Compact groups with a set of positive Haar measure satisfying a nilpotent law

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Abstract

The following question is proposed by Martino, Tointon, Valiunas and Ventura in [4, question 1.20]:
Let $G$ be a compact group, and suppose that

$$N_k(G) = \{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\}$$

has positive Haar measure in $G^{k+1}$. Does $G$ have an open $k$-step nilpotent subgroup?

We give a positive answer for $k = 2$.

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1. Introduction and results

Let $G$ be a (Hausdorff) compact group. Then $G$ has a unique normalised Haar measure denoted by $m_G$. The following question is proposed by Martino, Tointon, Valiunas and Ventura in [4, question 1.20].

**Question 1.1** [4, question 1.20]. Let $G$ be a compact group, and suppose that $N_k(G) = \{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\}$ has positive Haar measure in $G^{k+1}$. Does $G$ have an open $k$-step nilpotent subgroup?

Here $[x, y] := x^{-1}y^{-1}xy$ for elements $x, y$ of a group and $[x_1, \ldots, x_k, x_{k+1}]$ is a left normed commutator defined inductively as $[[x_1, \ldots, x_k], x_{k+1}]$ for $k \geq 2$.

A positive answer to Question 1.1 is known for $k = 1$ (see [3, theorem 1.2]). It follows from [4, theorem 1.19] that Question 1.1 has positive answer for arbitrary $k$ whenever we further assume that $G$ is totally disconnected (i.e., profinite). Here we give a positive answer to Question 1.1 for $k = 2$ (see Theorem 1.2 below).

**Theorem 1.2.** Let $G$ be a compact group, and suppose that $N_2(G) = \{(x_1, x_2, x_3) \in G \times G \times G \mid [x_1, x_2, x_3] = 1\}$ has positive Haar measure in $G \times G \times G$. Then $G$ has an open 2-step nilpotent subgroup.
2. A preliminary lemma

We need the following lemma in the proof of our main result.

**Lemma 2.1** Suppose that $x_1, x_2, x_3, g_1, g_2, g_3$ are elements of a group such that $[x_1 u_1, x_2 u_2, x_3 u_3] = 1$ for each triple of the following triples ($u_1, u_2, u_3$):

$$(1, 1, 1), (g_1, g_2, g_3), (g_1, g_2, 1), (g_1, 1, g_2);$$

$$(g_1, 1, 1), (g_1, 1, g_3), (1, 1, g_1), (1, g_2, g_1);$$

$$(1, g_2, 1), (1, 1, g_2), (1, g_2, g_3), (1, 1, g_3).$$

Then $[g_1, g_2, g_3] = 1$.

**Proof.** Note that $[x, y]$ denotes $x^{-1} y^{-1} x y$ and $[x, y, z] = [[x, y], z]$ for arbitrary elements $x, y, z$ of a group. We will throughout use famous commutator calculus identities: $[x y, z] = [x, z]^y [y, z]$ ($\dagger$) and $[x, y z] = [x, z] [x, y]^z$ ($\dagger\dagger$) for all elements $x, y, z$ of a group, where $g^h$ denotes $h^{-1} g h$. In the following (i) refers to the equality $[x_1 u_1, x_2 u_2, x_3 u_3] = 1$, where $(u_1, u_2, u_3)$ is the $i$th triple counting them from left to right starting at the top.

$$1 = [x_1 g_1, x_2 g_2, g_3] = [[x_1 g_1, g_2], [x_1 g_1, x_2]]^{g_3}, g_3]$$

by (\dagger\dagger), (2) and (3)

$$= [[x_1 g_1, g_2], [x_1 g_1, x_2]]^{g_3}, g_3]$$

by (4) and (5)

$$= [x_1 g_1, g_2, g_3] = [[x_1, g_2], [x_1, g_2]]^{g_1}, g_1], g_3]$$

by (\dagger), (5) and (6). (I)

On the other hand,

$$1 = [x_1, x_2 g_2, g_1]$$

by (8) and (9)

$$= [[x_1, g_2], [x_1, x_2]]^{g_2}, g_1] = [[x_1, g_2], [x_1, x_2], g_1]$$

by (\dagger\dagger), (1) and (10)

$$= [x_1, g_2, g_1]$$

by (1) and (7). (II)

Also,

$$1 = [x_1, x_2 g_2, g_3]$$

by (9) and (11)

$$= [[x_1, g_2], [x_1, x_2]]^{g_2}, g_3] = [[x_1, g_2], [x_1, x_2], g_3]$$

by (\dagger\dagger), (1) and (10)

$$= [x_1, g_2, g_3]$$

by (1) and (12). (III)

Now it follows from (I), (II) and (III) that $[g_1, g_2, g_3] = 1$.

**Remark.** The “left version” ($g_1 x_1$ instead of $x_1 g_1$) of Lemma 1.2 is not clear to hold. The validity of a similar result to Lemma 1.2 for commutators with length more than 3 is also under question.

3. Compact groups with many elements satisfying the 2-step nilpotent law

We need the “right version” of [5, theorem 2.3] as follows.

**Theorem 3.1** If $A$ is a measurable subset with positive Haar measure in a compact group $G$, then for any positive integer $k$ there exists an open subset $U$ of $G$ containing 1 such that $m_G(A \cap U u_1 \cap \cdots \cap U u_k) > 0$ for all $u_1, \ldots, u_k \in U$. 

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Proof. Since $m_G(A) = m_G(A^{-1})$, it follows from [5, theorem 2.3] that there exists an open subset $V$ of $G$ containing 1 such that

$$m_G(A^{-1} \cap v_1 A^{-1} \cap \cdots \cap v_k A^{-1}) > 0$$

for all $v_1, \ldots, v_k \in V$. Now take $U := V^{-1}$ which is an open subset of $G$ containing 1. Thus for all $u_1, \ldots, u_k \in U$

$$m_G(A \cap Au_1 \cap \cdots \cap Au_k) = m_G((A \cap Au_1 \cap \cdots \cap Au_k)^{-1}) = m_G(A^{-1} \cap u_1^{-1} A^{-1} \cap \cdots \cap u_k^{-1} A^{-1}) > 0$$

This completes the proof.

Now we can prove our main result.

Proof of Theorem 1.2. Let $X := N_2^0(G)$. It follows from Theorem 3.1 and [2, theorem 4.5] that there exists an open subset $U = U^{-1}$ of $G$ containing 1 such that

$$X \cap X\bar{u}_1 \cap \cdots \cap X\bar{u}_{11} \neq \emptyset$$

(\*)

for all $\bar{u}_1, \ldots, \bar{u}_{11} \in U \times U \times U$. Now take arbitrary elements $g_1, g_2, g_3 \in U$ and consider

$$\bar{u}_1 = (g_1^{-1}, g_2^{-1}, g_3^{-1}), \bar{u}_2 = (g_1^{-1}, g_2^{-1}, 1), \bar{u}_3 = (g_1^{-1}, 1, g_2^{-1})$$

$$\bar{u}_4 = (g_1^{-1}, 1, 1), \bar{u}_5 = (g_1^{-1}, 1, g_3^{-1}), \bar{u}_6 = (1, 1, g_1^{-1}), \bar{u}_7 = (g_2^{-1}, 1, g_1^{-1})$$

$$\bar{u}_8 = (1, g_2^{-1}, 1), \bar{u}_9 = (1, 1, g_2^{-1}), \bar{u}_10 = (1, g_2^{-1}, g_3^{-1}), \bar{u}_11 = (1, 1, g_3^{-1})$$

By (\*), there exists $(x_1, x_2, x_3) \in X$ such that all the following 3-tuples are in $X$:

$$(x_1g_1, x_2g_2, x_3g_3), (x_1g_1, x_2g_2, x_3), (x_1g_1, x_2, x_3g_2)$$

$$(x_1g_1, x_2, x_3), (x_1g_1, x_2, x_3g_3), (x_1, x_2, x_3g_1), (x_1, x_2g_2, x_3g_1)$$

$$(x_1, x_2g_2, x_3), (x_1, x_2, x_3g_2), (x_1, x_2g_2, x_3g_3), (x_1, x_2, x_3g_3).$$

Now Lemma 2.1 implies that $[g_1, g_2, g_3] = 1$. Therefore the subgroup $H := \langle U \rangle$ generated by $U$ is 2-step nilpotent. Since $H = \bigcup_{n \in \mathbb{N}} U^n$, $H$ is open in $G$. This completes the proof.

We finish with the following open question that would resolve Question 1.1 for arbitrary $k$:

**Question 3.2** Are there finitely many words $w_{ij} (1 \leq i \leq n, 1 \leq j \leq k + 1)$ in the free group on $k + 1$ generators such that if $G$ is a compact group, $(x_1, \ldots, x_{k+1}), u = (u_1, \ldots, u_{k+1}) \in G^{k+1}$ and $[x_1 w_{i1}(u), \ldots, x_{k+1} w_{i,k+1}(u)] = 1$ for all $i \in [1, \ldots, n]$ then $[u_1, \ldots, u_{k+1}] = 1$?

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