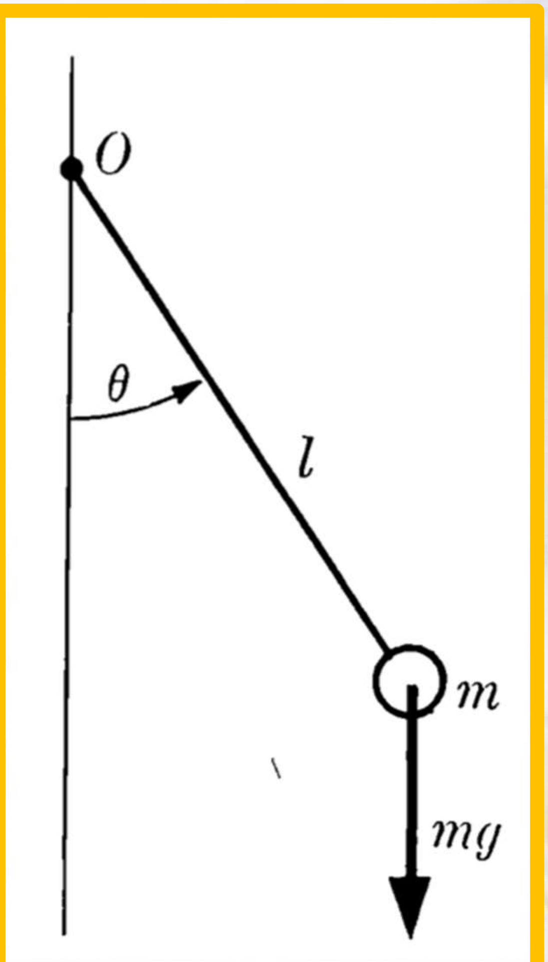
Rectilinear motion		*Rotation about a fixed axis*	
position:	x	angular position:	θ
velocity:	$v = \dot{x}$	angular velocity:	$\omega = \dot{\theta}$
acceleration:	$a = \ddot{x}$	angular acceleration:	$\alpha = \ddot{\theta}$
force:	F	torque:	N_z
mass:	m	moment of inertia:	I_z
potential energy:		potential energy:	
$V(x) = -\int_{x_s}^{x} F(x)\,dx$		$V(\theta) = -\int_{\theta_s}^{\theta} N_z(\theta)\,d\theta$	
$F(x) = -\dfrac{dV}{dx}$		$N_z(\theta) = -\dfrac{dV}{d\theta}$	
kinetic energy:	$T = \frac{1}{2}m\dot{x}^2$	kinetic energy:	$T = \frac{1}{2}I_z\dot{\theta}^2$
linear momentum:	$p = m\dot{x}$	angular momentum:	$L = I_z\dot{\theta}$

$$\boxed{\frac{dL_z}{dt} = N_z = I_z\ddot{\theta}}$$

مثال: آونگ ساده



$$L_z = I_z\dot{\theta}$$

$$I_z = ml^2$$

$$L_z = ml^2\dot{\theta}$$

$$\frac{dL_z}{dt} = N_z$$

$$\frac{dL_z}{dt} = ml^2\frac{d\dot{\theta}}{dt} = ml^2\ddot{\theta}$$

$$N_z = -mgl\sin\theta$$

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

معادله دیفرانسیل حاکم بر آونگ ساده

حل معادله حاکم بر آونگ ساده با فرض کوچک بودن زاویه

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

$$\theta \ll 1$$

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$\theta = Ae^{pt}$$

$$\ddot{\theta} = Ap^2e^{pt}$$

$$Ap^2e^{pt} = -\frac{g}{l}Ae^{pt}$$

$$p^2 = -\frac{g}{l}$$

$$p = \pm\sqrt{-\frac{g}{l}} = \pm i\sqrt{\frac{g}{l}} = \pm i\omega$$

$$\theta(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$\theta(t) = A\cos(\omega t + \theta)$$

$$\omega = \sqrt{\frac{g}{l}} = 2\pi\nu$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

$$\nu = \frac{1}{T}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

حل معادله حاکم بر آونگ ساده بدون فرض کوچک بودن زاویه

$$V(\theta) = -\int_{\theta_s}^{\theta} N_z(\theta)\, d\theta$$

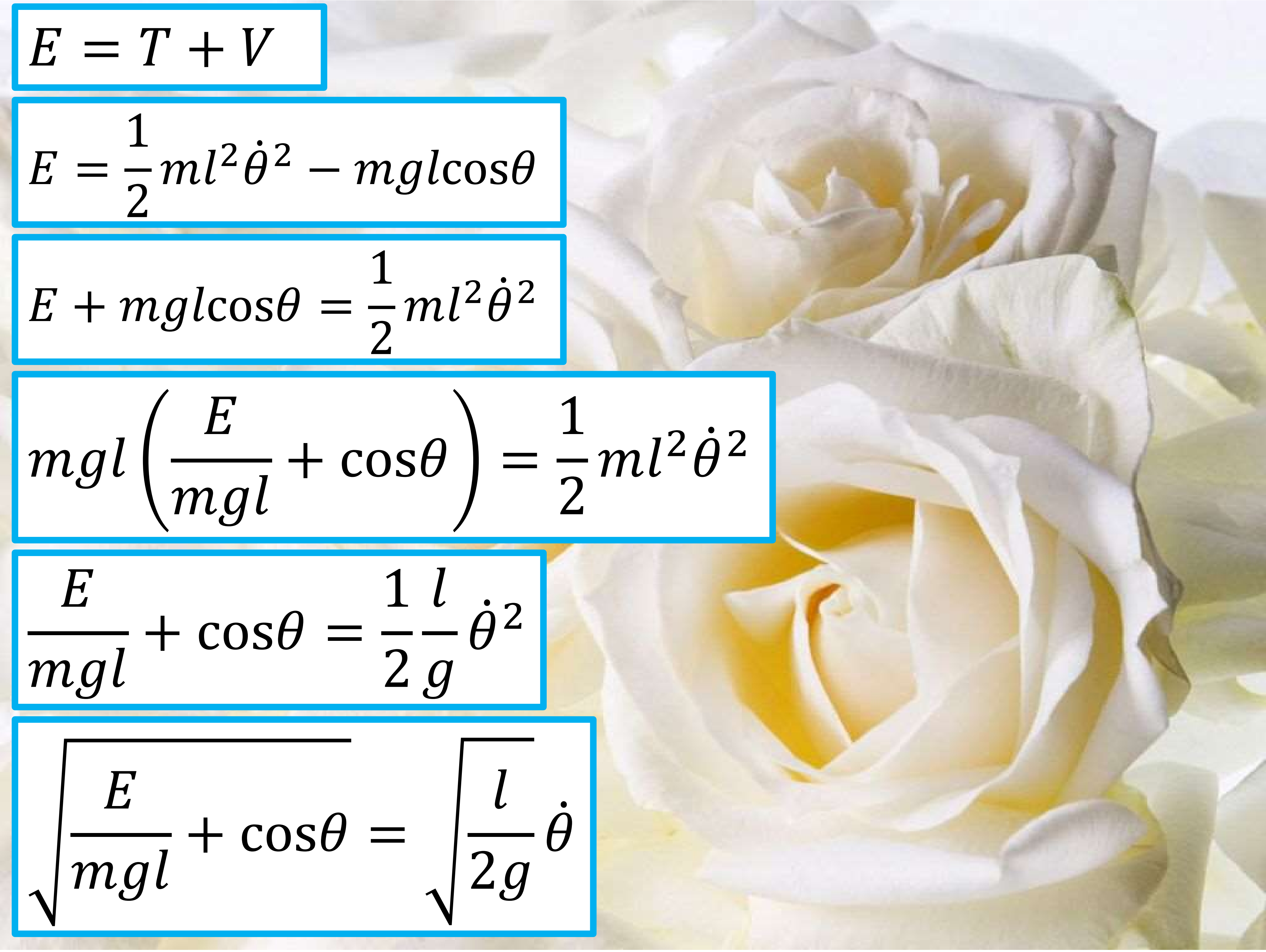
$$N_z = -mgl \sin\theta$$

$$V(\theta) = \int_{\theta_s}^{\theta} mgl \sin\theta \, d\theta$$

$$V(\theta) = -mgl(\cos\theta - \cos\theta_s)$$

$$\theta_s = \frac{\pi}{2}$$

$$V(\theta) = -mgl\cos\theta$$



$$E = T + V$$

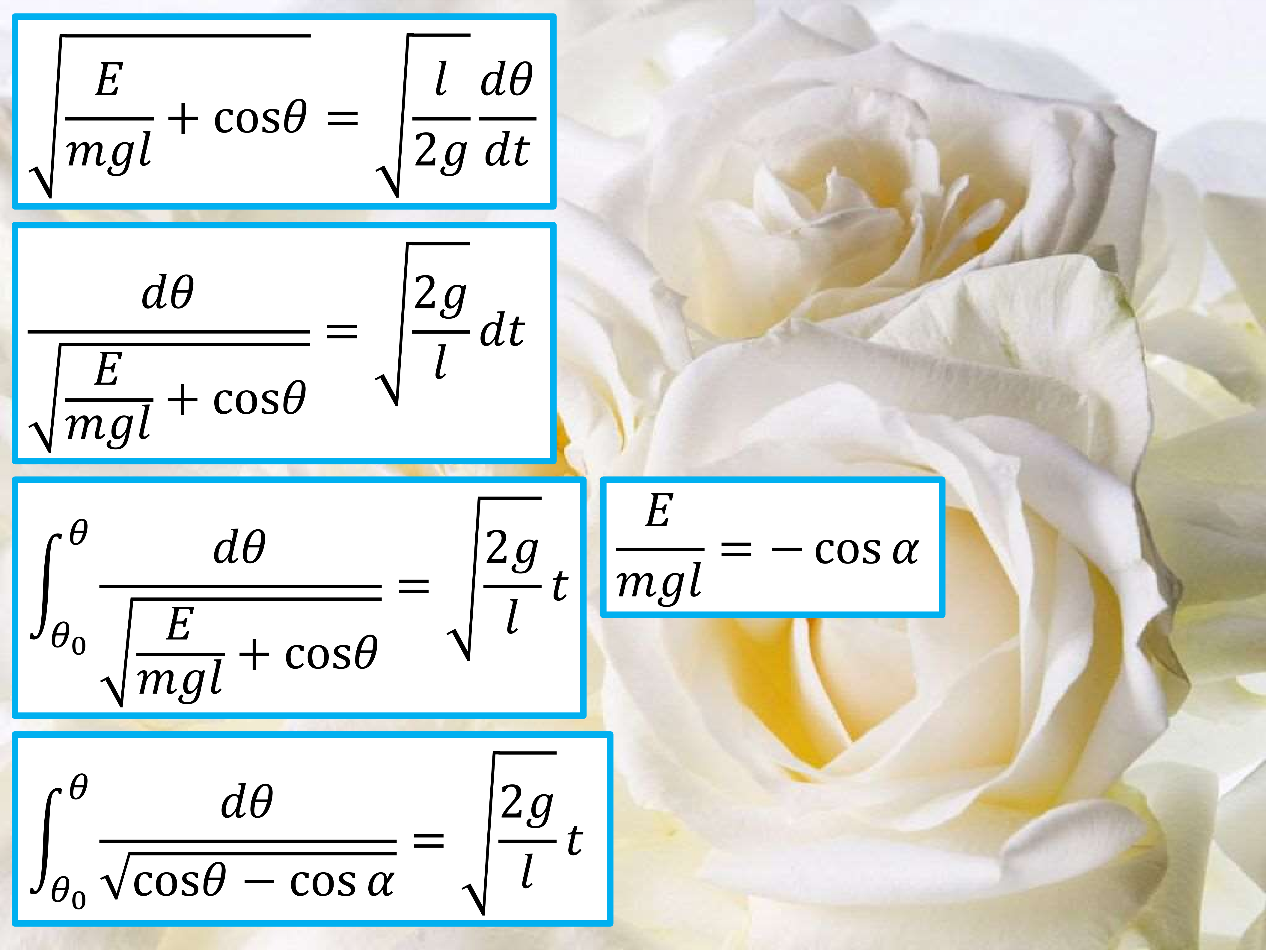
$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

$$E + mgl\cos\theta = \frac{1}{2}ml^2\dot{\theta}^2$$

$$mgl\left(\frac{E}{mgl} + \cos\theta\right) = \frac{1}{2}ml^2\dot{\theta}^2$$

$$\frac{E}{mgl} + \cos\theta = \frac{1}{2}\frac{l}{g}\dot{\theta}^2$$

$$\sqrt{\frac{E}{mgl} + \cos\theta} = \sqrt{\frac{l}{2g}}\,\dot{\theta}$$



$$\sqrt{\frac{E}{mgl} + \cos\theta} = \sqrt{\frac{l}{2g}}\frac{d\theta}{dt}$$

$$\frac{d\theta}{\sqrt{\frac{E}{mgl} + \cos\theta}} = \sqrt{\frac{2g}{l}}dt$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{E}{mgl} + \cos\theta}} = \sqrt{\frac{2g}{l}}t$$

$$\frac{E}{mgl} = -\cos\alpha$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}t$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{l}}\,t$$

$$\cos\theta - \cos\alpha = \left(1 - 2\sin^2\frac{\theta}{2}\right) - \left(1 - 2\sin^2\frac{\alpha}{2}\right)$$

$$\cos\theta - \cos\alpha = 2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)}} = \sqrt{\frac{2g}{l}}\,t$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} = 2\sqrt{\frac{g}{l}}\,t$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sin\frac{\alpha}{2}\sqrt{1-\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}\right)^2}} = 2\sqrt{\frac{g}{l}}\,t$$

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}} = \frac{1}{a}\sin\frac{\theta}{2}$$

$$a = \sin\frac{\alpha}{2}$$

$$\cos\varphi\, d\varphi = \frac{1}{a}\frac{1}{2}\cos\frac{\theta}{2}\, d\theta$$

$$d\theta = \frac{2a\cos\varphi}{\cos\frac{\theta}{2}}\, d\varphi$$

$$\int_{\theta_0}^{\theta} \frac{\frac{2a\cos\varphi}{\cos\frac{\theta}{2}}\, d\varphi}{a\sqrt{1-\sin^2\varphi}} = 2\sqrt{\frac{g}{l}}\,t$$

$$\int_{\theta_0}^{\theta} \frac{\cos\varphi\, d\varphi}{\cos\frac{\theta}{2}\sqrt{\cos^2\varphi}} = \sqrt{\frac{g}{l}}\,t$$

$$\int_{\theta_0}^{\theta} \frac{\cos\varphi\, d\varphi}{\sqrt{1-\sin^2\frac{\theta}{2}\cos\varphi}} = \sqrt{\frac{g}{l}}\,t$$

$$\int_{\theta_0}^{\theta} \frac{d\varphi}{\sqrt{1-a^2\sin^2\varphi}} = \sqrt{\frac{g}{l}}\,t$$

انتگرال بیضوی

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - a^2 \sin^2 \varphi}} = \sqrt{\frac{g}{l}}\, t$$

$$\int_0^{\varphi} \left(1 - a^2 \sin^2 \varphi\right)^{-\frac{1}{2}} d\varphi = \sqrt{\frac{g}{l}}\, t$$

$$\int_0^{\varphi} \left(1 - a^2 \sin^2 \varphi\right)^{-\frac{1}{2}} d\varphi = \sqrt{\frac{g}{l}}\, t$$

$$\int_0^{\varphi} \left(1 + \frac{1}{2} a^2 \sin^2 \varphi + \cdots\right) d\varphi = \sqrt{\frac{g}{l}}\, t$$

$$\int_0^{\varphi} d\varphi + \frac{1}{2} a^2 \int_0^{\varphi} \sin^2 \varphi \, d\varphi + \cdots = \sqrt{\frac{g}{l}}\, t$$

$$\varphi + \frac{1}{2} a^2 \int_0^{\varphi} \frac{1 - \cos 2\varphi}{2} d\varphi + \cdots = \sqrt{\frac{g}{l}}\, t$$

$$\varphi + \frac{1}{2}a^2\frac{1}{2}\int_0^{\varphi}(1 - \cos 2\varphi)\,d\varphi + \cdots = \sqrt{\frac{g}{l}}\,t$$

$$\varphi + \frac{1}{4}a^2\left(\varphi - \frac{1}{2}\sin 2\varphi\right) + \cdots = \sqrt{\frac{g}{l}}\,t$$

$$\varphi + \frac{1}{8}a^2(2\varphi - \sin 2\varphi) + \cdots = \sqrt{\frac{g}{l}}\,t$$

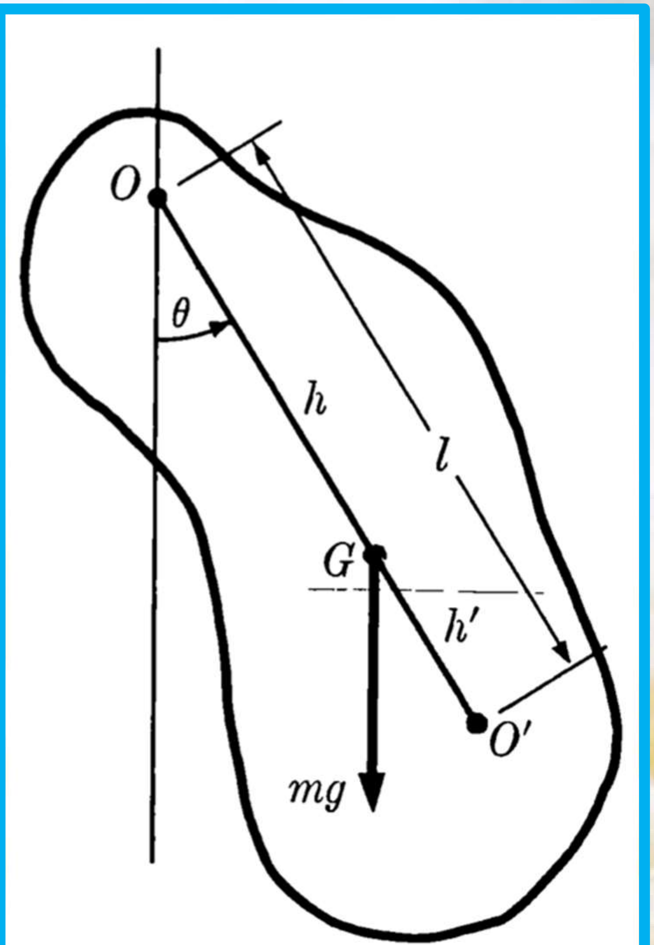
$$\varphi = 2\pi \rightarrow t = \tau$$

$$2\pi + \frac{1}{8}a^2(4\pi - 0) + \cdots = \sqrt{\frac{g}{l}}\,\tau$$

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{a^2}{4} + \cdots\right)$$

که بزرگتر از زمان تناوب آونگ ساده
با فرض کوچک بودن زاویه است

آونگ مرکب



$$l = h + h'$$

$$hl = h^2 + hh'$$

$$k_O^2 = hl$$

$$k_O^2 = h^2 + hh'$$

$$hh' = k_O^2 - h^2$$

$$k_O^2 = h^2 + hh'$$

$$I_z = I_O = mk_O^2 = mlh$$

$$I_O^2 = I_G^2 + mh^2$$

$$\frac{I_O^2}{m} = \frac{I_G^2}{m} + h^2$$

$$k_O^2 = k_G^2 + h^2$$

$$k_O^2 = hl$$

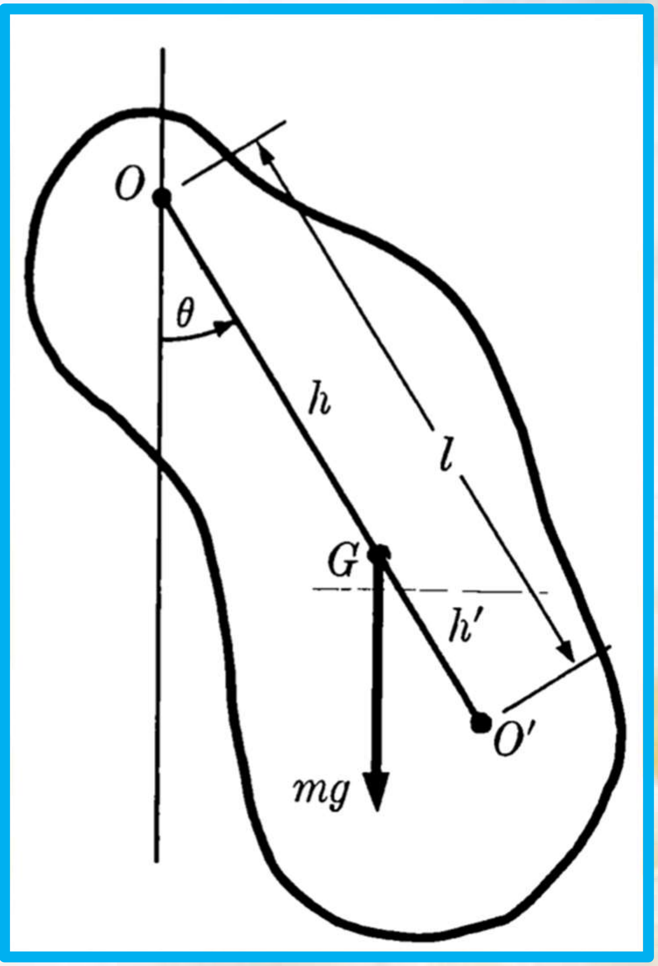
$$hl = k_G^2 + h^2$$

$$l = \frac{k_G^2}{h} + h$$

$$l = h + h'$$

$$\frac{k_G^2}{h} = h'$$

$$k_G^2 = hh'$$



$$L_z = I_z\dot{\theta}$$

$$I_z = mk_O^2 = mhl$$

$$L_z = mk_O^2\dot{\theta} = mhl\dot{\theta}$$

$$\frac{dL_z}{dt} = N_z$$

$$\frac{dL_z}{dt} = mk_O^2\frac{d\dot{\theta}}{dt} = mk_O^2\ddot{\theta} = mhl\frac{d\dot{\theta}}{dt} = mhl\ddot{\theta}$$

$$N_z = -mgh\sin\theta$$

$$mk_O^2\ddot{\theta} = -mgh\sin\theta$$

$$\ddot{\theta} = -\frac{gh}{k_O^2}\sin\theta$$

معادله دیفرانسیل حاکم بر آونگ مرکب

$$mlh\ddot{\theta} = -mgh\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

که مانند آونگ ساده قابل حل است

اگر از نقطه O' آونگ مرکب را آویزان کنیم به طور مشابه به دلیل تقارن خواهیم داشت

$$L_z = I_z\dot{\theta}$$

$$I_z = mk_{O'}^2 = mh'l$$

$$L_z = mk_{O'}^2\dot{\theta} = mh'l\dot{\theta}$$

$$\frac{dL_z}{dt} = N_z$$

$$\frac{dL_z}{dt} = mk_{O'}^2\frac{d\dot{\theta}}{dt} = mk_{O'}^2\ddot{\theta} = mh'l\frac{d\dot{\theta}}{dt} = mh'l\ddot{\theta}$$

$$N_z = -mgh'\sin\theta$$

$$mk_{O'}^2\ddot{\theta} = -mgh'\sin\theta$$

$$\ddot{\theta} = -\frac{gh'}{k_{O'}^2}\sin\theta$$

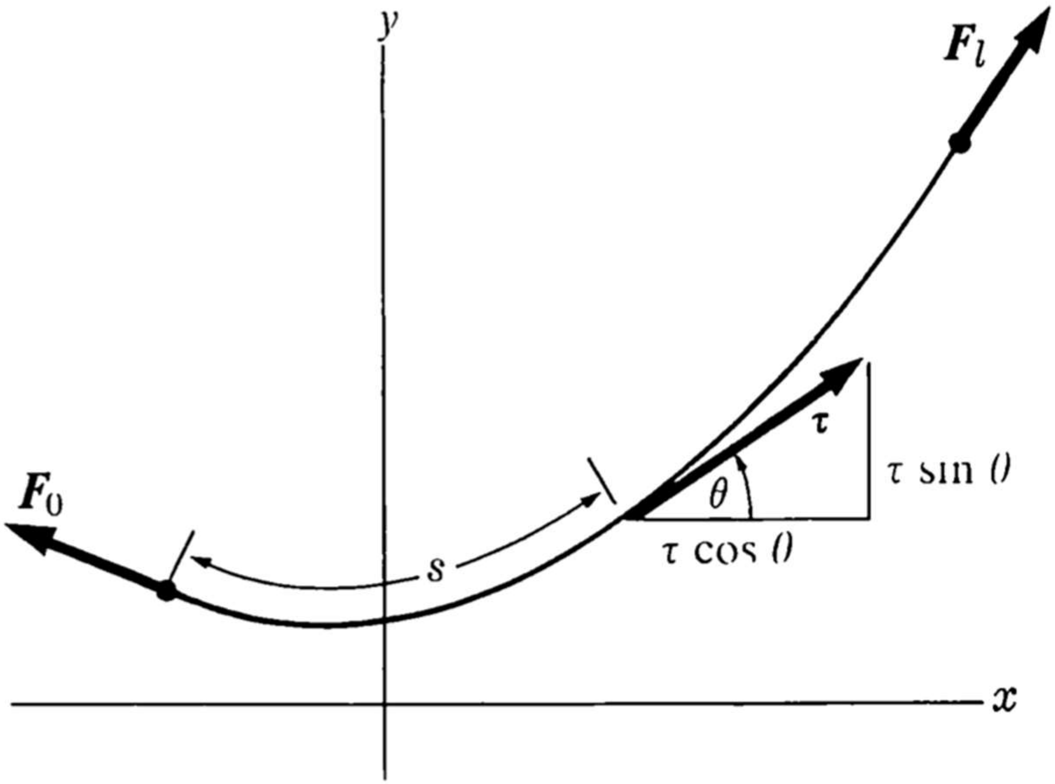
معادله دیفرانسیل حاکم بر آونگ مرکب

$$mlh'\ddot{\theta} = -mgh'\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

که به دلیل تقارن با نتیجه آویزان کردن از نقطه O یکی است و باز هم مانند آونگ ساده قابل حل است

معادله یک ریسمان یا کابل انعاف پذیر آویزان متصل به دو نقطه را به دست آورید



$$F_0 + \int_0^s f\,ds + \tau(s) = 0$$

$$\frac{d\tau(s)}{ds} = -f$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = dx\sqrt{1 + y'^2}$$

$$\frac{d(y')}{dx\sqrt{1 + y'^2}} = \frac{w}{C}$$

$$\frac{d(\tau \sin \theta)}{ds} = w$$

$$\frac{d(\tau \operatorname{co} \theta)}{ds} = 0$$

$$\tau = \frac{C}{\cos \theta}$$

$$\tau \operatorname{co} \theta = C$$

$$\frac{d\left(\frac{C}{\cos \theta}\sin \theta\right)}{ds} = w$$

$$\frac{d(\tan \theta)}{ds} = \frac{w}{C}$$

$$\tan \theta = \frac{dy}{dx} = y'$$

$$\frac{dy'}{\sqrt{1+y'^2}}=\frac{w}{C}dx$$

$$\int\frac{dy'}{\sqrt{1+y'^2}}=\frac{w}{C}\int dx$$

$$y'=\sinh u$$

$$dy'=\cosh u\,du$$

$$\int\frac{\cosh u\,du}{\sqrt{1+\sinh^2 u}}=\frac{w}{C}x$$

$$\int\frac{\cosh u\,du}{\cosh u}=\frac{w}{C}x+\alpha$$

$$\int du=\frac{w}{C}x+\alpha$$

$$u=\frac{w}{C}x+\alpha$$

$$\sinh^{-1}y'=\frac{w}{C}x+\alpha$$

$$y'=\sinh\left(\frac{w}{C}x+\alpha\right)$$

$$\frac{dy}{dx}=\sinh\left(\frac{w}{C}x+\alpha\right)$$

$$dy=\sinh\left(\frac{w}{C}x+\alpha\right)dx$$

$$\int dy=\int\sinh\left(\frac{w}{C}x+\alpha\right)dx$$

$$y=\beta+\frac{\mathrm{C}}{\mathrm{w}}\sinh\left(\frac{w}{c}x+\alpha\right)$$

معادله کابل انعاف پذیر آویزان

طول ریسمان یا کابل انعاف پذیر آویزان متصل به دو نقطه را به دست آورید

$$ds = dx\sqrt{1 + y'^2}$$

$$y' = \sinh\left(\frac{W}{C}x + \alpha\right)$$

$$l = \int_{x_0}^{x_l} ds = \int_{x_0}^{x_l} \sqrt{1 + y'^2}dx$$

$$l = \int_{x_0}^{x_l} \sqrt{1 + \sinh^2\left(\frac{W}{C}x + \alpha\right)}\,dx$$

$$l = \int_{x_0}^{x_l} \sqrt{\cosh^2\left(\frac{W}{C}x + \alpha\right)}\,dx$$

$$l = \int_{x_0}^{x_l} \cosh\left(\frac{W}{C}x + \alpha\right)dx$$

$$l = \left[\frac{C}{W}\sinh\left(\frac{W}{C}x + \alpha\right)\right]_{x_0}^{x_l}$$

$$l = \frac{C}{W}\left[\sinh\left(\frac{W}{C}x_l + \alpha\right) - \sinh\left(\frac{W}{C}x_0 + \alpha\right)\right]$$