Chapter Fourteen

عملگرها، ماتریسها و اسپین

Operators, Matrixes and Spins

اندازه حرکت زاویه ای متناظر با l=1/2 را اسپین گویند.

$$Y_{ll}(\theta,\varphi)=\sqrt{\frac{(2l+1)!}{4\pi}}\,\frac{1}{2^l\,l!}\sin^l\theta\, e^{il\varphi}$$

$$l=\frac{1}{2}\Rightarrow Y_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi)\propto \sin^{\frac{1}{2}}\theta\, e^{i\frac{1}{2}\varphi}$$

متاسفانه اسپین را نمی توان به روشهایی که تا کنون آموخته ایم (یعنی توابع موج و معادلات دیفرانسیل) توصیف کرد:

$$Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\varphi) \propto \sqrt{\sin\theta}\, e^{-i\frac{1}{2}\varphi}$$

$$Y_{l,-m}(\theta,\varphi) = (-1)^m\, Y^*_{l,m}(\theta,\varphi)$$

از طرفی می دانیم که:

$$L_-Y_{lm} = C_-(l,m)Y_{l,m-1}$$

تمرین : با استفاده از رابطه فوق و محاسبه رابطه زیر

$$L_-Y_{\frac{1}{2},\frac{1}{2}} \propto Y_{\frac{1}{2},-\frac{1}{2}}$$

$$L_-Y_{lm} = C_-(l,m)Y_{l,m-1}$$

$$L_- = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right)$$

نشان دهید که به تناقض زیر می رسیم:

$$L_-Y_{\frac{1}{2},\frac{1}{2}} \propto \frac{\cos\theta}{\sqrt{\sin\theta}}\, e^{-i\frac{1}{2}\varphi} \not\propto \sqrt{\sin\theta}\, e^{-i\frac{1}{2}\varphi} \text{ or } Y_{\frac{1}{2},-\frac{1}{2}}(\theta,\varphi)$$

$$L_- Y_{\frac{1}{2},\frac{1}{2}} \propto \frac{\cos\theta}{\sqrt{\sin\theta}} e^{-i\frac{1}{2}\varphi}$$

علاوه بر تناقض ذکر شده اسپین ½ دارای مشکلات دیگری نیز می باشد:

مثلاً تابع فوق در $\theta=0$ و $\theta=\pi$ دارای تکینگی می باشد که فیزیکی نیست.

بنابراین برای $l=1/2$ مشکلاتی وجود دارد که نمی توان آن را به روشهای قبلی مورد بررسی قرار داد.

از طرفی توصیف مناسب اتمها بدون در نظر گرفتن اسپین الکترونها امکان پذیر نیست.

اگر درجه آزادی جدید الکترون را که اسپین نام دارد در نظر نگیریم ممکن نیست که بتوانیم فیزیک اتمی و زیر اتمی را چنان که باید تشریح نماییم.

اسپین علی رغم نام فریبنده ای که دارد خاصیتی از الکترون است که همتای کلاسیکی ندارد.

تصویری از الکترون که حول محورش دوران وضعی انجام می دهد نادرست است.

زیرا الکترون یک نقطه ریاضی بدون بعد است که برای آن دوران وضعی بی معنی است.

بنابراین عملگر اندازه حرکت زاویه ای کل ($J=L+S$ نه L) همتای کلاسیکی ندارد و بررسی آن مستلزم رهیافت انتزاعی تری است.

از این پس به جای $l=1/2$ از $s=1/2$ استفاده می کنیم و نماد l را برای اندازه حرکت زاویه ای حفظ می نمایم.

اسپین با آزمایش کشف شد.

تحلیل الگوی خطوط طیفی با فیزیک پیش کوانتمی (فیزیک جدید) جنبه های ناشناخته زیادی داشت.

از آن جمله خطوط دوتایی اتمهای فلزات قلیایی برانگیخته در میدان مغناطیسی بود.

توجیه نیمه کلاسیکی اثر زیمان که نام کاشف خود را داشت نتوانست چنین شکافتگی ای را توضیح دهد .

تا این که پاولی در سال ١٩٢٤ (یعنی پیش از مکانیک کوانتمی) این راز را گشود و پیشنهاد کرد که الکترون در اتم فزون بر مجموعه اعداد کوانتمی (n, l, m) عدد کوانتمی دو مقداری غیر کلاسیکی دیگری نیز دارد.

سالی از این ماجرا نگذشته بود که ساموئل گدشمیت و جورج اولنباخ که در آن زمان دانشجویان دوره تحصیلات تکمیلی در هلند بودند پیشنهاد کردند که الکترون اندازه حرکت زاویه ای ذاتی دارد که مقدار آن $s=1/2$ است.

الکترون نقطه است و نقطه چیزی نیست که دوران کند.

خوشبختانه تا حدودی آمادگی لازم برای ترک توصیفی که قویا به فضای مختصات وابسته است را قبلا ایجاد کرده ایم.

در فصل هفت نوسانگر هماهنگ :

$$Hu_n = \hbar\omega(n + 1/2)u_n$$

$$u_n = \frac{1}{\sqrt{n!}}\left(\frac{A^\dagger}{\sqrt{\hbar^n}}\right)^n u_0$$

$$\boxed{\langle u_m | u_{\boldsymbol{n}} \rangle = \delta_{mn}}$$

$$\delta = \begin{array}{cl} \begin{array}{ccccc} n=0 & n=1 & n=2 & n=3 & \dots \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & \mathbf{1} & 0 & 0 & \dots \\ 0 & 0 & \mathbf{1} & 0 & \dots \\ 0 & 0 & 0 & \mathbf{1} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} & \begin{array}{l} m=0 \\ m=1 \\ m=2 \\ m=3 \\ \dots\dots \end{array} \end{array}$$

تفاوت برا و کت:

$$|\alpha\rangle = \begin{pmatrix} 2 \\ 5i+6 \\ 7i \end{pmatrix}_{3\times 1}$$

DC

$$\langle\alpha| = \begin{pmatrix} 2 & -5i+6 & -7i \end{pmatrix}_{1\times 3}$$

$$\mathrm{H}u_n = \hbar\omega(n + 1/2)u_n$$

$$H|u_n\rangle = \hbar\omega(n + 1/2)|u_n\rangle$$

$$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

$$\langle m|H|n\rangle = \langle m|\hbar\omega(n + 1/2)|n\rangle$$

$$\langle m|H|n\rangle = \hbar\omega(n + 1/2)\langle m|n\rangle$$

$$\langle m|H|n\rangle = \hbar\omega(n + 1/2)\delta_{mn}$$

$$\langle m|H|n\rangle = \frac{\hbar\omega}{2}(2n + 1)\delta_{mn}$$

$$\boldsymbol{H} = \frac{\hbar\omega}{2}\begin{array}{c} \overbrace{\begin{array}{ccccc} n{=}0 & n{=}1 & n{=}2 & n{=}3 & \ldots \end{array}} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \mathbf{3} & 0 & 0 & \ldots \\ 0 & 0 & \mathbf{5} & 0 & \ldots \\ 0 & 0 & 0 & \mathbf{7} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \end{array} \begin{array}{l} \\ m = 0 \\ m = 1 \\ m = 2 \\ m = 3 \\ \ldots\ldots\ldots \end{array}$$

$$A^{\dagger}|n\rangle = \sqrt{(n+1)}|n+1\rangle$$

$$\langle m|A^{\dagger}|n\rangle = \sqrt{(n+1)}\langle m|n+1\rangle$$

$$\langle m|A^{\dagger}|n\rangle = \sqrt{(n+1)}\delta_{m,n+1}$$

$$A^{\dagger} = \begin{array}{c} \overbrace{\begin{array}{ccccc} n{=}0 & n{=}1 & n{=}2 & n{=}3 & \ldots \end{array}} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ \sqrt{1} & 0 & 0 & 0 & \ldots \\ 0 & \sqrt{2} & 0 & 0 & \ldots \\ 0 & 0 & \sqrt{3} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \end{array} \begin{array}{l} \\ m = 0 \\ m = 1 \\ m = 2 \\ m = 3 \\ \ldots\ldots\ldots \end{array}$$

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$\langle m|A|n\rangle = \sqrt{n}\langle m|n-1\rangle$$

$$\left\langle m|A^{\dagger}|n\right\rangle = \sqrt{n}\delta_{m,n-1}$$

$$\boldsymbol{A} = \begin{array}{c} \overbrace{\begin{array}{ccccc} n{=}0 & n{=}1 & \boldsymbol{n}{=}2 & n{=}3 & \dots \end{array}} \\ \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \mathbf{0} & 0 & \sqrt{2} & 0 & \dots \\ 0 & \mathbf{0} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \mathbf{0} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array} \begin{array}{l} \\ m = 0 \\ m = 1 \\ m = 2 \\ m = 3 \\ \dots\dots\dots \end{array}$$

و در فصل ده هم مساله ویژه مقدار اندازه حرکت زاویه ای را مطرح کردیم:

$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$$

$$L_z Y_{lm} = m\hbar Y_{lm}$$

$$L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$$

$$L_z|l,m\rangle = m\hbar|l,m\rangle$$

$$\langle l',m'|L^2|l,m\rangle = l(l+1)\hbar^2\langle l',m'|l,m\rangle$$

$$\langle l' m'|L_z|l,m\rangle = m\hbar\langle l',m'|l,m\rangle$$

$$\langle l',m'|L^2|l,m\rangle = l(l+1)\hbar^2\delta_{l',l}\delta_{m',m}$$

$$\langle l',m'|L_z|l,m\rangle = m\hbar\delta_{l',l}\delta_{m',m}$$

$$l' = l \Rightarrow \delta_{l',l} = 1$$

$$\langle l,m'|L^2|l,m\rangle = l(l+1)\hbar^2\delta_{m',m}$$

$$\langle l,m'|L_z|l,m\rangle = m\hbar\delta_{m',m}$$

$$-l \le m \le l \quad \& -l' \le m' \le l' \Rightarrow -l \le m' \le l$$

$$l = l' = 1 \;\Rightarrow\; -1 \le m \le 1 \;\&\; -1 \le m' \le 1$$

$$m, m' = -1, 0, 1$$

$$\langle 1, m' | L_z | 1, m \rangle = \hbar m \delta_{m',m}$$

$$L_z = \hbar \begin{array}{cc} \overbrace{\begin{matrix} m{=}+1 & m{=}0 & m{=}-1 \end{matrix}} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} m' = +1 \\ m' = \;\;0 \\ m' = -1 \end{matrix} \end{array}$$

$$\langle 1, m' | L^2 | 1, m \rangle = 2\hbar^2 \delta_{m',m}$$

$$L^2 = 2\hbar^2 \begin{array}{cc} \overbrace{\begin{matrix} m{=}+1 & m{=}0 & m{=}-1 \end{matrix}} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} m' = +1 \\ m' = \;\;0 \\ m' = -1 \end{matrix} \end{array}$$

$$L_+|l,m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}|l,m+1\rangle$$

$$\langle l',m'|L_+|l,m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\langle l',m'|l,m+1\rangle$$

$$\langle l',m'|L_+|l,m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}\delta_{l',l}\delta_{m',m+1}$$

$$l = l' = 1 \;\Rightarrow \delta_{l',l} = 1 \,\&\, -1 \le m \le 1 \;\&\; -1 \le m' \le 1$$

$$\langle 1,m'|L_+|1,m\rangle = \hbar\sqrt{2-m(m+1)}\delta_{m',m+1}$$

$$L_+ = \hbar \begin{array}{cc} \overbrace{\begin{array}{ccc} \boldsymbol{m=+1} & \boldsymbol{m=0} & \boldsymbol{m=-1} \end{array}} & \\ \begin{pmatrix} \mathbf{0} & \sqrt{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sqrt{\mathbf{2}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{array}{l} m' = +1 \\ m' = 0 \\ m' = -1 \end{array} \end{array}$$

$$L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l,m-1\rangle$$

$$\langle l',m'|L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,\langle l',m'|l,m-1\rangle$$

$$\langle l',m'|L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,\delta_{l',l}\,\delta_{m',m-1}$$

$$l = l' = 1 \;\Rightarrow \delta_{l',l} = 1 \;\&\; -1 \le m \le 1 \;\;\&\;\; -1 \le m' \le 1$$

$$\langle 1,m'|L_-|1,m\rangle = \hbar\sqrt{2-m(m-1)}\,\delta_{m',m-1}$$

$$L_- = \hbar \begin{array}{cc} \overbrace{\begin{array}{ccc} \boldsymbol{m=+1} & \boldsymbol{m=0} & \boldsymbol{m=-1} \end{array}} & \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \sqrt{\mathbf{2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{\mathbf{2}} & \mathbf{0} \end{pmatrix} & \begin{array}{l} m' = +1 \\ m' = \;\;\;0 \\ m' = -1 \end{array} \end{array}$$

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

$$L_x = \frac{1}{2}\left(L_+ + L_-\right)$$

$$L_y = \frac{-i}{2}\left(L_+ - L_-\right)$$

$$L_+ = \hbar \begin{array}{c c} \overbrace{\begin{matrix} m=+1 & 0 & m=-1 \end{matrix}} & \\ \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} m' = +1 \\ m' = \ \ 0 \\ m' = -1 \end{matrix} \end{array}$$

$$L_- = \hbar \begin{array}{c c} \overbrace{\begin{matrix} m=+1 & 0 & m=-1 \end{matrix}} & \\ \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} & \begin{matrix} m' = +1 \\ m' = \ \ 0 \\ m' = -1 \end{matrix} \end{array}$$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[L_x, L_y] = i\frac{\hbar^2}{2}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$[L_x, L_y] = i\frac{\hbar^2}{2}\left[\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right] = i\frac{\hbar^2}{\cancel{2}}\begin{pmatrix} \cancel{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\cancel{2} \end{pmatrix}$$

$$[L_x, L_y] = i\hbar L_z$$

Consider the matrix

$$M_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Find the eigenvalues and the normalized eigenvectors of the matrix M_3; that is, find the values of α_i for which $(M_3)_{ij}\alpha_j = \lambda\alpha_i$.

(b) Find the unitary matrix U that diagonalizes M_3.

We begin by writing out the equation $(M_3)_{ij}\,\alpha_j = \lambda\alpha_i$:

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \lambda\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\lambda = 1, \quad \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix}; \qquad \lambda = 0, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \qquad \lambda = -1, \quad \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}$$

$$M_3|\alpha\rangle = \lambda|\alpha\rangle$$

$$M_3|\alpha\rangle - \lambda|\alpha\rangle = 0$$

$$(M_3 - \lambda I)|\alpha\rangle = 0$$

$$\left[\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) = 0$$

$$\lambda = 0 \; or \; \lambda^2 - 1 = 0$$

$$\lambda_1 = +1, \lambda_2 = 0, \lambda_3 = -1$$

$$\lambda_1 = +1$$

$$\begin{pmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & -i & 0 \\ i & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\alpha_1 - i\alpha_2 \\ i\alpha_1 - \alpha_2 \\ -\alpha_3 \end{pmatrix} = 0$$

$$\begin{aligned} -\alpha_1 &= i\alpha_2 \\ i\alpha_1 &= \alpha_2 \\ -\alpha_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \xrightarrow{\lambda_1=+1} |\lambda_1\rangle = \begin{pmatrix} \alpha_1 \\ i\alpha_1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \overset{DC}{\leftrightarrow} \langle\lambda_1| = \alpha_1^*(1 \quad -i \quad 0)$$

$$\langle\lambda_1|\lambda_1\rangle = 1 = \alpha_1^*\alpha_1(1 \quad -i \quad 0)\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = |\alpha_1|^2(1+1+0) = 2|\alpha_1|^2 \Rightarrow \alpha_1 = \frac{1}{\sqrt{2}}$$

$$|\lambda_1\rangle = \begin{pmatrix} \alpha_1 \\ i\alpha_1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\lambda_2 = 0$$

$$\begin{pmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -i\alpha_2 \\ i\alpha_1 \\ 0 \times \alpha_3 \end{pmatrix} = 0$$

$$\begin{aligned} -i\alpha_2 &= 0 \\ i\alpha_1 &= 0 \\ 0 \times \alpha_3 &= 0 \end{aligned}$$

$$\begin{aligned} \alpha_2 &= 0 \\ \alpha_1 &= 0 \\ \alpha_3 &= arbitrary \end{aligned}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \xrightarrow{\lambda_2 = 0} |\lambda_2\rangle = \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{DC}{\leftrightarrow} \langle\lambda_2| = \alpha_3^* (0 \quad 0 \quad 1)$$

$$\langle\lambda_2|\lambda_2\rangle = 1 = \alpha_3^* \alpha_3 (0 \quad 0 \quad 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |\alpha_3|^2 (0 + 0 + 1) = |\alpha_3|^2 \Rightarrow \alpha_3 = 1$$

$$|\lambda_2\rangle = \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

$$\lambda_3 = -1$$

$$\begin{pmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 - i\alpha_2 \\ i\alpha_1 + \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

$$\alpha_1 = i\alpha_2$$
$$i\alpha_1 = -\alpha_2$$
$$\alpha_3 = 0$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \xrightarrow{\lambda_3 = -1} |\lambda_3\rangle = \begin{pmatrix} \alpha_1 \\ -i\alpha_1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \overset{DC}{\leftrightarrow} \langle\lambda_3| = \alpha_1^* (1 \quad i \quad 0)$$

$$\langle\lambda_3|\lambda_3\rangle = 1 = \alpha_1^*\alpha_1 (1 \quad i \quad 0) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = |\alpha_1|^2(1+1+0) = 2|\alpha_1|^2 \Rightarrow \alpha_1 = \frac{1}{\sqrt{2}}$$

$$|\lambda_3\rangle = \begin{pmatrix} \alpha_1 \\ -i\alpha_1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$U = (|\lambda_1\rangle \quad |\lambda_2\rangle \quad |\lambda_3\rangle)$$

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ \sqrt{2}\end{pmatrix}$$

$$|\lambda_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ i & 0 & -i\\ 0 & \sqrt{2} & 0\end{pmatrix}$$

$$L_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$U^{+}M_3U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i & 0\\ 0 & 0 & \sqrt{2}\\ 1 & i & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ i & 0 & -i\\ 0 & \sqrt{2} & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} = \frac{L_z}{\hbar}$$

Use the procedure of the previous example to calculate the matrix representation of

$$e^{i\alpha M_3} = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} (M_3)^k$$

$$e^{i\hbar\alpha M_3} = \sum_{k=0}^{\infty} \frac{(i\hbar\alpha)^k}{k!} (M_3)^k$$

$$U^+ M_3 U = L_z/\hbar$$

$$\begin{aligned}(L_z/\hbar)^k &= (U^+ M_3 U)^k = U^+ M_3 U U^+ M_3 U \ldots U^+ M_3 U \\ &= U^+ (M_3)^k U\end{aligned}$$

$$(M_3)^k = U(L_z/\hbar)^k U^+$$

For even k

$$(L_z/\hbar)^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(L_z/\hbar)^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (L_z/\hbar)$$

For odd k

$$\sum_{k=0}^{\infty}\frac{(i\alpha M_3)^k}{k!} = U\sum_{k=0}^{\infty}\frac{(i\alpha L_z/\hbar)^k}{k!}U^+$$

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{(i\alpha L_z/\hbar)^k}{k!} &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{2!}\begin{pmatrix}\alpha^2&0&0\\0&0&0\\0&0&\alpha^2\end{pmatrix} + \frac{1}{4!}\begin{pmatrix}\alpha^4&0&0\\0&0&0\\0&0&\alpha^4\end{pmatrix} - \cdots \\
&\quad + i\left[\begin{pmatrix}\alpha&0&0\\0&0&0\\0&0&-\alpha\end{pmatrix} - \frac{1}{3!}\begin{pmatrix}\alpha^3&0&0\\0&0&0\\0&0&-\alpha^3\end{pmatrix} + \frac{1}{5!}\begin{pmatrix}\alpha^5&0&0\\0&0&0\\0&0&-\alpha^5\end{pmatrix} + \cdots\right] \\
&= \begin{pmatrix}\cos\alpha&0&0\\0&1&0\\0&0&\cos\alpha\end{pmatrix} + i\begin{pmatrix}\sin\alpha&0&0\\0&0&0\\0&0&-\sin\alpha\end{pmatrix} \\
&= \begin{pmatrix}e^{i\alpha}&0&0\\0&1&0\\0&0&e^{-i\alpha}\end{pmatrix}
\end{aligned}$$

$$U\begin{pmatrix}e^{i\alpha}&0&0\\0&1&0\\0&0&e^{-i\alpha}\end{pmatrix}U^+$$

تمرین: این ضربهای ماتریسی را انجام دهید

Eigenstates of spin 1/2

$$\langle s',m'_{\mathrm{s}} \,|\, \mathrm{S_z} \,|\, \mathrm{s},m_{\mathrm{s}} \rangle = m_{\mathrm{s}}\hbar\,\delta_{m_{\mathrm{s}}m'_{\mathrm{s}}}\delta_{ss'}$$

$$\mathrm{S_z} = \hbar \begin{array}{c} \overbrace{\begin{array}{cc} m_{\mathrm{s}}=+1/2 & m_{\mathrm{s}}=-1/2 \end{array}} \\ \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \end{array} \begin{array}{l} \\ m'_{\mathrm{s}} = +1/2 \\ m'_{\mathrm{s}} = -1/2 \end{array}$$

$$S_z|s,m_s\rangle = m_s\hbar|s,m_s\rangle$$

$$\langle s',m_s'|S_z|s,m_s\rangle = m_s\hbar\langle s',m_s'|s,m_s\rangle$$

$$\langle s',m_s'|S_z|s,m_s\rangle = m_s\hbar\delta_{s,s'}\delta_{m_s,m_s'}$$

$$s = s' = \frac{1}{2}$$

$$-\frac{1}{2} \le m_s \le \frac{1}{2}$$

$$-\frac{1}{2} \le m_s' \le \frac{1}{2}$$

$$\left\langle \frac{1}{2}, m_s' \,|S^2|\, \frac{1}{2}, m_s \right\rangle = m_s\hbar\delta_{m_s,m_s'}$$

$$S_z = \hbar \begin{array}{cc} \overbrace{\mathrm{m}_s{=}{+}1/2 \quad \mathrm{m}_s{=}{-}1/2}^{} & \\ \begin{pmatrix} \mathbf{1/2} & 0 \\ 0 & -\mathbf{1/2} \end{pmatrix} & \begin{array}{l} m_s' = +1/2 \\ m_s' = -1/2 \end{array} \end{array} = \mathbf{1/2}\hbar \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$$

$$S^2|s,m_s\rangle = s(s+1)\hbar^2|s,m_s\rangle$$

$$\langle s',m_s'|S^2|s,m_s\rangle = s(s+1)\hbar^2\langle s',m_s'|s,m_s\rangle$$

$$\langle s',m_s'|S^2|s,m_s\rangle = s(s+1)\hbar^2\delta_{s,s'}\delta_{m_s,m_s'}$$

$$s = s' = \frac{1}{2} \Rightarrow s(s+1) = \frac{3}{4}$$

$$-\frac{1}{2} \leq m_s \leq \frac{1}{2}$$

$$-\frac{1}{2} \leq m_s' \leq \frac{1}{2}$$

$$\left\langle \frac{1}{2}, m_s' \,|S^2|\, \frac{1}{2}, m_s \right\rangle = \frac{3}{4}\hbar^2\delta_{m_s,m_s'}$$

$$S^2 = \hbar^2 \begin{array}{cc} & \overbrace{\begin{array}{cc} \mathrm{m}_s{=}{+}1/2 & \mathrm{m}_s{=}{-}1/2 \end{array}} \\ \begin{array}{c} \mathrm{m'}_s = +1/2 \\ \mathrm{m'}_s = -1/2 \end{array} & \begin{pmatrix} \mathbf{3/4} & 0 \\ 0 & \mathbf{3/4} \end{pmatrix} \end{array} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_+ = \hbar \begin{array}{cc} & \overbrace{m_s{=}{+}1/2 \quad m_s{=}{-}1/2} & \\ & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{array}{l} m'_s = +1/2 \\ m'_s = -1/2 \end{array} \end{array}$$

$$\langle s, m'_s \mid S_+ \mid s, m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)}\,\delta_{m'_s, m_s+1}$$

$$S_- = \hbar \begin{array}{cc} & \overbrace{m_s{=}{+}1/2 \quad m_s{=}{-}1/2} & \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{array}{l} m'_s = +1/2 \\ m'_s = -1/2 \end{array} \end{array}$$

$$\langle s, m'_s \mid S_- \mid s, m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)}\,\delta_{m'_s, m_s-1}$$

$$S_+ = S_x + iS_y$$

$$S_- = S_x - iS_y$$

$$S_x = \frac{1}{2}(S_+ + S_-)$$

$$S_y = \frac{-i}{2}(S_+ - S_-)$$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{-i}{2}(S_+ - S_-) = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli Matrices

$$\mathbf{S} = \tfrac{1}{2}\hbar\boldsymbol{\sigma}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

خواص ماتریسهای پاولی

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$\left[S_i, S_j\right] = i\varepsilon_{ijk}\hbar S_k$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x$$
$$\sigma_z\sigma_x = -\sigma_x\sigma_z$$
$$\sigma_y\sigma_z = -\sigma_z\sigma_y$$

این دسته روابط آخر فقط برای s=1/2 صادقند و برای l=1 درست نیستند

تمرین: درستی همه روابط فوق را تحقیق کنید.

ویژه مقادیر و ویژه حالتهای عملگر Sz

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda = \pm\frac{\hbar}{2}$$

$$S_z\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\chi_{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\det\begin{pmatrix} \frac{\hbar}{2}-\lambda & 0 \\ 0 & -\frac{\hbar}{2}-\lambda \end{pmatrix} = 0$$

تمرین: ویژه مقادیر و ویژه حالتهای Sx، Sy و Sz را به دست آورید. ویژه مقادیر و ویژه حالتهای ماتریسهای پاولی را با آنها مقایسه کنید.

An arbitrary spinor can be expanded in this complete set:

$$\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix} = \alpha_+ \begin{pmatrix}1 \\ 0\end{pmatrix} + \alpha_- \begin{pmatrix}0 \\ 1\end{pmatrix}$$

And the expansion postulate yields the interpretation that $|\alpha_+|^2$ $|\alpha_-|^2$, when properly normalized, so that

$$|\alpha_+|^2 + |\alpha_-|^2 = 1$$

Yield the probabilities that a measurement of S_z on the arbitrary state yields $+1/2\hbar$ and $-1/2\hbar$, respectively.

بردار اصلی

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

بردارهای پایه یا
بردارهای یکه

ضرایب بسط یا مولفه های بردار اصلی یا تصویر بردار $|\alpha>$ در امتداد بردار پایه $|a'>$

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

$$\langle a|\alpha\rangle = \sum_{a'} c_{a'}\langle a|a'\rangle = \sum_{a'} c_{a'}\delta_{aa'} = c_a$$

$\vec{A}$ $\quad A_x, A_y, A_z \quad \hat{i}, \hat{j}, \hat{k}$

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1 \,\&\quad \mathbf{i.j} = \mathbf{i.k} = \mathbf{j.k} = 0$$

$$A_x = \mathbf{A.i} \,\&\quad A_y = \mathbf{A.j} \,\&\quad A_z = \mathbf{A.k}$$

Given an arbitrary ket $|\alpha\rangle$ in the ket space spanned by the eigenkets of A, let us attempt to expand it as follows:

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

$$\langle a''|\alpha\rangle = \sum_{a'} c_{a'}\langle a''|a'\rangle = \sum_{a'} c_{a'}\delta_{a'a''}$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$c_{a'} = \langle a'|\alpha\rangle \qquad = c_{a''}$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \Rightarrow \sum_{a'} |a'\rangle\langle a'| = 1$$

Completeness Relation or Closure

$$\langle\alpha|\alpha\rangle = \langle\alpha|\cdot\left(\sum_{a'}|a'\rangle\langle a'|\right)\cdot|\alpha\rangle$$

$$= \sum_{a'}|\langle a'|\alpha\rangle|^2$$

This, incidentally, shows that if $|\alpha\rangle$ is normalized, then the expansion coefficients in (1.3.7) must satisfy

$$\sum_{a'}|c_{a'}|^2 = \sum_{a'}|\langle a'|\alpha\rangle|^2 = 1$$

$$\left(|a'\rangle\langle a'|\right)\cdot|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle = c_{a'}|a'\rangle$$

We see that $|a'\rangle\langle a'|$ selects that portion of the ket $|\alpha\rangle$ parallel to $|a'\rangle$, so $|a'\rangle\langle a'|$ is known as the **projection operator** along the base ket $|a'\rangle$ and is denoted by $\Lambda_{a'}$:

$$\Lambda_{a'} \equiv |a'\rangle\langle a'|$$

The completeness relation (1.3.11) can now be written as

$$\sum_{a'} \Lambda_{a'} = 1$$

An operator X always acts on a bra from the right side

$$(\langle\alpha|)\cdot X = \langle\alpha|X$$

We define the symbol X^{+} as

$$X|\alpha\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha|X^{\dagger}$$

An operator X is said to be Hermitian if

$$X = X^{\dagger}$$

We also have

$$X(Y|\alpha\rangle) = (XY)|\alpha\rangle = XY|\alpha\rangle, \qquad (\langle\beta|X)Y = \langle\beta|(XY) = \langle\beta|XY$$

Notice that

$$(XY)^\dagger = Y^\dagger X^\dagger$$

Because

$$XY|\alpha\rangle = X(Y|\alpha\rangle) \overset{\text{DC}}{\leftrightarrow} (\langle\alpha|Y^\dagger)X^\dagger = \langle\alpha|Y^\dagger X^\dagger$$

Matrix Representations

$$X = \sum_{a''} \sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|$$

$$\underset{\text{row}}{\langle a''|} X \underset{\text{column}}{|a'\rangle}$$

$$X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \dots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where the symbol $\doteq$ stands for "is represented by."

$$\langle a''|X|a'\rangle = \langle a'|X^{\dagger}|a''\rangle^{*}$$

If an operator B is Hermitian, we have

$$\langle a''|B|a'\rangle = \langle a'|B|a''\rangle^{*}$$

The matrix representation of the outer product $|\beta\rangle\langle\alpha|$ is easily seen to be

$$|\beta\rangle\langle\alpha| \doteq \begin{pmatrix} \langle a^{(1)}|\beta\rangle\langle a^{(1)}|\alpha\rangle^* & \langle a^{(1)}|\beta\rangle\langle a^{(2)}|\alpha\rangle^* & \dots \\ \langle a^{(2)}|\beta\rangle\langle a^{(1)}|\alpha\rangle^* & \langle a^{(2)}|\beta\rangle\langle a^{(2)}|\alpha\rangle^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

The matrix representation of an observable A becomes particularly simple if the eigenkets of A themselves are used as the base kets. First, we

$$A = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|A|a'\rangle\langle a'|$$

But the square matrix <a``|A|a`> is obviously diagonal

$$\langle a''|A|a'\rangle = a'\langle a''|a'\rangle = a'\delta_{a'a''}$$

$$A = \sum_{a'}\sum_{a''}|a''\rangle\langle a''|A|a'\rangle\langle a'| = \sum_{a'}\sum_{a''}|a''\rangle a'\delta_{a'a''}\langle a'|$$

Therefore

$$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$$

Spin 1/2 Systems

The base kets used are $|S_z;\pm\rangle$, which we denote, for brevity, as $|\pm\rangle$.

$$\sum_{a'} |a'\rangle\langle a'| = 1$$

$$1 = |+\rangle\langle+| + |-\rangle\langle-|$$

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$$

$$S_z|\pm\rangle = \pm(\hbar/2)|\pm\rangle$$

$$S_z = (\hbar/2)\left[(|+\rangle\langle+|) - (|-\rangle\langle-|)\right]$$

$$S_z = (\hbar/2)\left[\left(|+\rangle\langle+|\right) - \left(|-\rangle\langle-|\right)\right]$$

$$S_z = \frac{\hbar}{2}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right]$$

$$S_z = \frac{\hbar}{2}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Matrix Representation of Spin 1/2 Systems

$$X = \sum_{a''} \sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|$$

$$S_z = |+\rangle\langle+|S_z|+\rangle\langle+| + |-\rangle\langle-|S_z|+\rangle\langle+| + |+\rangle\langle+|S_z|-\rangle\langle-| + |-\rangle\langle-|S_z|-\rangle\langle-|$$

$$S_z = \frac{\hbar}{2}|+\rangle\langle+|+\rangle\langle+| + \frac{\hbar}{2}|-\rangle\langle-|+\rangle\langle+| - \frac{\hbar}{2}|+\rangle\langle+|-\rangle\langle-| - \frac{\hbar}{2}|-\rangle\langle-|-\rangle\langle-|$$

$$S_z = \frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|$$

$$S_z = \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]$$

Matrix Representation of Spin 1/2 Systems

$$X = \sum_{a''} \sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|$$

$$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle +| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\langle -| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\mathrm{S}_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)$$

$$= \frac{\hbar}{2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\right)$$

$$= \frac{\hbar}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

$$|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|S_x;-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|S_y;+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$|S_y;-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$$

تمرین: با استفاده از رابطه فوق ثابت کنید که:

$$S_x = \frac{\hbar}{2}[(|+\rangle\langle -|) + (|-\rangle\langle +|)]$$

راهنمایی: ابتدا **Sx** را در پایه خودش به صورت زیر بنویسید:

$$S_x = \frac{\hbar}{2}[|S_x;+\rangle\langle S_x;+| - |S_x;-\rangle\langle S_x;-|]$$

سپس رابطه زیر را در آن قرار دهید و از تعامد استفاده کنید:

$$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

تمرین: به طور مشابه **Sy** را هم در پایه **Sz** به دست آورید

$$S_x = \frac{\hbar}{2}[(|+\rangle\langle -|) + (|-\rangle\langle +|)]$$

$$S_y = \frac{\hbar}{2}[-i(|+\rangle\langle -|) + i(|-\rangle\langle +|)]$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right] =$$
$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\left[-i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right] =$$
$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

We now define the S^2 operator as follows:

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

For spin ½ systems, the S^2 operator is a multiple of the identity operator

$$S^2 = \frac{3}{4}\hbar^2$$

For spins higher than 1/2, S^2 is no longer a multiple of the identity operator; however, the following relation still holds.

$$[S^2, S_i] = 0$$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the anticommutation relations

$$\{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}$$

$$S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2$$

$$[S^2, S_i] = [\frac{3}{4}\hbar^2, S_i] = 0$$

$$S_x^2 + S_x^2 = \frac{1}{2}\hbar^2 \qquad S_y^2 + S_y^2 = \frac{1}{2}\hbar^2 \qquad S_z^2 + S_z^2 = \frac{1}{2}\hbar^2$$

$$S_x^2 = \frac{1}{4}\hbar^2 \qquad S_y^2 = \frac{1}{4}\hbar^2 \qquad S_z^2 = \frac{1}{4}\hbar^2$$

where the commutator [,] and the anticommutator { , } are defined by

$$[A, B] \equiv AB - BA,$$
$$\{A, B\} \equiv AB + BA.$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{S}_+ = \hbar |+\rangle\langle-| = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathrm{S}_- = \hbar |-\rangle\langle+| = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_+ = \hbar\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_- = \hbar\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_+ \equiv \hbar|+\rangle\langle -|$$

$$S_+^\dagger = \hbar|-\rangle\langle +| = S_- \neq S_+$$

$$S_- \equiv \hbar|-\rangle\langle +|$$

$$S_-^\dagger = \hbar|+\rangle\langle -| = S_+ \neq S_-$$

Which are both non-Hermitian

$$S_+|+\rangle = \hbar|+\rangle\langle -|+\rangle = 0$$

The physical interpretation of S^+ is that it raises the spin component by one unit of ħ

$$S_+|-\rangle = \hbar|+\rangle\langle -|-\rangle = \hbar|+\rangle$$

$$S_-|+\rangle = \hbar|-\rangle\langle +|+\rangle = \hbar|-\rangle$$

The physical interpretation of S^- is that it lowers the spin component by one unit of ħ

$$S_-|-\rangle = \hbar|-\rangle\langle +|-\rangle = 0$$

MEASUREMENTS, OBSERVABLES

Quantum theory of measurement processes.

"A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured."

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

When the measurement is performed, the system is "thrown into" one of the eigenstates, say |a`> of observable A.

In other words:

$$|\alpha\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

Thus a measurement usually changes the state.

The only exception is when the state is already in one of the eigenstates of the observable being measured:

$$|a'\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

Transformation Matrix

$$U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$$

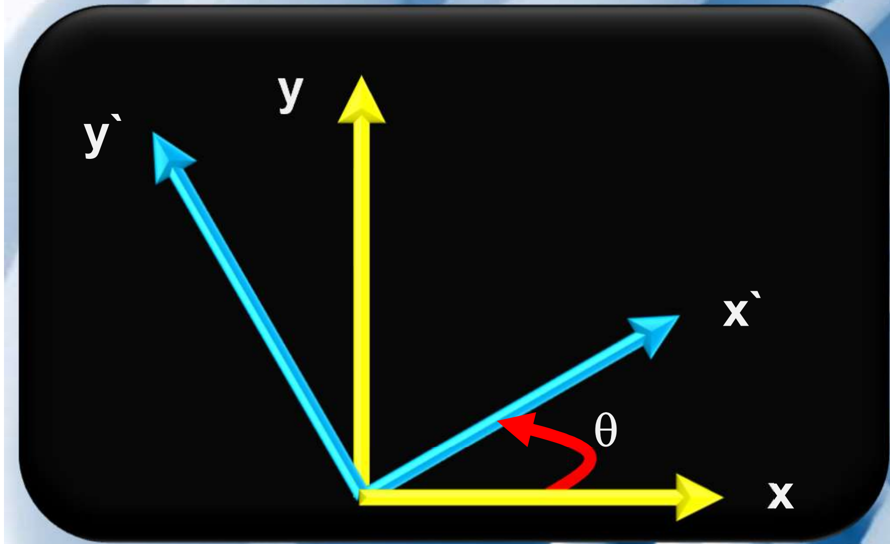
$$U|a^{(l)}\rangle = |b^{(l)}\rangle$$

$$\langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$$

In other words, the matrix elements of the U operator are built up of the inner products of old base bras and new base kets.

The square matrix made up of $\langle a^{(k)}|U|a^{(l)}\rangle$ is referred to as the transformation matrix from the $\{|a'\rangle\}$ basis to the $\{|b'\rangle\}$ basis.

We recall that the rotation matrix in three dimensions that changes one set of unit base vectors (**x, y, z**) into another set (**x', y', z'**) can be written as:



$$R = \begin{pmatrix} \hat{\mathbf{x}}\cdot\hat{\mathbf{x}}' & \hat{\mathbf{x}}\cdot\hat{\mathbf{y}}' & \hat{\mathbf{x}}\cdot\hat{\mathbf{z}}' \\ \hat{\mathbf{y}}\cdot\hat{\mathbf{x}}' & \hat{\mathbf{y}}\cdot\hat{\mathbf{y}}' & \hat{\mathbf{y}}\cdot\hat{\mathbf{z}}' \\ \hat{\mathbf{z}}\cdot\hat{\mathbf{x}}' & \hat{\mathbf{z}}\cdot\hat{\mathbf{y}}' & \hat{\mathbf{z}}\cdot\hat{\mathbf{z}}' \end{pmatrix}$$

$$\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$$

$$\hat{\mathbf{y}}' = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$$

$$\hat{\mathbf{x}}.\hat{\mathbf{x}}' = \cos\theta \qquad \hat{\mathbf{x}}.\hat{\mathbf{y}}' = -\sin\theta$$

$$\hat{\mathbf{y}}.\hat{\mathbf{x}}' = \sin\theta \qquad \hat{\mathbf{y}}.\hat{\mathbf{y}}' = \cos\theta$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Given an arbitrary ket |α> whose expansion coefficients <a'|α> are known in the old basis, how can we obtain <b'|α>, the expansion coefficients in the new basis?

$$|\alpha\rangle = \sum_l |a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle$$

$$(\text{New}) = (U^\dagger)(\text{old})$$

$$\langle b^{(k)}|\alpha\rangle = \sum_l \langle b^{(k)}|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle = \sum_l \langle a^{(k)}|U^\dagger|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle$$

$$U|a^{(l)}\rangle = |b^{(l)}\rangle$$

$$<a^{(l)}|U^\dagger = <b^{(l)}|$$

In matrix notation, it states that the column matrix for |α> in the new basis can be obtained just by applying the square matrix U^+ to the column matrix in the old basis:

The relationships between the old matrix elements and the new matrix elements are also easy to obtain:

$$\langle b^{(k)}|X|b^{(l)}\rangle = \sum_m \sum_n \langle b^{(k)}|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|b^{(l)}\rangle$$

$$= \sum_m \sum_n \langle a^{(k)}|U^\dagger|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|U|a^{(l)}\rangle$$

This is simply the well-known formula for a similarity transformation in matrix algebra,

$$X' = U^\dagger X U$$

The trace of an operator X is defined as the sum of diagonal elements:

$$\mathrm{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle$$

The trace of an operator X is independent of representation:

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{a'} \sum_{b'} \sum_{b''} \langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle$$

$$= \sum_{b'} \sum_{b''} \langle b''|b'\rangle\langle b'|X|b''\rangle = \sum_{b'} \langle b'|X|b'\rangle \checkmark$$

We can also prove that

$$\mathrm{tr}(XY) = \mathrm{tr}(YX)$$

$$\mathrm{tr}(\mathrm{XY}) = \sum_{a'} \langle a'|\mathrm{XY}|a'\rangle$$

$$= \sum_{a'b'} \langle a'|\mathrm{X}|b'\rangle\langle b'|\mathrm{Y}|a'\rangle$$

$$= \sum_{a'b'} \langle b'|\mathrm{Y}|a'\rangle\langle a'|\mathrm{X}|b'\rangle$$

$$= \sum_{b'} \langle b'|\mathrm{YX}|b'\rangle$$

$$= \sum_{a'} \langle a'|\mathrm{YX}|a'\rangle = \mathrm{tr}(\mathrm{YX})$$

$$\mathrm{tr}(U^{\dagger}XU) = \mathrm{tr}(X)$$

$$\mathrm{tr}(U^{\dagger}XU) = \sum_{a'} < a' | U^{\dagger}XU | a' >$$

$$= \sum_{a'b'b''} < a' | U^{\dagger} | b' >< b' | X | b'' >< b'' | U | a' >$$

$$= \sum_{a'b'b''} < b'' | U | a' >< a' | U^{\dagger} | b' >< b' | X | b'' >$$

$$= \sum_{b'b''} < b'' | U^{\dagger}U | b' >< b' | X | b'' >$$

$$= \sum_{b'b''} < b'' | b' >< b' | X | b'' >$$

$$= \sum_{b'} < b' | X | b' > = \mathrm{tr}(X) \checkmark$$

$$\mathrm{tr}\left(|a'\rangle\langle a''|\right) = \delta_{a'a''}$$

$$\mathrm{tr}(|a'><a''|) = \sum_{b'} <b'|a'><a''|b'> = <a''|a'> = \delta_{a'a''}$$ ✓

$$\mathrm{tr}\left(|b'\rangle\langle a'|\right) = \langle a'|b'\rangle$$

$$\mathrm{tr}(|b'><a'|) = \sum_{c'} <c'|b'><a'|c'> = \sum_{c'} <a'|c'><c'|b'>$$

$$= <a'|b'>$$ ✓

تمرین: ویژه مقادیر و ویژه حالتهای عملگر $S_x\cos\varphi + S_y\sin\varphi$ را به دست آورید:

$$(S_x \cos\phi + S_y \sin\phi)\begin{pmatrix} u \\ v \end{pmatrix} = \tfrac{1}{2}\hbar\lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 0 & \cos\phi - i\sin\phi \\ \cos\phi + i\sin\phi & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

$$v\, e^{-i\phi} = \lambda\, u$$
$$u\, e^{i\phi} = \lambda v$$

$$uv(\lambda^2 - 1) = 0$$
$$\lambda = \pm 1$$

$$\lambda = 1$$
$$v = e^{i\phi}\, u$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$$

We take advantage of the fact that we can multiply a state vector by an arbitrary phase factor, which we choose to be $e^{-i\phi/2}$. This yields

$$u_{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}$$

and similarly, the eigenstate corresponding to $\lambda = -1$ can be written in the form

$$u_{-} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi/2} \\ -e^{i\phi/2} \end{pmatrix}$$

which is easily seen to be orthogonal to u_+:

$$u_+^* u_- = \frac{1}{2}(e^{i\phi/2}, e^{-i\phi/2})\begin{pmatrix} e^{-i\phi/2} \\ -e^{i\phi/2} \end{pmatrix}$$
$$= 0$$

Given an arbitrary state α, the expectation value of $\mathbf{S}$ can be calculated. We have

$$\langle\alpha|\mathbf{S}|\alpha\rangle = \sum_i \sum_j \langle\alpha|i\rangle\langle i|\mathbf{S}|j\rangle\langle j|\alpha\rangle$$

or, equivalently,

$$(\alpha_+^*, \alpha_-^*)\,\mathbf{S}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}$$

$$\begin{aligned}\langle S_x\rangle &= (\alpha_+^*, \alpha_-^*)\,\tfrac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}\\ &= \tfrac{1}{2}\hbar(\alpha_+^*, \alpha_-^*)\begin{pmatrix}\alpha_-\\ \alpha_+\end{pmatrix} = \tfrac{1}{2}\hbar(\alpha_+^*\alpha_- + \alpha_-^*\alpha_+)\end{aligned}$$

$$\langle S_y \rangle = \tfrac{1}{2}\hbar(\alpha_+^*, \alpha_-^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

$$= \tfrac{1}{2}\hbar(\alpha_+^*, \alpha_-^*)\begin{pmatrix} -i\alpha_- \\ i\alpha_+ \end{pmatrix} = -\frac{i\hbar}{2}(\alpha_+^*\alpha_- - \alpha_-^*\alpha_+)$$

$$\langle S_z \rangle = \tfrac{1}{2}\hbar(\alpha_+^*, \alpha_-^*)\begin{pmatrix} \alpha_+ \\ -\alpha_- \end{pmatrix} = \tfrac{1}{2}\hbar(|\alpha_+|^2 - |\alpha_-|^2)$$

Note that all of these are real, as expected for hermitian operators.

A measurement of S_x for a spin 1/2 system leads to the eigenvalue $+\hbar/2$. Subsequently, a measurement of $S_x \cos\phi + S_y \sin\phi$ is carried out. What is the probability that the result is $+\hbar/2$?

After the initial measurement, the system must be in the eigenstate of S_x with eigenvalue $+\hbar/2$.

$$\frac{\hbar}{2}\sigma_x \begin{pmatrix} u \\ v \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Since $|u|^2 + |v|^2 = 1$ results from normalization, it is convenient to choose the numbers u and v to be real. Thus

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This state may be expanded in eigenstates of the operator $S_x \cos\phi + S_y \sin\phi$, which were obtained before. They are given in (10-17) and (10-18). Writing

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \alpha_+ \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\phi/2}\\e^{i\phi/2}\end{pmatrix} + \alpha_- \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\phi/2}\\-e^{i\phi/2}\end{pmatrix}$$

$$\alpha_+ = \frac{1}{2}(e^{i\phi/2}\;\; e^{-i\phi/2})\begin{pmatrix}1\\1\end{pmatrix} = \cos\frac{\phi}{2}$$

The result is that $P_+ = |\alpha_+|^2 = \cos^2 \phi/2.$

THE INTRINSIC MAGNETIC MOMENT OF SPIN 1/2 PARTICLES

$$\mathbf{M} = -\frac{eg}{2m_e}\mathbf{S}$$

$$H = -\mathbf{M} \bullet \mathbf{B} = \frac{eg\hbar}{4m_e}\boldsymbol{\sigma} \bullet \mathbf{B}$$

The Schrödinger equation for the state $\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$ is

$$i\hbar \frac{d}{dt}\psi(t) = \frac{eg\hbar}{4m_e}\boldsymbol{\sigma} \cdot \mathbf{B}\psi(t)$$

If $\mathbf{B}$ is taken to define the z-axis, and if we write $\psi(t) = e^{-i\omega t}\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}$, then the equation becomes

$$\hbar\omega\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix} = \frac{eg\hbar B}{4m_e}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}$$

One eigenvalue is $\omega_0 = egB/4m_e$ and the corresponding solution is $\begin{pmatrix}1 \\ 0\end{pmatrix}$. The other eigenvalue is $\omega_0 = -egB/4m_e$ and the eigenvector is $\begin{pmatrix}0 \\ 1\end{pmatrix}$. If the initial state is

$$\psi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

then the state at a later time will be

$$\psi(t) = \begin{pmatrix} ae^{-i\omega_0 t} \\ be^{i\omega_0 t} \end{pmatrix}$$

Suppose that at time $t = 0$ the spin is in an eigenstate of S_x with eigenvalue $\hbar/2$, so that it "points in the x-direction." This means that

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$$

$$\langle S_x \rangle = \frac{\hbar}{2} \frac{1}{\sqrt{2}} (e^{i\omega_0 t} \quad e^{-i\omega_0 t}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{2} \cos 2\omega_0 t$$

Similarly,

$$\langle S_y \rangle = \frac{\hbar}{2} \frac{1}{\sqrt{2}} (e^{i\omega_0 t} \quad e^{-i\omega_0 t}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{2} \sin 2\omega_0 t$$

Thus the spin precesses about the direction of B with a frequency

$$2\omega_0 = \frac{egB}{2m_e} \equiv g\omega_c$$

$$\frac{d\mathbf{S}}{dt} = \frac{i}{\hbar}[H, \mathbf{S}(t)]$$

$$H = \frac{eg}{2m_e}\mathbf{S}\cdot\mathbf{B} \equiv \gamma\mathbf{S}\cdot\mathbf{B}$$

$$\begin{aligned} \frac{dS_1(t)}{dt} &= \frac{i\gamma}{\hbar}\left\{B_1[S_1, S_1] + B_2[S_2, S_1] + B_3[S_3, S_1]\right\} \\ &= \frac{i\gamma}{\hbar}\left\{-i\hbar B_2 S_3 + i\hbar B_3 S_2\right\} = \gamma(\mathbf{B}\times\mathbf{S})_1 \end{aligned}$$

Similar calculation for the other components yields

$$\frac{d\mathbf{S}(t)}{dt} = \gamma(\mathbf{S} \times \mathbf{B})$$

This, however, is exactly the equation for the precession of a classical angular momentum vector (associated with a magnetic dipole) subject to a torque due to a magnetic field. In this form we easily see that the precession is not restricted to the case of spin 1/2. Given that the magnetic moment of a system with angular momentum $\mathbf{J}$ is $\gamma\mathbf{J}$, the same equation holds.