Compact groups with many elements of bounded order

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Abstract. Lévai and Pyber proposed the following as a conjecture: Let $G$ be a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure. Then $G$ has an open subgroup $H$ and an element $t$ such that all elements of the coset $tH$ have order dividing $n$ (see [V. D. Mazurov and E. I. Khukhro, Unsolved Problems in Group Theory. The Kourovka Notebook. No. 19, Russian Academy of Sciences, Novosibirsk, 2019; Problem 14.53]). The validity of the conjecture has been proved in [L. Lévai and L. Pyber, Profinite groups with many commuting pairs or involutions, Arch. Math. (Basel) 75 (2000), 1–7] for $n = 2$. Here we study the conjecture for compact groups $G$ which are not necessarily profinite and $n = 3$; we show that in the latter case the group $G$ contains an open normal 2-Engel subgroup.

1 Introduction and results

Every compact Hausdorff topological group $G$ admits a unique normalized Haar measure $m_G$. If $X$ is a measurable subset of $G$ such that $m_G(X) > 0$, then it is not true in general that $X$ contains a non-empty open subset; the latter is not even true for profinite groups, i.e., compact totally disconnected Hausdorff topological groups; see e.g. [5]. However, as far as we know, there is no counterexample for the latter question for certain subsets of positive Haar measure defined by words in profinite groups. In this direction, the following conjecture has been proposed by Lévai and Pyber in [5].

Conjecture 1.1 ([5, Conjecture 3], [7, Problem 14.53]). Let $G$ be a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure. Then $G$ has an open subgroup $H$ and an element $t$ such that all elements of the coset $tH$ have order dividing $n$.

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The validity of Conjecture 1.1 was confirmed for \( n = 2 \) in [5]. We generalize the latter for compact (Hausdorff) groups for the set of elements inverting by an automorphism (not necessarily continuous) of a compact group; see Theorem 3.1 below. Here we also study Conjecture 1.1 for \( n = 3 \). However, we were not able to settle Conjecture 1.1 for \( n = 3 \); we prove that compact groups (not necessarily profinite) in which the set of elements of order dividing 3 is of positive Haar measure, contain open normal 2-Engel subgroups; see Corollary 4.6 below; actually, we prove a more general result that the latter is valid for compact groups; see Theorem 4.4 below.

Our main tool (Theorem 2.3 below) to deal with compact groups is proved in Section 2. Theorem 2.3 is a general result of independent interest on measurable subsets of compact groups with positive Haar measure. Theorem 2.3 in particular shows that the latter subsets are “relatively \( k \)-large sets” in compact groups (see [2, 3]; for definition of “\( k \)-large sets” and for some results on them, see [6]).

## 2 Subsets of compact groups with positive Haar measure are large

Throughout, all topological groups are Hausdorff. Let \( G \) be a compact group with the unique normalized Haar measure \( m_G \); usually, the index \( G \) is dropped if the group is known from the context. It follows from [4, Corollary 20.17] that, for measurable subsets \( A \) and \( B \) of \( G \) with \( m(A) > 0 \) and \( m(B) > 0 \), the map from \( G \) to \( \mathbb{R}^\geq 0 \) defined by \( x \mapsto m(A \cap xB) \) is non-zero and continuous. We prove here a slight generalization of the latter result as Theorem 2.3, below. Actually, Theorem 2.3 plays a key role in the proofs given in the next sections. We found that Theorem 2.3 can be further improved as Theorem 2.2, although we will not use it here in its full generality.

We need the following lemma in the proof of Theorem 2.2.

**Lemma 2.1.** Let \( \xi \) be in \( L^2(G) \). Then the map

\[
G \to L^2(G), \quad x \mapsto L_x \xi
\]

is continuous.

**Proof.** It is a special case of [1, (2.41) Proposition]. \( \square \)

**Theorem 2.2.** Let \( G \) be a compact group, and let \( \xi_1, \ldots, \xi_n \) be elements in the unit ball of \( L^\infty(G) \). Then the map

\[
\Psi: G \times \cdots \times G \to \mathbb{C}, \quad (x_1, \ldots, x_n) \mapsto \int_G \left( \prod_{k=1}^n L_{x_k} \xi_k(g) \right) \, dm(g)
\]

is continuous.
Proof. The Cauchy–Schwarz inequality in the Hilbert space \( L^2(G) \) can be stated in the following integral form:

\[
\left| \int_G \xi(g) \overline{\eta(g)} \, d\mu(g) \right| \leq \|\xi\|_2 \|\eta\|_2 \quad (\xi, \eta \in L^2(G)).
\] (2.1)

The left translate of a function \( f \) on \( G \) by an \( x \in G \), \( L_x f \), is defined by

\[
L_x f(g) = f(x^{-1}g) \quad (g \in G).
\]

For \( x_k, y_k, k = 1, \ldots, n \), in \( G \), and \( 1 \leq s \leq n \), put

\[
F_s(g) := \left( \prod_{k<s} L_{y_k} \xi_k(g) \right) \left( \prod_{k>s} L_{x_k} \xi_k(g) \right) \quad (g \in G).
\]

If the index of the latter product is empty, we put 1 instead of the product. By the latter convention,

\[
\Psi(x_1, \ldots, x_n) - \Psi(y_1, \ldots, y_n) = \sum_{s=1}^n \int_G (L_{x_s} \xi_s(g) - L_{y_s} \xi_s(g)) F_s(g) \, d\mu(g).
\]

It follows from the Cauchy–Schwarz inequality, (2.1), and the fact that \( \|F_s\|_2 \leq 1 \),

\[
|\Psi(x_1, \ldots, x_n) - \Psi(y_1, \ldots, y_n)| \leq \sum_{s=1}^n \|L_{x_s} \xi_s - L_{y_s} \xi_s\|_2,
\]

whence by Lemma 2.1 the continuity of \( \Psi \) follows. \( \square \)

**Theorem 2.3.** Let \( G \) be a compact group. Suppose that \( A_1, \ldots, A_n \) are measurable subsets of \( G \) with positive Haar measure. Then the map

\[
\Lambda : G \times \cdots \times G \rightarrow \mathbb{R}^{\geq 0}, \quad (x_1, \ldots, x_k) \mapsto m(x_1 A_1 \cap \cdots \cap x_n A_n)
\]

is non-zero and continuous. In particular, if \( A \) is a measurable subset with positive Haar measure, then for any positive integer \( k \) there exists an open subset \( U \) of \( G \) containing \( 1 \) such that \( m_G(A \cap u_1 A \cap \cdots \cap u_k A) > 0 \) for all \( u_1, \ldots, u_k \in U \).

**Proof.** Applying Theorem 2.2, we conclude that \( \Lambda \) is continuous. Take \( \xi_k \) as \( \chi_{A_k} \) for \( k = 1, \ldots, n \) in Theorem 2.2, and note that, by Fubini’s theorem, we have

\[
\int_{\prod_{i=1}^n G} m(x_1 A_1 \cap \cdots \cap x_n A_n) \, d\mu_{\prod_{i=1}^n G}(x_1, \ldots, x_n) = m(A_1) \cdots m(A_n) > 0.
\]
Since $\epsilon := m_G(A) > 0$, it follows from the continuity of the map $\Lambda$ corresponding to $A_i := A (i = 1, \ldots, k + 1)$ that there exists an open subset $U$ of $G \times \cdots \times G$ containing $(1, 1, \ldots, 1)$ (k+1)-times such that $m_G(A \cap u_1 A \cap \cdots \cap u_k A) \geq \epsilon > 0$ for all $(1, u_1, \ldots, u_k) \in U$. Now $U := U_0 \cap U_1 \cap \cdots \cap U_k$, where $U_0 \times U_1 \times \cdots \times U_k \subseteq U$ and $U_i$ $(i = 0, 1, \ldots, k)$ are open subsets of $G$ containing $1$, has the required property.

**Remark 2.4.** The statement of Theorem 2.3 is conjectured by the authors and proposed in [10] by the second author. We are guided by the comments of other people on [10] not only to write a detailed proof for Theorem 2.3 but also to give Theorem 2.2.

**Remark 2.5.** Following [2], a subset $X$ of a group $G$ is called *large* if $\bigcap_{a \in F} aX$ is not empty for any finite non-empty subset $F \subseteq G$. We call a subset $X$ of a group $G$ relatively $k$-large with respect to a subset $M$ of $G$ for some $k \in \mathbb{N}$ if $\bigcap_{a \in F} aX$ is not empty for any subset $F \subseteq M$ with $|F| = k$. Thus, by Theorem 2.3, for every $k \in \mathbb{N}$, each measurable subset of a compact group with positive Haar measure is relatively $k$-large with respect to an open subset $U_k$ containing $1$.

### 3 Compact groups with an automorphism inverting many elements

In this section, we generalize [5, Corollary 6]. Throughout, we use the fact that, in any compact group $G$, given a closed subgroup $H$, the following are equivalent: (a) $H$ has positive measure; (b) $H$ has finite index in $G$; (c) $H$ is open.

**Theorem 3.1.** Let $G$ be a compact group having an automorphism (not necessarily continuous) $\alpha$ such that the set $X = \{x \in G \mid x^\alpha = x^{-1}\}$ is measurable and of positive Haar measure. Then $G$ contains an open normal abelian subgroup. In particular, there exists an open abelian subgroup $A$ of $G$ such that $tA \subseteq X$ for some $t \in X$.

**Proof.** It follows from Theorem 2.3 that there exists an open subset $U$ of $G$ containing $1$ such that $X \cap u_1 X \cap u_2 X \cap u_3 X$ is of positive Haar measure for all $u_1, u_2, u_3 \in U$. Since $U$ is open and $1 \in U$, it follows from [4, Theorem 4.5] that there exists an open subset $V \subseteq U$ such that $1 \in V$, $V = V^{-1} = \{v^{-1} \mid v \in V\}$ and $V^2 := \{v_1 v_2 \mid v_1, v_2 \in V\} \subseteq U$ (see [4, Theorem 4.5]). Now suppose that $a, b$ are arbitrary elements of $V$. Thus

$X \cap b^{-1} X \cap a^{-1} X \cap b^{-1} a^{-1} X$
has positive Haar measure, and in particular, it is non-empty. It follows that there exists \( x \in X \) such that \( bx, ax, abx \) are all in \( X \). Let \( \beta \) be the automorphism \( y \mapsto y^a x^{-1} \). The fact that \( \alpha \) is an automorphism that inverts \( x \) and \( bx \) gives us

\[
(bx)\alpha = b^a x^a = b^a x^{-1} = x^{-1} b^{-1},
\]

so \( b\beta = b^{-1} \). Similarly, \( a\beta = a^{-1} \) and \( (ab)\beta = (ab)^{-1} \). Hence

\[
ab = ((ab)^\beta)^{-1} = (a\beta b\beta)^{-1} = (a^{-1} b^{-1})^{-1} = ba,
\]

showing that \( a \) and \( b \) commute, and so the subgroup \( H \) generated by \( V \) is abelian. Since \( H \) is a subgroup with non-empty interior, \( H \) is open. Now take the core \( K \) of \( H \) in \( G \), which is open normal and abelian. This completes the proof of the first part.

For the second part, since \( |G : K| \) is finite, \( tK \cap X \) has positive Haar measure for some \( t \in X \). Thus \( A := \{ a \in K \mid ta \in X \} \) has positive Haar measure. The set \( A \) is the subset of elements of \( K \) inverted by the automorphism \( \gamma \) of \( G \), where \( \gamma : y \mapsto t^{-1} y^a t \); it follows that \( A \) is a subgroup; for, given \( a, b \in A \), then

\[
(ab^{-1})\gamma = a\gamma (b\gamma)^{-1} = a^{-1} b,
\]

and the last expression is the inverse of \( ab^{-1} \) since \( a \) and \( b \) commute, so \( ab^{-1} \in A \). Since \( X \) is closed, \( A \) is closed, and it is open since \( A \) has positive Haar measure. This completes the proof.

\[ \square \]

4 Compact groups with many elements of order 3

In this section, we study Conjecture 1.1 for \( n = 3 \).

The following lemma is used to prove for a relatively 8-large set with respect to an open subgroup \( U \) containing 1 that \( U \) is 2-Engel.

**Lemma 4.1.** Let \( a, b \) and \( x \) be elements of a group such that

\[
x^3 = (bx)^3 = (ax)^3 = (a^{-1} x)^3 = (ab^{-1} x)^3 = (ba^{-1} x)^3 = (abx)^3 = (b^{-1} a^{-1} x)^3 = 1.
\]

Then \( [a, b, b] = 1 \).

**Proof.** See the proof of [3, Proposition 4.3] and the proof of [6, Proposition 1]. \[ \square \]

The following two lemmas give a sufficient condition on a symmetric subset of a group to generate a 2-Engel subgroup.
**Lemma 4.2.** There exists a positive integer \( k \) such that every group generated by a symmetric subset \( X = X^{-1} \) containing 1 satisfying the property \([x, y, y] = 1\) for all \( x, y \in X^k := \{x_1 \cdots x_k \mid x_1, \ldots, x_k \in X\}\) is nilpotent of class at most 3.

**Proof.** It is enough to show that every 4-element subset of \( X \) generates a nilpotent group of class at most 3. Since 2-Engel groups are nilpotent of class at most 3 [9, Corollary 3, page 45], \( \langle [a, b, b] \mid a, b \in F_4 \rangle \) is nilpotent of class at most 3, where \( F_4 \) is the free group of rank 4 on the free generators \( x_1, x_2, x_3, x_4 \). Now it follows from [8, Lemma 1.43, page 32 and its Corollary, page 33] that \( \langle [a, b, b] \mid a, b \in F_4 \rangle \) is the normal closure of a finite number of its elements. Then there exist elements \( a_1, \ldots, a_s, b_1, \ldots, b_s \in F_4 \) such that
\[
\langle [a, b, b] \mid a, b \in F_4 \rangle = \langle [a_1, b_1, b_1], \ldots, [a_s, b_s, b_s] \rangle^{F_4}.
\]
Let \( k \) be a positive integer such that
\[
\{a_1, \ldots, a_s, b_1, \ldots, b_s\} \subseteq \{1, x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}\}^k.
\]
Therefore, any group satisfying the laws \([a_i, b_i, b_i] = 1\) for all \( i = 1, \ldots, s \) is nilpotent of class at most 3, and since \( a_i \)'s and \( b_i \)'s are the product of at most \( k \) free generators \( x_j \)s and their inverses, the proof is now complete. \( \square \)

**Lemma 4.3.** There exists a positive integer \( \ell \) such that every group generated by a symmetric subset \( X = X^{-1} \) containing 1 satisfying the property \([x, y, y] = 1\) for all \( x, y \in X^\ell \) is a 2-Engel group.

**Proof.** Let \( \ell := \max\{k, 2\} \), where \( k \) is a positive integer mentioned in the statement of Lemma 4.2. By Lemma 4.2, \( G := \langle X \rangle \) is nilpotent of class at most 3. Suppose that \( g \) and \( h \) are arbitrary elements of \( G \). Then \( g = x_1 \cdots x_t \) and \( h = y_1 \cdots y_t \) for some \( x_i, y_j \in X \). Now we may write \( [g, h, h] \) as follows:
\[
\prod_{i=1}^{t} \left( \prod_{j < k, i, j = 1}^{t} ([x_i, y_j, y_k][x_i, y_k, y_j]) \right).
\]
This follows from the facts that \( G \) is nilpotent of class at most 3 and \([x, y, y] = 1\) for all \( x, y \in X \). Now, since \( \ell \geq 2 \), \([x, yz, yz] = 1\) for all \( x, y, z \in X \). It follows that \([x, y, z][x, z, y] = 1\). Therefore, \([g, h, h] = 1\), and so \( G \) is 2-Engel. \( \square \)

**Theorem 4.4.** Let \( G \) be a compact group, and let \( \alpha \) be a not necessarily continuous automorphism of \( G \) such that \( \alpha^3 = 1 \) and the set \( X = \{x \in G \mid x^{\alpha^2} x^{\alpha} x = 1\} \) is measurable. If \( X \) has positive Haar measure, then \( G \) contains an open normal 2-Engel subgroup.
Proof. By Lemma 2.3, there exists an open subset $U$ of $G$ such that
$$X \cap u_1 X \cap \cdots \cap u_7 X$$
is of positive Haar measure for all $u_1, \ldots, u_7 \in U$. Since $U$ is open and $1 \in U$, it follows from [4, Theorem 4.5] that there exists an open subset $V \subseteq U$ such that $1 \in V$, $V = V^{-1}$ and $V^{2\ell} \subseteq U$. Now suppose that $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ are arbitrary elements of $V$. Let $a = a_1 \cdots a_\ell$ and $b = b_1 \cdots b_\ell$. Thus
$$X \cap b^{-1} X \cap a X \cap a^{-1} X \cap ab^{-1} X \cap ba^{-1} X \cap abX \cap b^{-1} a^{-1} X$$
has positive Haar measure, and in particular, it is non-empty. It follows that there exists $x \in X$ such that $bx, ax, a^{-1} x, ab^{-1} x, ba^{-1} x, abx, b^{-1} a^{-1} x$ are all in $X$. Working in the semidirect product $G \rtimes \langle \alpha \rangle$, since $\alpha^3 = 1$, it follows that $a \in X$ if and only if $(a\alpha)^3 = 1$. Therefore, $(x\alpha)^3 = (bx\alpha)^3 = 1$ and
$$\begin{align*}
(ax\alpha)^3 &= (a^{-1} x\alpha)^3 = (ab^{-1} x\alpha)^3 = (ba^{-1} x\alpha)^3 \\
&= (abx\alpha)^3 = (b^{-1} a^{-1} x\alpha)^3 = 1.
\end{align*}$$
Now Lemma 4.1 implies that $[a, b, b] = 1$, and it follows from Lemma 4.3 that $H := \langle V \rangle$ is 2-Engel. Since $V$ is open, $H$ is an open subgroup. Now take the core of $H$ in $G$, which is normal open and 2-Engel. This completes the proof. \hfill \Box

Remark 4.5. However, the statements of Theorem 4.4 and [6, Theorem 1] are not directly comparable as “largely splitting of order 3” has been replaced with “splitting of order 3 on a set of positive Haar measure”, but thanks to Theorem 2.3, the proof has quite a similar structure.

Corollary 4.6. Let $G$ be a compact group such that the set $X = \{x \in G \mid x^3 = 1 \}$ has positive Haar measure. Then $G$ contains an open normal 2-Engel subgroup.

Proof. Take $\alpha$ as the identity automorphism in Theorem 4.4. \hfill \Box

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Bibliography


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