Magnetic Dipole Moment

- Any loop of current subject to a torque in a magnetic field
  - Define a vector **A** from area and direction of current (using RHR)
  - Then can rewrite the torque:

    $\boldsymbol{\mu} \equiv I\mathbf{A}$

    $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$

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    $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$

  - Describes a compass needle too!
    - Measure the magnetic moment experimentally

Potential Energy

- The loop wants to be twisted till its dipole is aligned with the field
  - Can put this in terms of energy

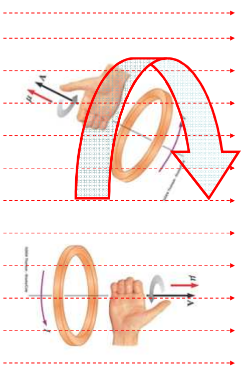
    $W = U_b - U_a = \int_a^b \mathbf{F} \cdot d\mathbf{x} = \int_a^b \tau d\theta$

    $= \int_a^b \mu B \sin\theta d\theta = -\mu B \cos\theta \Big|_a^b$

  - Choose $U = 0$ at 90°

    $U = -\boldsymbol{\mu} \cdot \mathbf{B}$

  - Lowest energy when aligned!



Motion in a Uniform Field

- This was our thought experiment, but there is more to talk about
  - Movement in a helix, what's the radius?

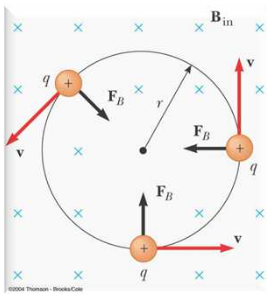
    $F = |q\mathbf{v} \times \mathbf{B}| = qv_{\perp}B = \frac{mv_{\perp}^2}{R}$

    $R = \frac{mv_{\perp}}{qB} = \frac{p_{\perp}}{qB}$

  - What's the frequency (of the orbit)?

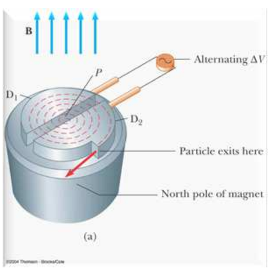
    $\omega = \frac{v_{\perp}}{R} = \frac{qB}{m}$

    - This is the "cyclotron frequency"



What's a Cyclotron?

- The cyclotron is a particle accelerator
  - The uniform magnetic field makes particles go in circles (only half shown)
  - If there is a voltage difference between D_1 and D_2, the particle is accelerated or decelerated
  - If we switch the voltage after each crossing, can just accelerate!



Generating Magnetic Fields

- The origins of magnetic fields were a mystery until Oersted's 1819 discovery that an electric current generates a magnetic field
  - He noticed the deflection of compass needles during a demonstration with electricity

Generating Magnetic Fields

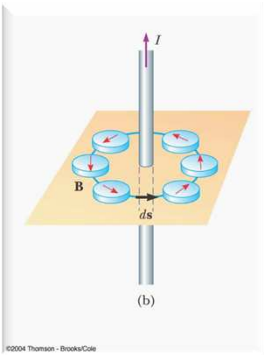
- The origins of magnetic fields were a mystery until Oersted's 1819 discovery that an electric current generates a magnetic field
  - He noticed the deflection of compass needles during a demonstration with electricity
- A current generates a magnetic field, but *what's the form* (i.e. what's the equation)?

Magnetic field about a wire

- If a current generates a magnetic field, let's look at the field lines about a current carrying wire

Magnetic field about a wire

- If a current generates a magnetic field, let's look at the field lines about a current carrying wire
  - Remember, each iron filing acts like a little magnet, and the magnets line up along the magnetic field



Magnetic field about a wire

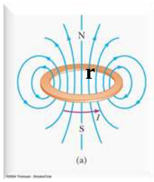
- If a current generates a magnetic field, let's look at the field lines about a current carrying wire
  - So the field is perpendicular to both the wire (the current) and the direction from the wire (**r**)
    $\mathbf{B} = f(r)(\mathbf{I} \times \mathbf{r})$
  - But a current isn't point-like, it's linear;
    Where's **r** measured from?

Magnetic field about a wire

- Problem: not sure about **r**
  - Solution: make *r* the same for everything!

Magnetic field of a loop

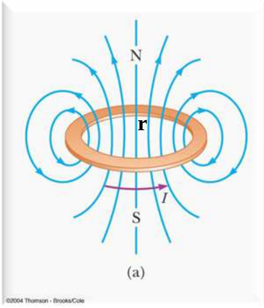
- Problem: not sure about **r**
  - Solution: make *r* the same for everything!
  - Measure the field by measuring the torque on a compass at the center!



Magnetic field of a loop

- Problem: not sure about **r**
  - Solution: make *r* the same for everything!
  - Measure the field by measuring the torque on a compass at the center!
  - Then make ring bigger and see how the torque changes, find

$$|\mathbf{B}| = \text{const} \times \frac{I}{r}$$



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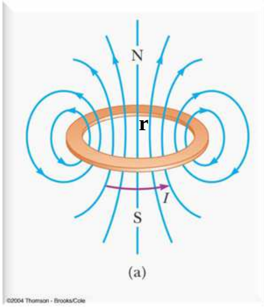


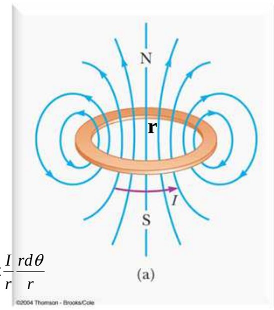
Magnetic field of a loop

- Magnetic field at the center of a loop

$$|\mathbf{B}| = \text{const} \times \frac{I}{r}$$

  - But a bigger loop (same current) has *more* charges in motion
    - Bigger circles have more little segments of current
    - Each segment gives

$$|d\mathbf{B}| = \text{const} \times \frac{I}{r}\,d\theta = \text{const} \times \frac{I}{r}\frac{r\,d\theta}{r}$$
$$= \text{const} \times \frac{Ids}{r^2}$$



The Biot-Savart Law

- Put it all together, have

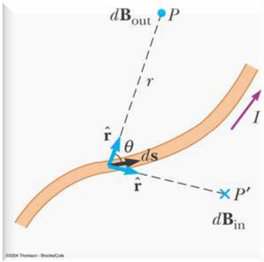
$$\boxed{d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{s}\times\hat{\mathbf{r}}}{r^2}}$$

  - μ_0 is the "permeability of free space"

$$\mu_0 = 4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$$

- Always have to integrate along whole wire (current) to get actual field

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\int\frac{Id\mathbf{s}\times\hat{\mathbf{r}}}{r^2}}$$



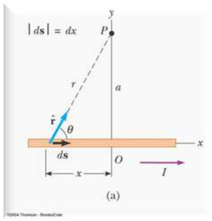
Example: Field of a Line Current

- What's the field about a long (infinite) straight wire, with current?

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{s}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\hat{\mathbf{k}}\int\frac{\sin\theta}{r^2}dx$$

Switch units from *x* to θ

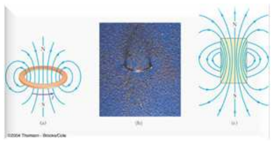
$$r = \frac{a}{\sin\theta} = a\csc\theta; \quad x = \frac{-a}{\tan\theta} = -a\cot\theta$$
$$dx = a\csc^2\theta\, d\theta$$

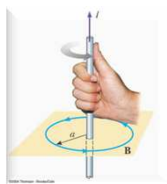


$$B = \frac{\mu_0 I}{4\pi}\int_0^\pi \frac{\sin\theta}{(a\csc\theta)^2}a\csc^2\theta\, d\theta$$
$$= \frac{\mu_0 I}{4\pi a}\int_0^\pi \sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\pi - \cos 0) = \frac{\mu_0 I}{2\pi a}$$

Magnetic Field Lines

- The generated magnetic field is always perpendicular to **r** from the source
  - This means that magnetic field lines don't start or end (like electric field lines)
  - Magnetic field lines just form loops





Force Between Two Parallel Wires

- Can now calculate the force between two long parallel wires
  - Magnetic force on one wire (1) due to field of the other (2)

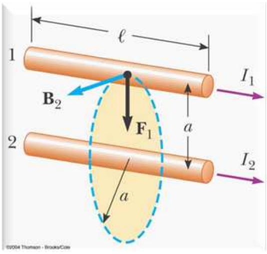
$$\mathbf{F}_1 = l\mathbf{I}_1 \times \mathbf{B}_2$$

$$|\mathbf{F}_1| = lI_1 B_2$$

  - Field of other wire (2)

$$\mathbf{B}_2 = \frac{\mu_0}{2\pi a}\mathbf{I}_2 \times \hat{\mathbf{a}}$$

$$|\mathbf{B}_2| = \frac{\mu_0 I_2}{2\pi a}$$



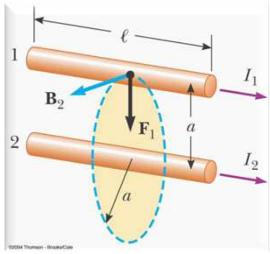
Force Between Two Parallel Wires

- Can now calculate the force between two long parallel wires
  - Putting it together

$$F_1 = lI_1 B_2; \quad B_2 = \frac{\mu_0 I_2}{2\pi a}$$

$$F_{12} = \frac{\mu_0 I_1 I_2}{2\pi a} l$$

  - *Attractive* for currents in the same direction
  - This is used to define the amp as a unit (and the coulomb)



Ampere's Law

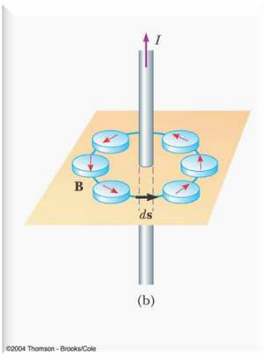
- Since magnetic field lines form loops about their sources, lets integrate around the loop
  - For an infinite wire:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \frac{\mu_0}{2\pi a} \oint (\mathbf{I} \times \hat{\mathbf{a}}) \cdot d\mathbf{s}$$

$$= \frac{\mu_0 I}{2\pi a} \oint ds$$

$$= \frac{\mu_0 I}{2\pi a} 2\pi a$$

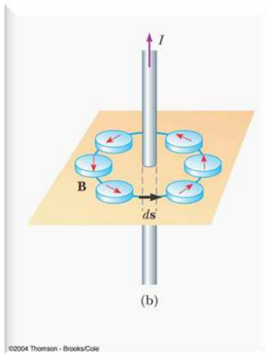
$$= \mu_0 I$$



Ampere's Law

- This is true in general, Ampere's Law:

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{thru}}}$$



Example: An extended wire

- Apply Ampere's Law to a real (finite diameter) wire
  - Outside the wire ($r > R$)

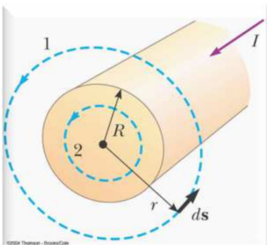
$$\oint \mathbf{B} \cdot d\mathbf{s} = 2\pi r B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

  - Inside wire ($r < R$)

$$\oint \mathbf{B} \cdot d\mathbf{s} = 2\pi r B = \mu_0 \left(\frac{r}{R}\right)^2 I$$

$$B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$



Field of a Loop of Current

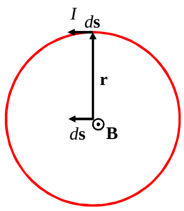
- What's at the center of a loop of current?

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{s} \times \hat{\mathbf{r}}}{r^2}$$

$$B = \frac{\mu_0}{4\pi} \frac{I}{r^2} \int ds$$

$$= \frac{\mu_0}{4\pi} \frac{I}{r^2} 2\pi r$$

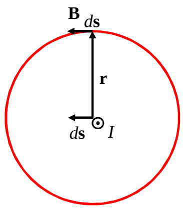
$$= \frac{\mu_0 I}{2r}$$



Line Integral of Field about a Loop

- Switch it around
- What's the line integral of the field going around a line current?

$$\oint \mathbf{B}\cdot d\mathbf{s} = \oint\left(\frac{\mu_0}{2\pi r}\mathbf{I}\times\hat{\mathbf{r}}\right)\cdot d\mathbf{s}$$
$$= \frac{\mu_0}{2\pi r}\oint(\mathbf{I}\times\hat{\mathbf{r}})\cdot d\mathbf{s}$$
$$= \frac{\mu_0 I}{2\pi r}\oint ds$$
$$= \frac{\mu_0 I}{2\pi r}2\pi r$$
$$= \mu_0 I$$



Ampere's Law

- This is true actually for any loop
  - Radial segments don't contribute
  - Tangential segments contribute by angle
- Could add another current
  - Superposition works for magnetic field too!
- General result is Ampere's Law:

$$\boxed{\oint_{\text{loop}} \mathbf{B}\cdot d\mathbf{s} = \mu_0 I_{\text{thru}}}$$

  - Use it like Gauss's Law in electrostatics

Example: An extended wire

- Apply Ampere's Law to a real (finite diameter) wire
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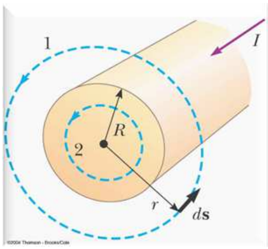
$$\oint \mathbf{B}\cdot d\mathbf{s} = 2\pi r B = \mu_0 I$$
$$B = \frac{\mu_0 I}{2\pi r}$$

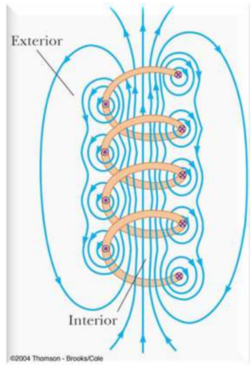
  - Inside wire ($r < R$)

$$\oint \mathbf{B}\cdot d\mathbf{s} = 2\pi r B = \mu_0\left(\frac{r}{R}\right)^2 I$$
$$B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}$$



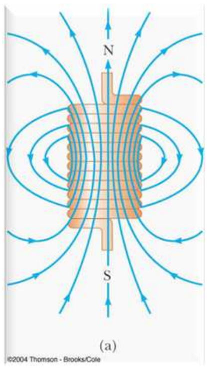
Example: Field Inside a Solenoid

- A solenoid is a wire wrapped in a helix



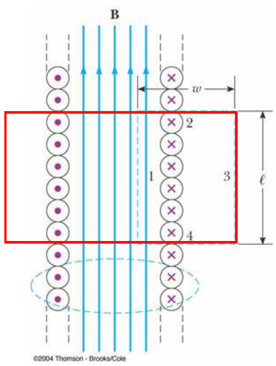
Example: Field Inside a Solenoid

- A solenoid is a wire wrapped in a helix
- Ideally, the coil spacing is tight…



Example: Field Inside a Solenoid

- A solenoid is a wire wrapped in a helix
- Ideally, the coil spacing is tight…
- …and it goes on forever
- Field outside is small
  - Using Ampere's Law: no net current thru loop



Example: Field Inside a Solenoid

- Inside:

$$\oint \mathbf{B}\cdot d\mathbf{s} = \sum_i \int_i \mathbf{B}\cdot d\mathbf{s}$$

$$\int_{2,4} \mathbf{B}\cdot d\mathbf{s} = \int B\cos\tfrac{\pi}{2}\, ds = 0$$

$$\int_{3} \mathbf{B}\cdot d\mathbf{s} = \int 0\, ds = 0$$

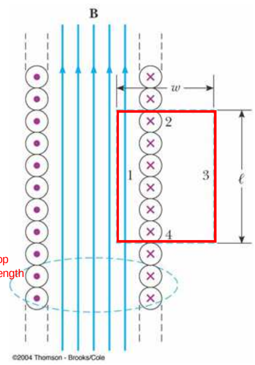
$$\int_{1} \mathbf{B}\cdot d\mathbf{s} = Bl$$

$$\oint \mathbf{B}\cdot d\mathbf{s} = Bl = \mu_0 NI$$

N turns inside loop

n turns per unit length

$$B = \mu_0 I \frac{N}{l} = \mu_0 nI$$



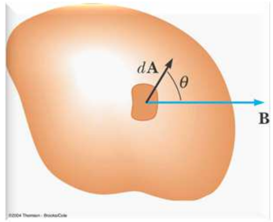
Magnetic Flux

- Just as with the electric field, we define the flux of the magnetic field thru a surface *S*

$$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{A}$$

  - $d\mathbf{A}$ is a vector perpendicular to the surface
  - Units of flux: Weber (Wb)

$$1\,\mathrm{Wb} \equiv 1\,\mathrm{T}\cdot\mathrm{m}^2$$



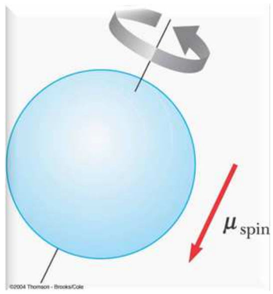
Gauss's Law in Magnetism

- Since all magnetic field lines form loops (no magnetic charges), net flux thru any closed surface is zero!

$$\boxed{\oint_S \mathbf{B}\cdot d\mathbf{A} = 0}$$

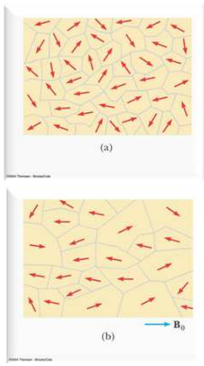
Permanent Magnets

- Each electron acts like a spinning top
  - This means that it also acts like a tiny loop of electric current
  - Each electron has a magnetic dipole moment



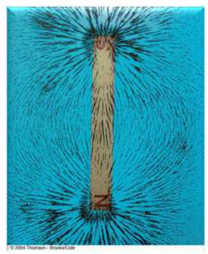
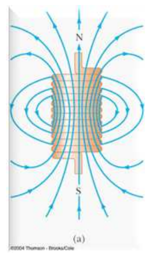
Permanent Magnets

- Each electron acts like a spinning top
  - This means that it also acts like a tiny loop of electric current
  - Each electron has a magnetic dipole moment
- In regular materials, the electron spins are randomly oriented
- In a magnet, can get spins to line up by applying an external field



Permanent Magnets

- In a bar magnet, the aligned spins give a field just like that of a solenoid

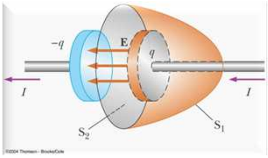
Displacement Current

- Consider the magnetic field around a capacitor being charged

$$\oint_{S_1} \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

$$\oint_{S_2} \mathbf{B} \cdot d\mathbf{s} = 0$$

  - But S_1 and S_2 have the same boundary!



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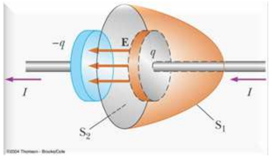
  - But S_1 and S_2 have the same boundary!
  - Have to add something to make these consistent



Displacement Current

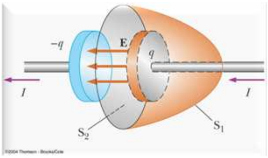
- If there is a current flowing, the field (and electric flux) in the capacitor is changing!

$$\Phi_E = \int_{S_2} \mathbf{E} \cdot d\mathbf{A}$$

$$= \int_C \frac{q}{\varepsilon_0 A_C} dA_C + 0$$

$$= \frac{q}{\varepsilon_0}$$

$$\frac{d\Phi_E}{dt} = \frac{1}{\varepsilon_0}\frac{dq}{dt} = \frac{I}{\varepsilon_0}$$



Define the Displacement Current

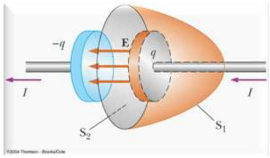
$$\boxed{I_D \equiv \varepsilon_0 \frac{d\Phi_E}{dt}}$$

Displacement Current

- If there is a current flowing, the field (and electric flux) in the capacitor is changing!
- Use displacement current to make the two integrals consistent

$$\oint_{S_1} \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

$$\oint_{S_2} \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_D$$



Define the Displacement Current

$$\boxed{I_D \equiv \varepsilon_0 \frac{d\Phi_E}{dt}}$$

General Form of Ampere's Law

- In general Ampere's Law (Ampere-Maxwell's Law) should be

$$\boxed{\begin{aligned}\oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{s} &= \mu_0 \left(I_{\text{thru}} + I_{\text{disp}}\right) \\ &= \mu_0 I_{\text{thru}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}\end{aligned}}$$