral. Thus, as $G_T \in \mathcal{WJ}(G_{T'}, nK_1)$, by Theorem 3.6, $\text{Cay}(S_n, T')$ is integral. This proves one direction. Conversely, suppose that $\text{Cay}(S_n, T')$ is integral, and v is a complete vertex. Then, again as $G_T \in \mathcal{WJ}(G_{T'}, nK_1)$, Theorem 3.6 implies that $\text{Cay}(S_n, T)$ is integral.

(b) We may assume that G_T has at least three vertices. Suppose that $\text{Cay}(S_n, T)$ is integral. Then, as G_T is a tree, by part (a), G_T has a complete vertex, and so it is $K_{1,n-1}$. The converse direction is clear by Proposition 2.3. $\square$

Definition 3.10. Let Γ be a graph.

- a) We call Γ a *generalized complete multipartite graph of type 0* if Γ is a disjoint union of complete multipartite graphs (maybe K_1),

- b) We say Γ is a *generalized complete multipartite graph of type 1* if we can present $\Gamma = \bigcup_{i=1}^l \Gamma^i$, where for every $1 \leq i \leq l$, there exist generalized complete multipartite graphs $\Gamma_1^i, \dots, \Gamma_k^i$ of type 0 with the same vertex set such that $\Gamma^i \in \mathcal{WJ}(\Gamma_1^i, \dots, \Gamma_k^i)$.
- c) For $n \geq 2$, we say that the graph Γ is a *generalized complete multipartite graph of type n* , if $\Gamma = \bigcup_{i=1}^l \Gamma^i$, where for every $1 \leq i \leq l$, $\Gamma^i \in \mathcal{WJ}(\Gamma_1^i, \dots, \Gamma_k^i)$ for some graphs $\Gamma_1^i, \dots, \Gamma_k^i$ with the same vertex set, and the graph Γ_j^i is a generalized complete multipartite graph of type at most $n - 1$ for every $j = 1, 2, \dots, k$, and for some $1 \leq r \leq k$, Γ_r^i is a generalized complete multipartite graph of type exactly $n - 1$.

We wish to present a well-know class of generalized complete multipartite graphs of given type $m \geq 1$. A threshold graph independently introduced by Chvátal and Hammer [7], and Henderson and Zalcstein [10] in 1977, is a graph which is achieved through an iterative process that starts with an isolated vertex, and where, at each step, either a new isolated or complete vertex is added. An alternative way to describe the above inductive construction is to associate a threshold graph of order n with a binary sequence $(b_1 \cdots b_n)$ of length n , where b_i is 0, if in i th step an isolated vertex is added, and b_i is 1 if in i th step a complete vertex is added. For instance a threshold graph with the binary sequence $(0^m 1^n)$ is $mK_1 * K_n$.

The Laplacian eigenvalues of a threshold graph is well-know as follows.

Proposition 3.11. ([4, Proposition 3.11.1]) *Let Γ be a threshold graph with Laplacian eigenvalues (in non-increasing order) $v_1 \geq v_2 \geq \dots \geq v_n = 0$. Then*

$$v_j = |\{u \in V(\Gamma) \mid \deg_\Gamma(u) \geq j\}|.$$

For more details on the structure of these graphs and related topics, we encourage the reader to see [17].

Example 3.12. Suppose that Γ_m is the threshold graph of order at least 2 with the binary sequence $(0^{r_1} 1^{s_1} \dots 0^{r_m} 1^{s_m})$, where r_i 's and s_i 's are positive integers. We claim that Γ_m is a generalized complete multipartite graph of type $m - 1$. We prove our claim by induction on m . If $m = 1$ then it is easy to see that Γ_1 is a complete multipartite graph and so it is a generalized complete multipartite graph of type 0. Let $m \geq 2$ and assume that the threshold graph Γ_{m-1} with the binary sequence $(0^{r_1} 1^{s_1} \dots 0^{r_{m-1}} 1^{s_{m-1}})$ is a generalized complete multipartite graph of type $m - 2$. Let $\Gamma_m^1 = \Gamma_{m-1} \cup (r_m + s_m)K_1$ and $\Gamma_m^2 = K_{s_m} \cup (r_m + \sum_{i=1}^{m-1} (r_i + s_i))K_1$. By induction hypothesis, Γ_m^1 is a generalized complete multipartite graph of type $m - 2$ and it is clear that Γ_m^2 is a generalized complete multipartite graph of type 0. Therefore, as $\Gamma_m \in \mathcal{WJ}(\Gamma_m^1, \Gamma_m^2)$, the graph Γ_m is a generalized complete multipartite graph of type $m - 1$.

In what follows, for short, we write “gcm graph of type k ” instead of “generalized complete multipartite graph of type k ”.

Proposition 3.13. *Let T be a set of transpositions in S_n such that G_T is a gcm graph of type k , for some non-negative integer k . Then, $\text{Cay}(S_n, T)$ is integral.*

Proof. We prove by induction on k . If $k = 0$, the integrality of $\text{Cay}(S_n, T)$ is clear by Proposition 2.3. So we may assume that $k \geq 1$. Since G_T is a gcm graph of type k , there exist gcm graphs $\Gamma_1, \dots, \Gamma_m$ of type at most $k - 1$, and at least one of them is of type exactly $k - 1$, with the same vertex set $V(G_T)$, such that $G_T \in \mathcal{WJ}(\Gamma_1, \dots, \Gamma_m)$. For every $i = 1, \dots, m$, there exists a unique set of transpositions T_i on the set $V(G_T) = \{1, \dots, n\}$ such that $\Gamma_i \cong G_{T_i}$.

Suppose that the result is true for every gcm graph of type at most $k - 1$. We wish to prove the integrality of $\text{Cay}(S_n, T)$. By induction hypothesis, for every $i = 1, \dots, m$, the graph $\text{Cay}(S_n, T_i)$ is integral. Now, by Corollary 3.7, the graph $\text{Cay}(S_n, T)$ is integral, as desired. $\square$

By the above result in the sequel, we suggest the following conjecture and try to show that this conjecture is true for some classes of Cayley graphs of symmetric groups generated by transpositions.

Conjecture 3.14. *Let T be a set of transpositions in S_n . Then, $\text{Cay}(S_n, T)$ is integral if and only if for some positive integer k , the graph G_T is a gcm graph of type k .*

In what follows, we give some families of graphs supporting the above conjecture.

Corollary 3.15. *Let T be a set of transpositions in S_n such that G_T is a connected threshold graph of order at least two. Then*

- (a) $\text{Cay}(S_n, T)$ is integral,
 (b) for every $1 \leq i \leq n$, $|T| - v_i$ is an eigenvalue of $\text{Cay}(S_n, T)$, where $v_i = |\{j \in [n] \mid \deg_{G_T}(j) \geq i\}|$.

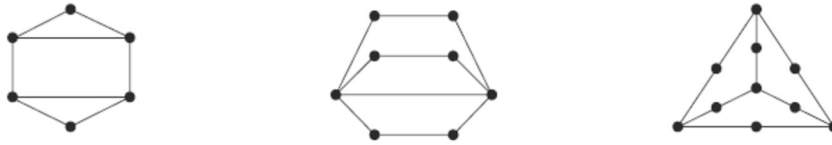


Fig. 1. The Laplacian integral tricyclic graphs Γ_1 , Γ_2 and Γ_3 .

Proof. Part (a) follows from Example 3.12 and Proposition 3.13. By Proposition 3.11 and Theorem 3.1, part (b) follows. $\square$

Proposition 3.16. Let T be a set of transpositions in S_m such that $G_T \cong C_m$. Then $\text{Cay}(S_m, T)$ is integral if and only if $m = 3$ or 4 .

Proof. It is well-known that the only integral cycles are C_3 , C_4 and C_6 , see for example [3, p. 43]. Since cycles are regular graphs, the above cycles are the only Laplacian integral cycles. If $\text{Cay}(S_m, T)$ is integral, Theorem 3.1 and Remark 3.2 imply that $m = 3$ or $m = 4$. The converse direction is clear by Proposition 2.3. $\square$

Proposition 3.17. Let T be a set of transpositions in S_n such that G_T is a connected cubic graph. Then $\text{Cay}(S_n, T)$ is integral if and only if G_T is isomorphic to one of the graphs K_4 or $K_{3,3}$.

Proof. Since G_T is a connected cubic integral graph, G_T is one of the thirteen graphs given in [19]. Suppose that $\text{Cay}(S_n, T)$ is integral. Then, G_T is isomorphic to K_4 or $K_{3,3}$, by GAP [20], Proposition 2.1 and Theorem 3.1. The converse direction is clear by Proposition 2.3. $\square$

We call any connected graph with n vertices and m edges k -cyclic, where $k = m - n + 1$. Clearly any tree is a 0-cyclic graph. In the following result, we give a characterization of integral Cayley graphs $\text{Cay}(S_n, T)$, where G_T is a k -cyclic graph and $k = 1, 2, 3$ that confirms Conjecture 3.14.

Proposition 3.18. Let T be a set of transpositions in S_n such that G_T is a k -cyclic graph.

- (a) If $k = 1$ then $\text{Cay}(S_n, T)$ is integral if and only if G_T is one of the graphs K_3 , $K_{2,2}$ or $(K_2 \cup (n-3)K_1) * K_1$.
- (b) If $k = 2$ then $\text{Cay}(S_n, T)$ is integral if and only if G_T is one of the graphs $K_{2,3}$, $K_2 * (2K_1)$, $(2K_2 \cup (n-5)K_1) * K_1$ or $(K_{1,2} \cup (n-4)K_1) * K_1$.
- (c) If $k = 3$ then $\text{Cay}(S_n, T)$ is integral if and only if G_T is one of the graphs $(3K_2) * K_1$, $(K_{1,2} \cup K_2) * K_1$, $K_{2,4}$, $K_{1,3} * K_1$, K_4 , $K_{2,3}$, $(3K_2 \cup (n-7)K_1) * K_1$, $(K_{1,2} \cup K_2 \cup (n-6)K_1) * K_1$, $(K_3 \cup (n-4)K_1) * K_1$ or $(K_{1,3} \cup (n-5)K_1) * K_1$.

Proof. Suppose that $\text{Cay}(S_n, T)$ is integral. Then, by Theorem 3.1, G_T is Laplacian integral. If $k = 1$, then [15, Theorem 3.2] and Remark 3.2 imply that G_T is isomorphic to one of the graphs K_3 , $K_{2,2}$ or $(K_2 \cup (n-3)K_1) * K_1$.

If $k = 2$ then by [15, Theorem 3.3], G_T is one of the graphs $(2K_2 \cup (n-5)K_1) * K_1$, $(K_{1,2} \cup (n-4)K_1) * K_1$, $K_{2,3}$, $K_2 * (2K_1)$ or $K_{1,2} \square K_2$, where $\square$ is the direct product operation of graphs. In the last case, one can easily check that $\text{Cay}(S_6, T)$ is not integral, by GAP [20] and Proposition 2.1.

If $k = 3$, then by [11, Theorem 4.1 and Theorem 5.1], G_T is one of the thirteen graphs $(3K_2) * K_1$, $(K_{1,2} \cup K_2) * K_1$, $K_{2,4}$, $K_{1,3} * K_1$, K_4 , $K_{2,3}$, $(3K_2 \cup (n-7)K_1) * K_1$, $(K_{1,2} \cup K_2 \cup (n-6)K_1) * K_1$, $(K_3 \cup (n-4)K_1) * K_1$, $(K_{1,3} \cup (n-5)K_1) * K_1$, or Γ_1 , Γ_2 and Γ_3 shown in Fig. 1. We checked, by GAP [20] and Proposition 2.1, that in the last three cases $\text{Cay}(S_n, T)$ is not integral. The converse direction is clear by Proposition 3.13. This completes the proof. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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Data availability

No data was used for the research described in the article.

References

- [1] A. Abdollahi, E. Vatandoost, Which Cayley graphs are integral?, *Electron. J. Comb.* 16 (1) (2009) 122.
- [2] M. Arezoomand, B. Taeri, On the characteristic polynomial of n -Cayley digraphs, *Electron. J. Comb.* 20 (3) (2013) 57.
- [3] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, A survey on integral graphs, *Publ. Elektroteh. Fak. Univ. Beogr., Mat.* 13 (2002) 42–65.
- [4] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.
- [5] F. Cesi, On the eigenvalues of Cayley graphs on the symmetric group generated by a complete multipartite set of transpositions, *J. Algebr. Comb.* 32 (2) (2010) 155–185.
- [6] G. Chapuy, V. Feray, A note on a Cayley graph of S_n , <https://arxiv.org/abs/1202.4976>, 2012.
- [7] V. Chvátal, P.L. Hammer, Aggregation of inequalities in integer programming, in: P.L. Hammer, et al. (Eds.), *Studies in Integer Programming*, in: *Ann. Discrete Math.*, vol. 1, North-Holland, Amsterdam, 1977, pp. 145–162.
- [8] C. Dalfó, M.A. Fiol, Spectra and eigenspaces from regular partitions of Cayley (di)graphs of permutation groups, *Linear Algebra Appl.* 597 (2020) 94–112.
- [9] P. Diaconis, M. Shahshahani, Generating a random permutation with random transpositions, *Z. Wahrscheinlichk.theor. Verw. Geb.* 57 (2) (1981) 159–179.
- [10] P.B. Henderson, Y. Zalcstein, A graph-theoretic characterization of the PV class of synchronizing primitives, *SIAM J. Comput.* 6 (1977) 88–108.
- [11] X. Huang, Q. Huang, F. Wen, On the Laplacian integral tricyclic graphs, *Linear Multilinear Algebra* 63 (7) (2015) 1356–1371.
- [12] G. James, A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley Publishing Company, 1981.
- [13] S. Kirkland, A bound on the algebraic connectivity of a graph in terms of the number of cut points, *Linear Multilinear Algebra* 47 (1) (2000) 93–103.
- [14] R. Krakovski, B. Mohar, Spectrum of Cayley graphs on the symmetric group generated by transpositions, *Linear Algebra Appl.* 437 (3) (2012) 1033–1039.
- [15] M. Liu, B. Liu, Some results on the Laplacian spectrum, *Comput. Math. Appl.* 59 (2010) 3612–3616.
- [16] X. Liu, S. Zhou, Eigenvalues of Cayley graphs, *Electron. J. Comb.* 29 (2) (2022) 2.9.
- [17] N.V.R. Mahadev, U.N. Peled, *Threshold Graphs and Related Topics*, Elsevier, 1995.
- [18] J.P. Serre, *Linear Representations of Finite Groups*, Translated from the second French edition by L. Scott, Graduate Text in Mathematics, vol. 42, Springer, New York, 1997.
- [19] A.J. Schwenk, Exactly thirteen connected cubic graphs have integral spectra, in: *Theory and Applications of Graphs, Proc. Internat. Conf.*, Western Mich. Univ., Kalamazoo, Mich., 1976, in: *Lecture Notes in Math.*, vol. 642, Springer, Berlin, 1978, pp. 516–533.
- [20] The GAP Group, GAP – groups, algorithms, and programming, version 4.12.2, <https://www.gap-system.org>, 2022.