تانسور لختی دورانی

اندازه حرکت زاویه ای تک ذره

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}$$

اندازه حرکت زاویه ای مجموعه ای از ذرات

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k \boldsymbol{r_k} \times \boldsymbol{v_k}$$

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k \boldsymbol{r_k} \times (\boldsymbol{\omega} \times \boldsymbol{r_k})$$

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k \big(\boldsymbol{\omega}(\boldsymbol{r_k}.\boldsymbol{r_k}) - \boldsymbol{r_k}(\boldsymbol{r_k}.\boldsymbol{\omega})\big)$$

$$\boldsymbol{r_k r_k} = \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix} \begin{pmatrix} x_k & y_k & z_k \end{pmatrix}$$

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k \big(\boldsymbol{\omega} r_k^2 - \boldsymbol{r_k r_k}.\boldsymbol{\omega}\big)$$

$$\boldsymbol{r_k r_k} = \begin{pmatrix} x_k^2 & x_k y_k & x_k z_k \\ y_k x_k & y_k^2 & y_k z_k \\ z_k x_k & z_k y_k & z_k^2 \end{pmatrix}$$

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k\left(\boldsymbol{\omega} r_k^2 - \boldsymbol{r_k}\boldsymbol{r_k}.\boldsymbol{\omega}\right)$$

$$\boldsymbol{L} = \left(\sum_{k=1}^{N} m_k r_k^2\right)\boldsymbol{\omega} - \left(\sum_{k=1}^{N} m_k \boldsymbol{r_k}\boldsymbol{r_k}\right).\boldsymbol{\omega}$$

$$\boldsymbol{L} = \left[\left(\sum_{k=1}^{N} m_k r_k^2 \mathbf{1}_{3\times3}\right) - \left(\sum_{k=1}^{N} m_k \boldsymbol{r_k}\boldsymbol{r_k}\right)\right].\boldsymbol{\omega}$$

$$\mathbf{1}_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{L} = \left(\sum_{k=1}^{N} m_k (r_k^2 \mathbf{1}_{3\times3} - \boldsymbol{r_k}\boldsymbol{r_k})\right).\boldsymbol{\omega}$$

$$\boldsymbol{r_k}\boldsymbol{r_k} = \begin{pmatrix} x_k^2 & x_k y_k & x_k z_k \\ y_k x_k & y_k^2 & y_k z_k \\ z_k x_k & z_k y_k & z_k^2 \end{pmatrix}$$

$$\boldsymbol{L} = \boldsymbol{I}.\boldsymbol{\omega}$$

$$\boldsymbol{I} = \sum_{k=1}^{N} m_k (r_k^2\, \mathbf{1}_{3\times3} - \boldsymbol{r_k}\boldsymbol{r_k})$$

تانسور لختی دورانی مجموعه ای از ذرات گسسته

مولفه های تانسور لختی دورانی مجموعه ای از ذرات گسسته

$$\boldsymbol{I} = \sum_{k=1}^{N} m_k (r_k^2\, \mathbf{1}_{3\times 3} - \boldsymbol{r}_k \boldsymbol{r}_k)$$

$$I_{ij} = \sum_{k=1}^{N} m_k \left(r_k^2 \delta_{i,j} - x_{i,k} x_{j,k}\right)$$

$$I_{ij} = \int dm \left(r^2 \delta_{i,j} - x_i x_j\right)$$

$$I_{11} = I_{xx} = \int dm \left((x^2 + y^2 + z^2)\delta_{1,1} - xx\right) = \int dm \left(y^2 + z^2\right)$$

$$I_{12} = I_{xy} = \int dm \left((x^2 + y^2 + z^2)\delta_{1,2} - xy\right) = \int dm \left(-xy\right)$$

$$I_{13} = I_{xz} = \int dm \left((x^2 + y^2 + z^2)\delta_{1,3} - xz\right) = \int dm \left(-xz\right)$$

$$I_{21} = I_{yx} = \int dm\left((x^2 + y^2 + z^2)\delta_{2,1} - yx\right) = \int dm\,(-yx) = I_{12}$$

$$I_{22} = I_{yy} = \int dm\left((x^2 + y^2 + z^2)\delta_{2,2} - yy\right) = \int dm\,(x^2 + z^2)$$

$$I_{23} = I_{yz} = \int dm\left((x^2 + y^2 + z^2)\delta_{2,3} - yz\right) = \int dm\,(-yz) = I_{32}$$

$$I_{31} = I_{zx} = \int dm\left((x^2 + y^2 + z^2)\delta_{3,1} - zx\right) = \int dm\,(-zx) = I_{13}$$

$$I_{32} = I_{zy} = \int dm\left((x^2 + y^2 + z^2)\delta_{3,2} - zy\right) = \int dm\,(-zy) = I_{23}$$

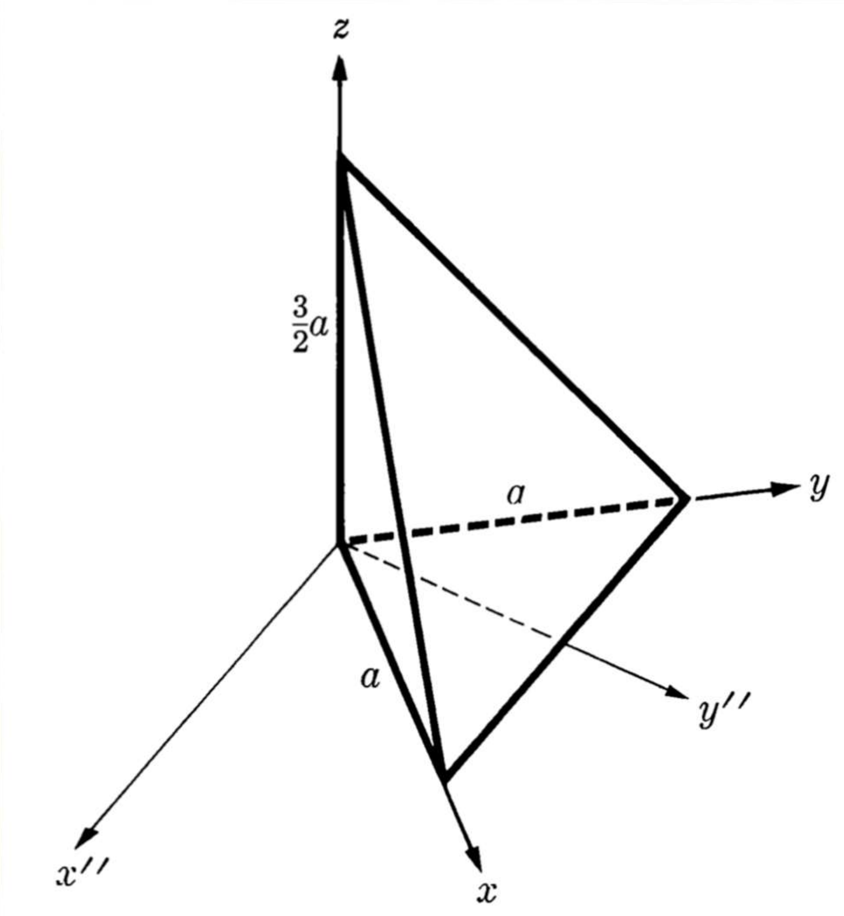
$$I_{33} = I_{zz} = \int dm\left((x^2 + y^2 + z^2)\delta_{3,3} - zz\right) = \int dm\,(x^2 + y^2)$$

$$\boldsymbol{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}$$

$$\boldsymbol{I} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int xzdm \\ -\int yxdm & \int (x^2 + z^2) dm & -\int yzdm \\ -\int zxdm & -\int zydm & \int (x^2 + y^2) dm \end{pmatrix}$$

$$\boldsymbol{I} = \int \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix} dm$$

مثال: تانسور گشتاور لختی دورانی شکل مقابل را در دستگاه x,y,z به دست آورید



$$\boldsymbol{I} = \int \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} dm$$

$$\boldsymbol{I} = \int_{z=0}^{\frac{3}{2}a} \int_{y=0}^{a-\frac{2}{3}z} \int_{x=0}^{a-y-\frac{2}{3}z} \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} \rho dV$$

$$\boldsymbol{I} = \begin{pmatrix} J_1+J_2 & -J_3 & -J_4 \\ -J_3 & J_1+J_2 & -J_4 \\ -J_4 & -J_4 & 2J_1 \end{pmatrix}$$

$$J_1 = \int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z} x^2\rho dV = \int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z} y^2\rho dV$$

$$J_1 = \rho\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z} x^2 dxdydz = \rho\frac{1}{3}\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\left[\left(a-y-\frac{2}{3}z\right)^3\right]dydz$$

$$= \rho\frac{1}{3}\left(-\frac{1}{4}\right)\int_{z=0}^{\frac{3}{2}a}\left[\left(a-\left(a-\frac{2}{3}z\right)-\frac{2}{3}z\right)^4-\left(a-\frac{2}{3}z\right)^4\right]dz$$

$$= \rho\frac{1}{3}\left(\frac{1}{4}\right)\int_{z=0}^{\frac{3}{2}a}\left(a-\frac{2}{3}z\right)^4 dz = \rho\frac{1}{3}\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\left(-\frac{3}{2}\right)\left[\left(a-\frac{2}{3}\frac{3}{2}a\right)^5-a^5\right] = \rho\frac{1}{40}a^5$$

$$M = \rho\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z} dxdydz = \rho\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\left(a-y-\frac{2}{3}z\right)dydz$$

$$= \rho\left(-\frac{1}{2}\right)\int_{z=0}^{\frac{3}{2}a}\left[\left(a-\left(a-\frac{2}{3}z\right)-\frac{2}{3}z\right)^2-\left(a-\frac{2}{3}z\right)^2\right]dz$$

$$= \rho\left(\frac{1}{2}\right)\int_{z=0}^{\frac{3}{2}a}\left(a-\frac{2}{3}z\right)^2 dz = \rho\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(-\frac{3}{2}\right)\left[\left(a-\frac{2}{3}\frac{3}{2}a\right)^3-a^3\right] = \rho\frac{1}{4}a^3$$

$$J_1 = \rho\frac{1}{40}a^5 = \rho\frac{1}{4}a^3\frac{1}{10}a^2 = \frac{1}{10}Ma^2$$

تمرین: به طور مشابه با انتگرل J_1 سایر انتگرالهای J_2,J_3,J_4 را به صورت زیر به دست آورید

$$J_1=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}x^2\rho dV=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}y^2\rho dV=\frac{1}{10}Ma^2$$

$$J_2=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}z^2\rho dV=\frac{9}{40}Ma^2$$

$$J_3=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}xy\rho dV=\frac{1}{20}Ma^2$$

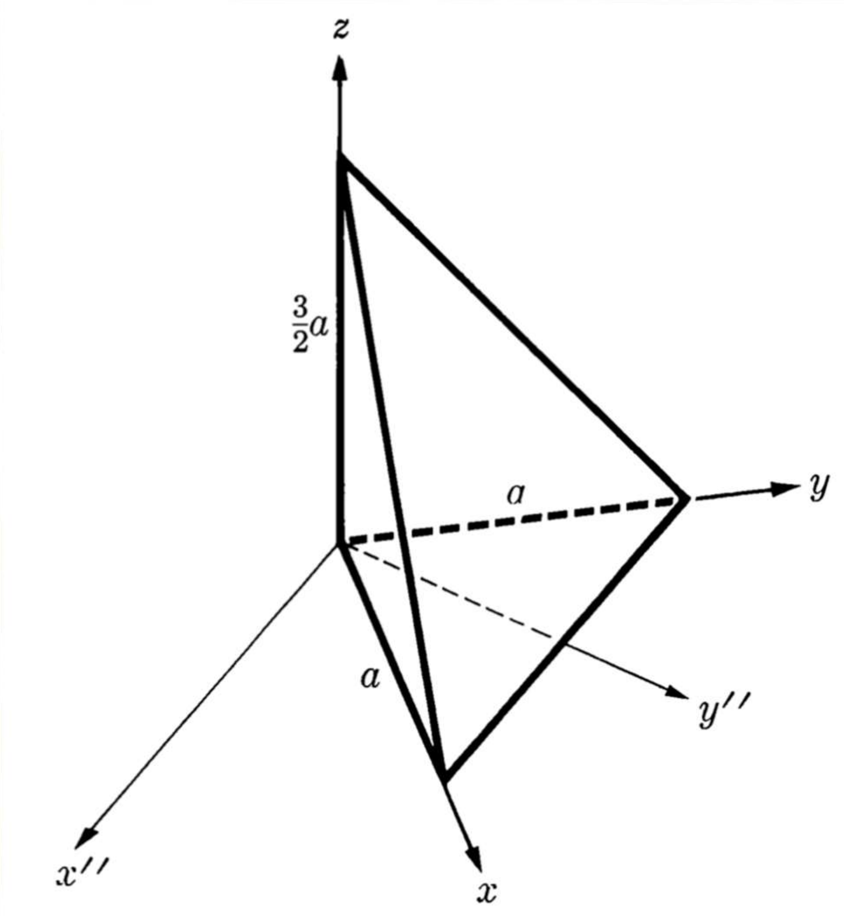
$$J_4=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}xz\rho dV=\int_{z=0}^{\frac{3}{2}a}\int_{y=0}^{a-\frac{2}{3}z}\int_{x=0}^{a-y-\frac{2}{3}z}yz\rho dV=\frac{3}{40}Ma^2$$

$$\boldsymbol{I}=\begin{pmatrix} J_1+J_2 & -J_3 & -J_4 \\ -J_3 & J_1+J_2 & -J_4 \\ -J_4 & -J_4 & 2J_1 \end{pmatrix}=\begin{pmatrix} 13 & -2 & -3 \\ -2 & 13 & -3 \\ -3 & -3 & 8 \end{pmatrix}\frac{Ma^2}{40}$$

مثال: تانسور گشتاور لختی دورانی شکل مقابل را در دستگاه x'',y'',z'' به دست آورید

برای این منظور نیاز به یک ماتریس تبدیل داریم

این ماتریس تبدیل را با استفاده از پایه های دو فضا به صورت زیر می سازیم



$$R=\begin{pmatrix}\hat{x}.\hat{x}'' & \hat{x}.\hat{y}'' & \hat{x}.\hat{z}''\\ \hat{y}.\hat{x}'' & \hat{y}.\hat{y}'' & \hat{y}.\hat{z}''\\ \hat{z}.\hat{x}'' & \hat{z}.\hat{y}'' & \hat{z}.\hat{z}''\end{pmatrix}$$

$$\hat{x}=(1,0,0)$$

$$\hat{x}''=(\cos\theta\,,-\sin\theta\,,0)$$

$$\hat{y}=(0,1,0)$$

$$\hat{y}''=(\sin\theta\,,\cos\theta\,,0)$$

$$\hat{z}=(0,0,1)$$

$$\hat{z}''=(0,0,1)$$

$$R=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$R=\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$I'' = R^T I R$$

پس از ساخت ماتریس تبدیل به صورت زیر عمل می کنیم

$$I''=\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}13 & -2 & -3\\ -2 & 13 & -3\\ -3 & -3 & 8\end{pmatrix}\frac{Ma^2}{40}\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$I''=\frac{Ma^2}{40}\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}(13+2)\frac{\sqrt{2}}{2} & (13-2)\frac{\sqrt{2}}{2} & -3\\ (-2-13)\frac{\sqrt{2}}{2} & (-2+13)\frac{\sqrt{2}}{2} & -3\\ (-3+3)\frac{\sqrt{2}}{2} & (-3-3)\frac{\sqrt{2}}{2} & 8\end{pmatrix}$$

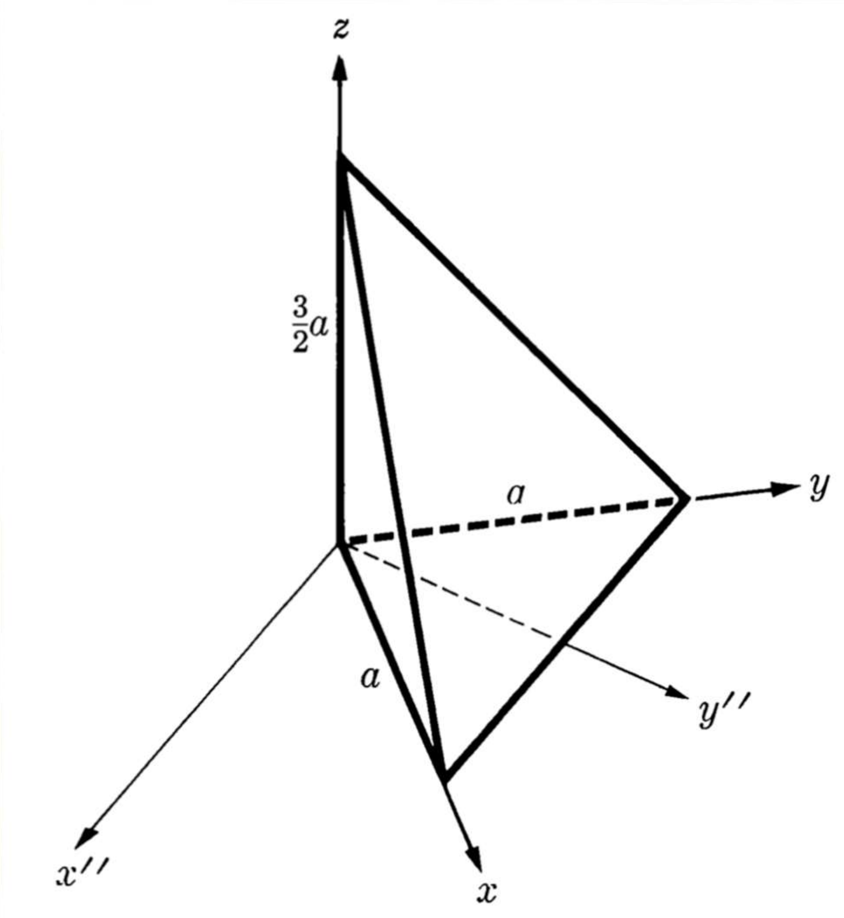
$$I''=\frac{Ma^2}{40}\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}15\times\frac{\sqrt{2}}{2} & 11\times\frac{\sqrt{2}}{2} & -3\\ -15\times\frac{\sqrt{2}}{2} & 11\times\frac{\sqrt{2}}{2} & -3\\ 0\times\frac{\sqrt{2}}{2} & -6\times\frac{\sqrt{2}}{2} & 8\end{pmatrix}$$

$$\boldsymbol{I}'' = \frac{Ma^2}{40}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 15\times\frac{\sqrt{2}}{2} & 11\times\frac{\sqrt{2}}{2} & -3 \\ -15\times\frac{\sqrt{2}}{2} & 11\times\frac{\sqrt{2}}{2} & -3 \\ 0\times\frac{\sqrt{2}}{2} & -6\times\frac{\sqrt{2}}{2} & 8 \end{pmatrix}$$

$$\boldsymbol{I}'' = \frac{Ma^2}{40}\begin{pmatrix} 15\times\frac{2}{4}+15\times\frac{2}{4} & 11\times\frac{2}{4}-11\times\frac{2}{4} & -3\frac{\sqrt{2}}{2}+3\frac{\sqrt{2}}{2} \\ 15\times\frac{2}{4}-15\times\frac{2}{4} & 11\times\frac{2}{4}+11\times\frac{2}{4} & -3\frac{\sqrt{2}}{2}-3\frac{\sqrt{2}}{2} \\ 1\times 0\times\frac{\sqrt{2}}{2} & 1\times\left(-6\times\frac{\sqrt{2}}{2}\right) & 1\times 8 \end{pmatrix}$$

$$\boldsymbol{I}'' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{pmatrix}$$

مثال: تانسور گشتاور لختی دورانی شکل مقابل را در دستگاه محورهای اصلی x',y',z' بیابید که در آن دستگاه تانسور گشتاور لختی دورانی قطری باشد



$$I'' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{pmatrix}$$

برای یافتن محورهای اصلی دستگاه x',y',z' کافی است که بردارهای ویژه ماتریس فوق را به دست آوریم

برای یافتن بردارهای ویژه این ماتریس ابتدا مقادیر ویژه آن را به دست آوریم

$$\begin{vmatrix} 15-\lambda & 0 & 0 \\ 0 & 11-\lambda & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8-\lambda \end{vmatrix} = 0$$

$$(15-\lambda)\left((11-\lambda)(8-\lambda) - \left(3\sqrt{2}\times 3\sqrt{2}\right)\right) = 0$$

$$(15-\lambda)\left(88 - 11\lambda - 8\lambda + \lambda^2 - 18\right) = 0$$

$$(15-\lambda)\left(\lambda^2 - 19\lambda + 70\right) = 0$$

$$\lambda_1 = 15$$

$$\lambda_{2,3} = \frac{19 \pm \sqrt{19^2 - 4\times 70}}{2}$$

$$\lambda_{2,3} = \frac{19 \pm \sqrt{361 - 280}}{2} = \frac{19 \pm \sqrt{81}}{2} = \frac{19 \pm 9}{2}$$

$$\lambda_2 = \frac{19-9}{2} = \frac{10}{2} = 5$$

$$\lambda_3 = \frac{19+9}{2} = \frac{28}{2} = 14$$

پس مقادیر ویژه ماتریس زیر عبارتند از:

$$I'' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{pmatrix}$$

$$\lambda_{x'} = \frac{Ma^2}{40}\lambda_1 = \frac{Ma^2}{40} \times 15 = \frac{3}{8}Ma^2$$

$$\lambda_{y'} = \frac{Ma^2}{40}\lambda_2 = \frac{Ma^2}{40} \times 5 = \frac{1}{8}Ma^2$$

$$\lambda_{z'} = \frac{Ma^2}{40}\lambda_3 = \frac{Ma^2}{40} \times 14 = \frac{7}{20}Ma^2$$

حال بردارهای ویژه متناظر با مقادیر ویژه فوق را به دست می آوریم

$$\frac{Ma^2}{40}\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \lambda_{x'}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \frac{3}{8}Ma^2\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = 15\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{cases} 15u = 15u \\ 11v - 3\sqrt{2}w = 15v \\ -3\sqrt{2}v + 8w = 15w \end{cases}$$

$$\begin{cases} u = u \\ -\dfrac{3\sqrt{2}}{4}w = v \\ v = -\dfrac{7\sqrt{2}}{6}w \end{cases}$$

$$v = -\frac{3\sqrt{2}}{4}w = -\frac{7\sqrt{2}}{6}w \Rightarrow \left(\frac{7}{3} - \frac{3}{2}\right)\frac{\sqrt{2}}{2}w = \frac{14-9}{6}\frac{\sqrt{2}}{2}w$$
$$= 0 \Rightarrow \frac{5}{6}\frac{\sqrt{2}}{2}w = 0 \Rightarrow w = 0$$

$$v = -\frac{3\sqrt{2}}{4}w = -\frac{3\sqrt{2}}{4} \times 0 = 0$$

$$\hat{\boldsymbol{x}}' = \begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}u\\ 0\\ 0\end{pmatrix} = u\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$$u^2(1 \quad 0 \quad 0)\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = 1$$
$$\Rightarrow u^2(1+0+0) = u^2$$
$$= 1 \Rightarrow u = 1$$

$$\hat{\boldsymbol{x}}' = u\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$$\frac{Ma^2}{40}\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \lambda_{y'}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \frac{1}{8}Ma^2\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = 5\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{cases}15u = 5u\\ 11v - 3\sqrt{2}w = 5v\\ -3\sqrt{2}v + 8w = 5w\end{cases}$$

$$\begin{cases}3u = u\\ \dfrac{3\sqrt{2}}{6}w = v\\ v = \dfrac{3}{3\sqrt{2}}w\end{cases}$$

$$\begin{cases}3u - u = 0 \Rightarrow 2u = 0 \Rightarrow u = 0\\ \dfrac{\sqrt{2}}{2}w = v\\ v = \dfrac{1}{\sqrt{2}}w = \dfrac{\sqrt{2}}{2}w\end{cases}$$

$$\hat{\mathbf{y}}' = \begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}0\\ \frac{\sqrt{2}}{2}w\\ w\end{pmatrix} = w\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\ 1\end{pmatrix}$$

$$w^2\begin{pmatrix}0 & \frac{\sqrt{2}}{2} & 1\end{pmatrix}\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\ 1\end{pmatrix} = 1 \Rightarrow w^2\left(0 + \frac{1}{2} + 1\right) = w^2\left(\frac{3}{2}\right) = 1$$

$$\Rightarrow w^2 = \frac{2}{3} \Rightarrow w = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

$$\hat{\mathbf{y}}' = w\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\ 1\end{pmatrix} = \frac{\sqrt{6}}{3}\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\ 1\end{pmatrix} = \begin{pmatrix}0\\ \frac{\sqrt{12}}{6}\\ \frac{\sqrt{6}}{3}\end{pmatrix} = \begin{pmatrix}0\\ \frac{2\sqrt{3}}{6}\\ \frac{\sqrt{6}}{3}\end{pmatrix} = \begin{pmatrix}0\\ \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3}\end{pmatrix}$$

$$\frac{Ma^2}{40}\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \lambda_{z'}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \frac{7}{20}Ma^2\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{pmatrix}15 & 0 & 0\\ 0 & 11 & -3\sqrt{2}\\ 0 & -3\sqrt{2} & 8\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = 14\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$\begin{cases} 15u = 14u \\ 11v - 3\sqrt{2}w = 14v \\ -3\sqrt{2}v + 8w = 14w \end{cases}$$

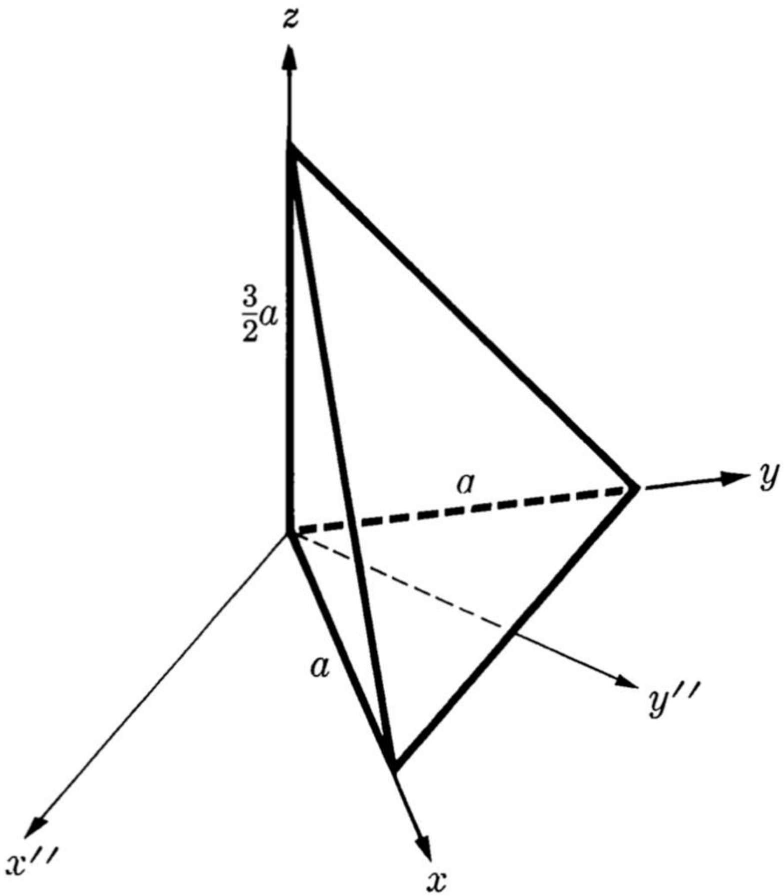
$$\begin{cases} (15-14)u = 0 \Rightarrow u = 0 \\ -3\sqrt{2}w = 3v \Rightarrow -\sqrt{2}w = v \\ 6w = -3\sqrt{2}v \Rightarrow -\dfrac{6}{3\sqrt{2}}w = v \Rightarrow -\dfrac{6\sqrt{2}}{3\times 2}w = v \Rightarrow -\sqrt{2}w = v \end{cases}$$

$$\hat{\mathbf{z}}' = \begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}0\\ -\sqrt{2}w\\ w\end{pmatrix} = w\begin{pmatrix}0\\ -\sqrt{2}\\ 1\end{pmatrix}$$

$$w^2\begin{pmatrix}0 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\ -\sqrt{2}\\ 1\end{pmatrix} = 1$$

$$\Rightarrow w^2(0+2+1) = 3w^2 = 1 \Rightarrow w = \frac{\sqrt{3}}{3}$$

$$\hat{\mathbf{z}}' = w\begin{pmatrix}0\\ -\sqrt{2}\\ 1\end{pmatrix} = \frac{\sqrt{3}}{3}\begin{pmatrix}0\\ -\sqrt{2}\\ 1\end{pmatrix} = \begin{pmatrix}0\\ -\dfrac{\sqrt{6}}{3}\\ \dfrac{\sqrt{3}}{3}\end{pmatrix}$$



$$\hat{\boldsymbol{x}}' = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\hat{\boldsymbol{y}}' = \begin{pmatrix}0\\ \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3}\end{pmatrix}$$

$$\hat{\boldsymbol{z}}' = \begin{pmatrix}0\\ -\frac{\sqrt{6}}{3}\\ \frac{\sqrt{3}}{3}\end{pmatrix}$$

$$\hat{\boldsymbol{x}}' = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\hat{\boldsymbol{y}}' = \frac{1}{3}\begin{pmatrix}0\\ \sqrt{3}\\ \sqrt{6}\end{pmatrix}$$

$$\hat{\boldsymbol{z}}' = \frac{1}{3}\begin{pmatrix}0\\ -\sqrt{6}\\ \sqrt{3}\end{pmatrix}$$

$$\hat{\boldsymbol{x}}'' = \frac{1}{2}\left(\sqrt{2}, -\sqrt{2}, 0\right)$$

$$\hat{\boldsymbol{y}}'' = \frac{1}{2}\left(-\sqrt{2}, \sqrt{2}, 0\right)$$

$$\hat{\boldsymbol{z}}'' = (0, 0,1)$$

$$\hat{\boldsymbol{x}} = (1,0,0)$$

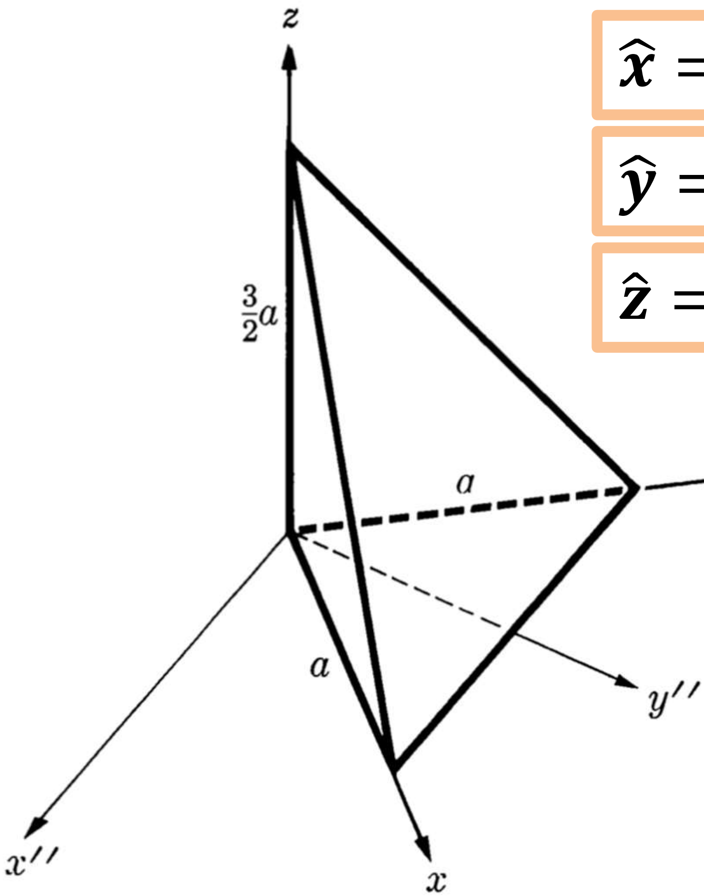
$$\hat{\boldsymbol{y}} = (0,1,0)$$

$$\hat{\boldsymbol{z}} = (0,0,1)$$

$$\hat{\boldsymbol{x}}'' = (\cos\theta\,, -\sin\theta\,, 0)$$

$$\hat{\boldsymbol{y}}'' = (\sin\theta\,, \cos\theta\,, 0)$$

$$\hat{\boldsymbol{z}}'' = (0, 0,1)$$



$$\hat{x} = (1,0,0)$$
$$\hat{y} = (0,1,0)$$
$$\hat{z} = (0,0,1)$$

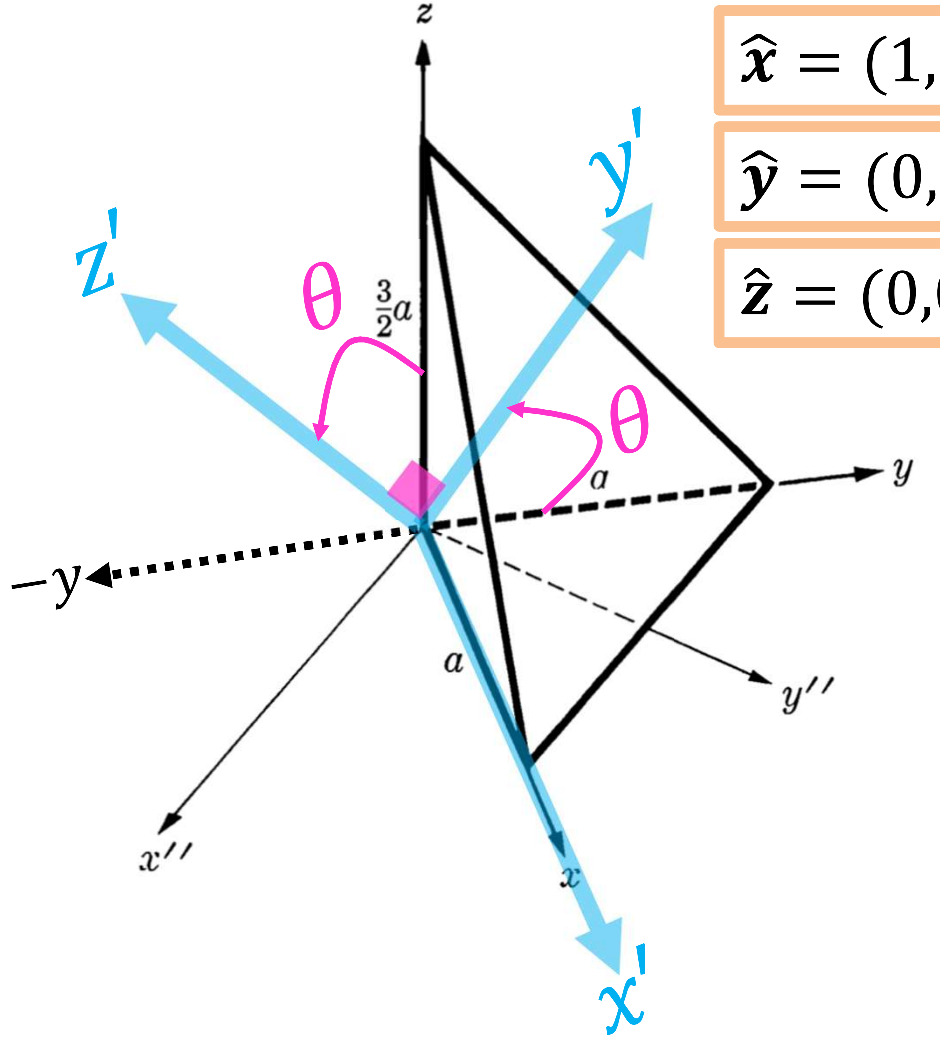
$$\hat{x}'' = \frac{1}{2}\left(\sqrt{2}, -\sqrt{2}, 0\right)$$
$$\hat{y}'' = \frac{1}{2}\left(\sqrt{2}, \sqrt{2}, 0\right)$$
$$\hat{z}'' = (0,0,1)$$

$$\hat{x}' = (1,0,0)$$
$$\hat{y}' = \frac{1}{3}\left(0, \sqrt{3}, \sqrt{6}\right)$$
$$\hat{z}' = \frac{1}{3}\left(0, -\sqrt{6}, \sqrt{3}\right)$$

$$\hat{x}'' = \frac{1}{2}\left(\sqrt{2}, -\sqrt{2}, 0\right) = \frac{\sqrt{2}}{2}\hat{x} - \frac{\sqrt{2}}{2}\hat{y}$$
$$\hat{y}'' = \frac{1}{2}\left(\sqrt{2}, \sqrt{2}, 0\right) = \frac{\sqrt{2}}{2}\hat{x} + \frac{\sqrt{2}}{2}\hat{y}$$
$$\hat{z}'' = \hat{z}$$

$$\hat{x}' = (1,0,0) = \hat{x}$$
$$\hat{y}' = \frac{1}{3}\left(0, \sqrt{3}, \sqrt{6}\right) = \frac{\sqrt{3}}{3}(0,1,0) + \frac{\sqrt{6}}{3}(0,0,1) = \frac{\sqrt{3}}{3}\hat{y} + \frac{\sqrt{6}}{3}\hat{z}$$
$$\hat{z}' = \frac{1}{3}\left(0, -\sqrt{6}, \sqrt{3}\right) = -\frac{\sqrt{6}}{3}(0,1,0) + \frac{\sqrt{3}}{3}(0,0,1) = -\frac{\sqrt{6}}{3}\hat{y} + \frac{\sqrt{3}}{3}\hat{z}$$

محورهای اصلی دستگاه که در آن تانسور گشتاور لختی دستگاه به صورت زیر قابل قطری شدن است



دستگاه محورهای اصلی جسم صلب
در شکل با محورهای آبی کم رنگ
نشان داده شده اند

$$\hat{\boldsymbol{x}} = (1,0,0)$$

$$\hat{\boldsymbol{y}} = (0,1,0)$$

$$\hat{\boldsymbol{z}} = (0,0,1)$$

$$\hat{\boldsymbol{x}}' = (1,0,0) = \hat{\boldsymbol{x}}$$

$$\hat{\boldsymbol{y}}' = \frac{1}{3}\left(0, \sqrt{3}, \sqrt{6}\right) = \frac{\sqrt{3}}{3}\hat{\boldsymbol{y}} + \frac{\sqrt{6}}{3}\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{z}}' = \frac{1}{3}\left(0, -\sqrt{6}, \sqrt{3}\right) = -\frac{\sqrt{6}}{3}\hat{\boldsymbol{y}} + \frac{\sqrt{3}}{3}\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{y}}.\hat{\boldsymbol{y}}' = |\hat{\boldsymbol{y}}'||\hat{\boldsymbol{y}}| \cos\theta$$

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 \\ \sqrt{3} \\ \sqrt{6} \end{pmatrix} = 1 \times 1 \cos\theta$$

$$\frac{1}{3}\left(0 \times 0 + \sqrt{3} + 0 \times \sqrt{6}\right) = \cos\theta$$

$$\frac{\sqrt{3}}{3} = \cos\theta$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{3}$$

$$\theta = 54.736^{\circ}$$

$$\hat{\boldsymbol{x}}' = (1,0,0)$$

$$\hat{\boldsymbol{y}}' = \frac{1}{3}\left(0, \sqrt{3}, \sqrt{6}\right)$$

$$\hat{\boldsymbol{z}}' = \frac{1}{3}\left(0, -\sqrt{6}, \sqrt{3}\right)$$

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}$$

$$\boldsymbol{I}'' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{pmatrix}$$

$$\boldsymbol{I}' = \boldsymbol{R}^T\boldsymbol{I}''\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}\frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}$$

$$\boldsymbol{I}' = \boldsymbol{R}^T\boldsymbol{I}''\boldsymbol{R} = \frac{Ma^2}{40}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11\times\frac{\sqrt{3}}{3} - 2\sqrt{3} & -11\times\frac{\sqrt{6}}{3} - \sqrt{6} \\ 0 & -\sqrt{6} + \frac{8}{3}\times\sqrt{6} & 2\sqrt{3} + \frac{8}{3}\times\sqrt{3} \end{pmatrix}$$

$$\boldsymbol{I}' = \boldsymbol{R}^T\boldsymbol{I}''\boldsymbol{R} = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11\times\frac{1}{3} - 2 - 2 + \frac{16}{3} & -11\times\frac{\sqrt{2}}{3} - \sqrt{2} + 2\sqrt{2} + \frac{8}{3}\sqrt{2} \\ 0 & -11\times\frac{\sqrt{2}}{3} + 2\sqrt{2} - \sqrt{2} + \frac{8}{3}\sqrt{2} & 11\times\frac{2}{3} + 2 + 2 + \frac{8}{3} \end{pmatrix}$$

$$\boldsymbol{I}' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 11\times\frac{1}{3} - 2 - 2 + \frac{16}{3} & -11\times\frac{\sqrt{2}}{3} - \sqrt{2} + 2\sqrt{2} + \frac{8}{3}\sqrt{2} \\ 0 & -11\times\frac{\sqrt{2}}{3} + 2\sqrt{2} - \sqrt{2} + \frac{8}{3}\sqrt{2} & 11\times\frac{2}{3} + 2 + 2 + \frac{8}{3} \end{pmatrix}$$

$$\boldsymbol{I}' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & \frac{11 - 12 + 16}{3} & \frac{(-11+8)\sqrt{2}}{3} + \sqrt{2} \\ 0 & \frac{(-11+8)\sqrt{2}}{3}\sqrt{2} & \frac{22 + 12 + 8}{3} \end{pmatrix}$$

$$= \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & \frac{15}{3} & \frac{-3\sqrt{2}}{3} + \sqrt{2} \\ 0 & \frac{-3\sqrt{2}}{3} + \sqrt{2} & \frac{42}{3} \end{pmatrix}$$

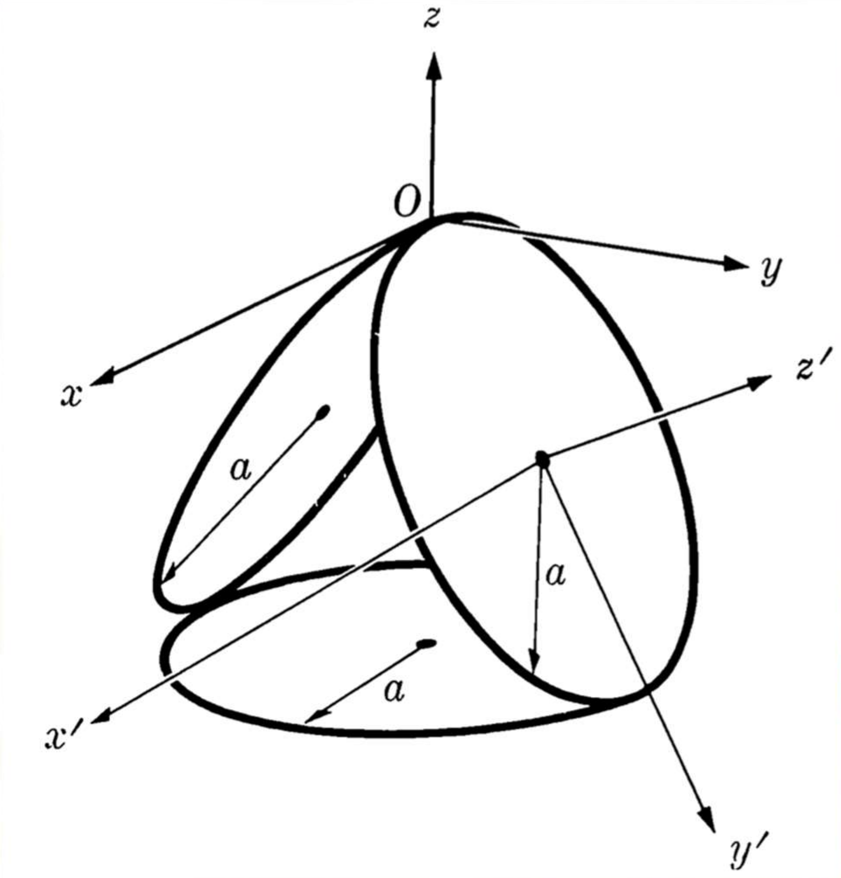
$$\boldsymbol{I}' = \frac{Ma^2}{40}\begin{pmatrix} 15 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

$$\lambda_1 = 15$$

$$\lambda_2 = 5$$

$$\lambda_3 = 14$$

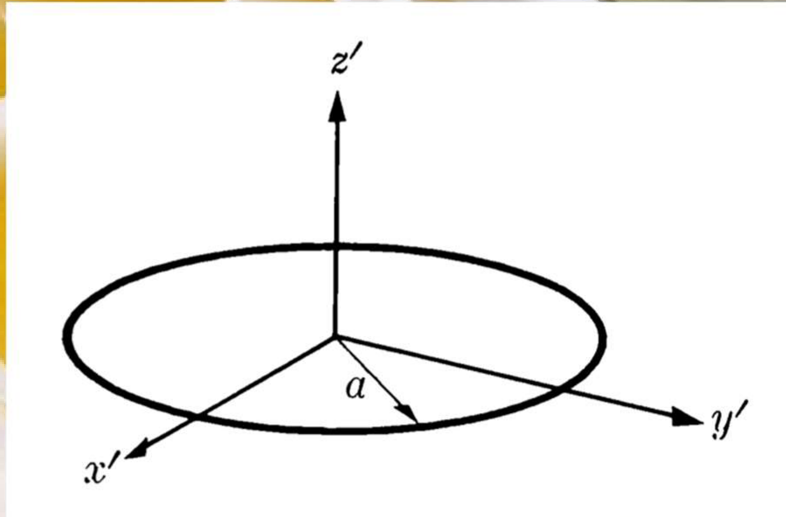
مثال: تانسور گشتاور لختی دورانی دستگاه نشان داده شده در شکل را حول نقطه O به دست آورید



بنابر تقارن محورهای اصلی دستگاه منطق بر x,y,z است

ابتدا تانسور یک دیسک تکی را نسبت به محورهای اصلی خودش یعنی x',y',z' که مبدا آن بر مرکز جرم دیسک منطبق است به دست می آوریم

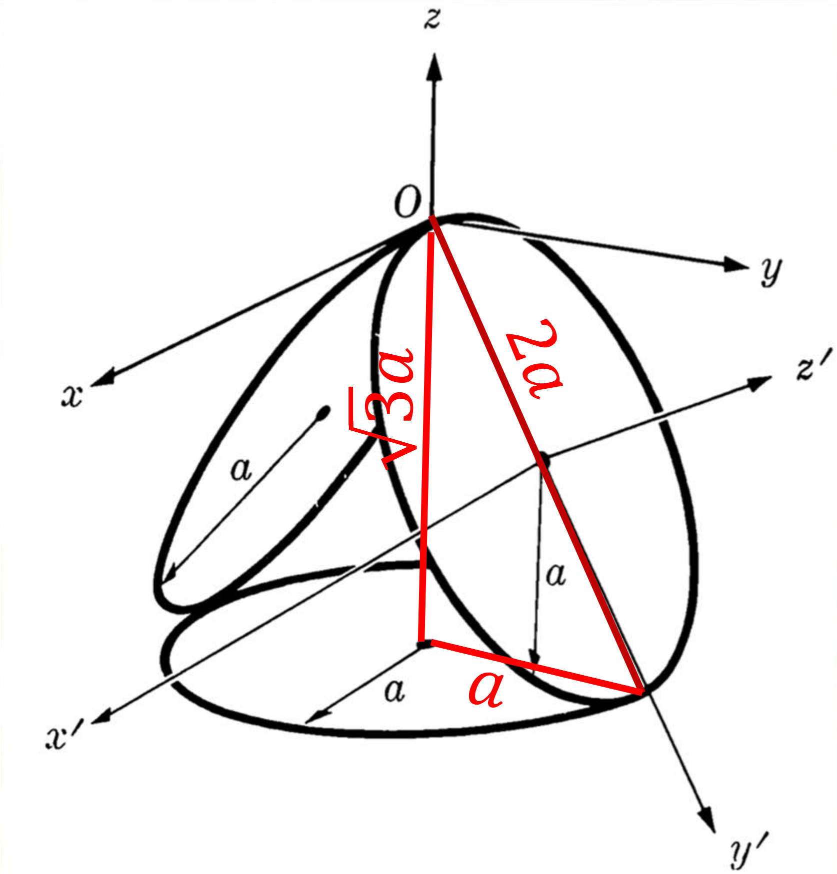
این مساله را قبلا حل کرده ایم



$$\boldsymbol{I_G} = \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

حال برای دیسکی که در کف واقع شده است به صورت زیر عمل می کنیم

$$\boldsymbol{I}_{B,O(xyz)} = \boldsymbol{I_G} + M(R^2\mathbf{1} - \boldsymbol{RR})$$



$$\boldsymbol{I}_{B,O(xyz)} = \boldsymbol{I}_{\boldsymbol{G}} + M(R^2\mathbf{1} - \boldsymbol{R}\boldsymbol{R})$$

$$R^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2$$

$$\boldsymbol{R} = \sqrt{3}a\hat{\mathbf{z}}$$

$$\boldsymbol{R}\boldsymbol{R} = 3a^2\hat{\mathbf{z}}\hat{\mathbf{z}}$$

$$\hat{\mathbf{z}}\hat{\mathbf{z}} = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

$$\boldsymbol{I}_{\boldsymbol{G}} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$$

$$\boldsymbol{I}_{B,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix} + M\left(3a^2\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} - 3a^2\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}\right)$$

$$\boldsymbol{I}_{B,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + M\left(3a^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - 3a^2\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

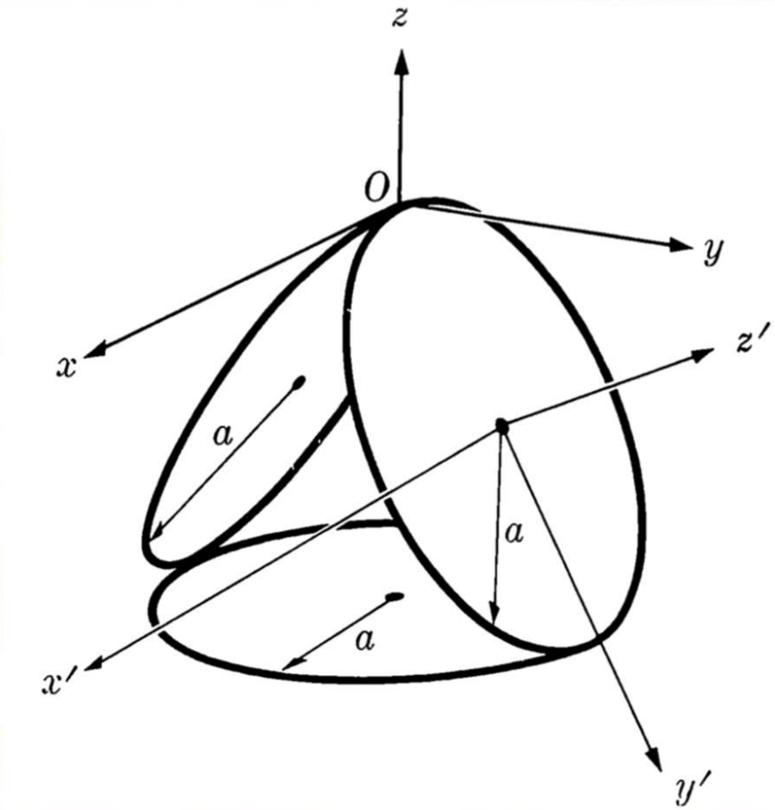
$$\boldsymbol{I}_{B,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + M\left(3a^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\right)$$

$$\boldsymbol{I}_{B,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + \frac{12Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\boxed{\boldsymbol{I}_{B,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix}13 & 0 & 0\\ 0 & 13 & 0\\ 0 & 0 & 2\end{pmatrix}}$$

گشتاور لختی دورانی دیسک واقع در کف حول نقطه O نسبت به محورهای x,y,z

حال برای دیسکی که در سمت راست واقع شده است به صورت زیر عمل می کنیم



$$\boldsymbol{I}_{R,O(x'y'z')} = \boldsymbol{I}_{\boldsymbol{G}} + M(R^2\mathbf{1} - \boldsymbol{R}\boldsymbol{R})$$

$$R^2 = a^2$$

$$\boldsymbol{R} = a\hat{\mathbf{y}}'$$

$$\boldsymbol{R}\boldsymbol{R} = a^2\hat{\mathbf{y}}'\hat{\mathbf{y}}'$$

$$\hat{\mathbf{y}}'\hat{\mathbf{y}}' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{I}_{\boldsymbol{G}} = \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + M\left(a^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + M\left(a^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - a^2\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\right)$$

$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + Ma^2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

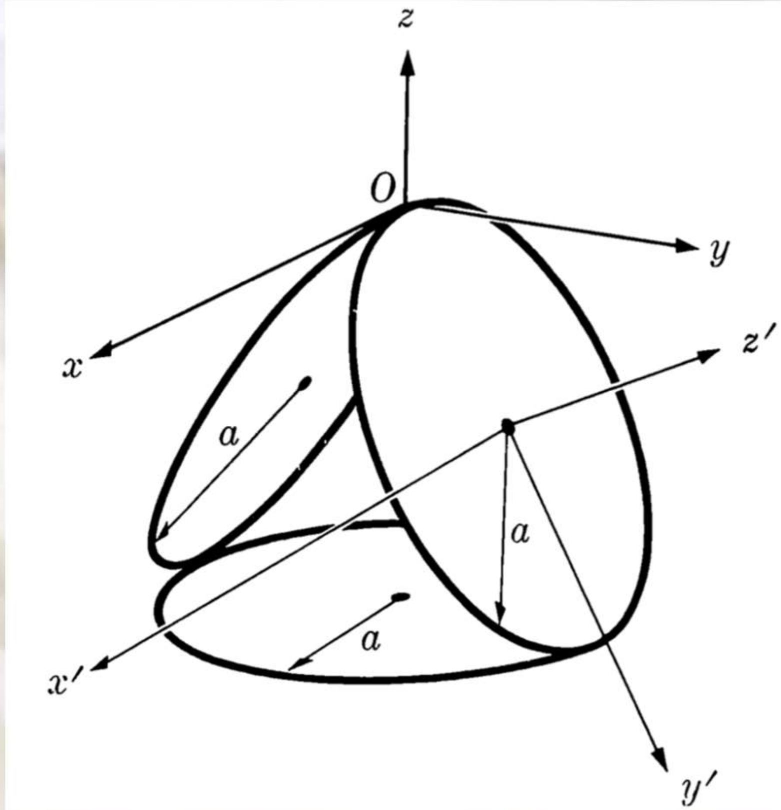
$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + \frac{4Ma^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 6\end{pmatrix}$$

گشتاور لختی دورانی دیسک سمت راست حول نقطه O نسبت به محورهای x',y',z'

حال می خواهیم که گشتاور لختی دورانی دیسک سمت راست حول نقطه O نسبت به محورهای x,y,z را به دست آوریم

برای این کار ماتریس تبدیل بین دو دستگاه را به صورت زیر به دست می آوریم



$$\boldsymbol{R} = \begin{pmatrix} \hat{\boldsymbol{x}}.\hat{\boldsymbol{x}}' & \hat{\boldsymbol{x}}.\hat{\boldsymbol{y}}' & \hat{\boldsymbol{x}}.\hat{\boldsymbol{z}}' \\ \hat{\boldsymbol{y}}.\hat{\boldsymbol{x}}' & \hat{\boldsymbol{y}}.\hat{\boldsymbol{y}}' & \hat{\boldsymbol{y}}.\hat{\boldsymbol{z}}' \\ \hat{\boldsymbol{z}}.\hat{\boldsymbol{x}}' & \hat{\boldsymbol{z}}.\hat{\boldsymbol{y}}' & \hat{\boldsymbol{z}}.\hat{\boldsymbol{z}}' \end{pmatrix}$$

$$\hat{\boldsymbol{x}} = (1,0,0)$$

$$\hat{\boldsymbol{y}} = (0,1,0)$$

$$\hat{\boldsymbol{z}} = (0,0,1)$$

$$\hat{\boldsymbol{x}}' = (1,0,0)$$

$$\hat{\boldsymbol{y}}' = \left(0, \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$$

$$\hat{\boldsymbol{z}}' = \left(0, -\sin\frac{\pi}{3}, \cos\frac{\pi}{3}\right)$$

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boldsymbol{I}_{R,O(x'y'z')} = \frac{Ma^2}{4}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boldsymbol{I}_{R,O(xyz)} = R^T \boldsymbol{I}_{\boldsymbol{R},O(x'y'z')} \mathrm{R}$$

$$\boldsymbol{I}_{R,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boldsymbol{I}_{R,O(xyz)} = \frac{Ma^2}{4}\begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{19}{4} & \frac{4}{5}\sqrt{3} \\ 0 & \frac{4}{5}\sqrt{3} & \frac{9}{4} \end{pmatrix}$$

تمرین: به طور مشابه با دیسک سمت راستی گشتاور لختی دورانی دیسک سمت چپ را حول نقطه O نسبت به محورهای x,y,z به دست آورید:

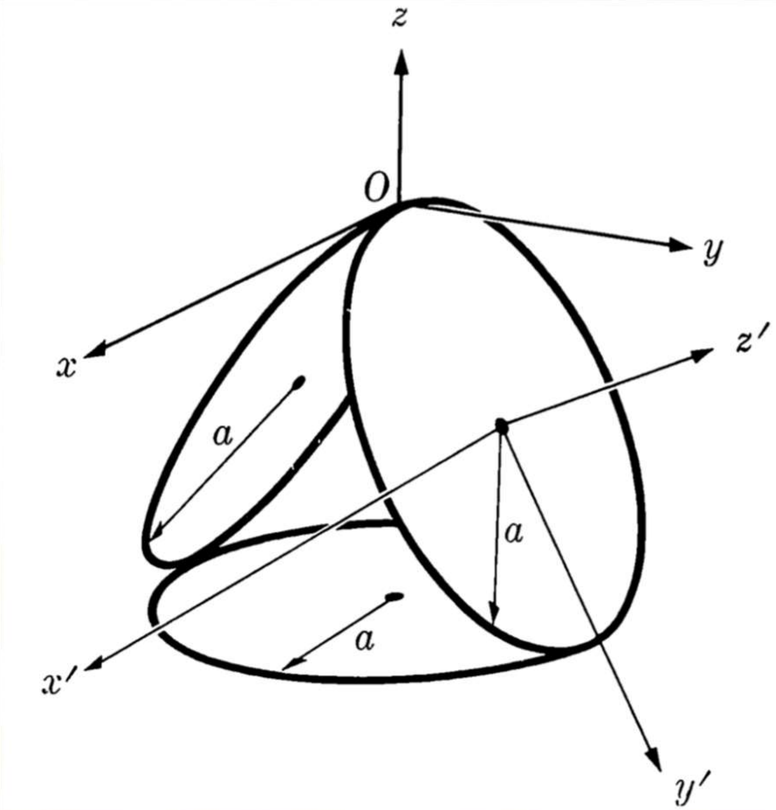
$$\boldsymbol{I}_{L,O(xyz)}=\frac{Ma^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & \frac{19}{4} & -\frac{4}{5}\sqrt{3}\\ 0 & -\frac{4}{5}\sqrt{3} & \frac{9}{4}\end{pmatrix}$$

$$\boldsymbol{I}_{O(xyz)}=\boldsymbol{I}_{B,O(xyz)}+\boldsymbol{I}_{R,O(xyz)}+\boldsymbol{I}_{L,O(xyz)}$$

$$\boldsymbol{I}_{O(xyz)}=\frac{Ma^2}{4}\begin{pmatrix}13 & 0 & 0\\ 0 & 13 & 0\\ 0 & 0 & 2\end{pmatrix}+\frac{Ma^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & \frac{19}{4} & \frac{4}{5}\sqrt{3}\\ 0 & \frac{4}{5}\sqrt{3} & \frac{9}{4}\end{pmatrix}+\frac{Ma^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & \frac{19}{4} & -\frac{4}{5}\sqrt{3}\\ 0 & -\frac{4}{5}\sqrt{3} & \frac{9}{4}\end{pmatrix}$$

$$\boldsymbol{I}_{O(xyz)}=\frac{Ma^2}{4}\begin{pmatrix}23 & 0 & 0\\ 0 & \frac{45}{2} & 0\\ 0 & 0 & \frac{26}{4}\end{pmatrix}=\frac{Ma^2}{4}\begin{pmatrix}23 & 0 & 0\\ 0 & 22\frac{1}{2} & 0\\ 0 & 0 & 6\frac{1}{2}\end{pmatrix}$$

مثال: گشتاور لختی دورانی کل دستگاه حول نقطه O نسبت به محور y' را به دست آورید:



$$\boldsymbol{I}_{Oy'} = \hat{\boldsymbol{y}}'.\boldsymbol{I}_{O(xyz)}.\hat{\boldsymbol{y}}'$$

$$\hat{\boldsymbol{y}}' = \left(0, \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$$

$$\boldsymbol{I}_{Oy'} = \left(0, \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right).\boldsymbol{I}_{O(xyz)}.\begin{pmatrix} 0 \\ \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \end{pmatrix}$$

$$\boldsymbol{I}_{Oy'} = \frac{Ma^2}{4}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\begin{pmatrix} 23 & 0 & 0 \\ 0 & \frac{45}{2} & 0 \\ 0 & 0 & \frac{13}{2} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\boldsymbol{I}_{Oy'} = \frac{Ma^2}{4}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\begin{pmatrix} 0 \\ \frac{45}{4} \\ \frac{13\sqrt{3}}{4} \end{pmatrix} = \frac{Ma^2}{4}\left(0 + \frac{45}{8} + 13 \times \frac{3}{8}\right)$$

$$\boldsymbol{I}_{Oy'} = \left(\frac{45+39}{8}\right)\frac{Ma^2}{4} = \frac{84}{8}\frac{Ma^2}{4} = 10.5\frac{Ma^2}{4} = 10\frac{1}{2}\frac{Ma^2}{4}$$

انرژی جنبشی دورانی

انرژی جنبشی دورانی را می توان برحسب گشتاور لختی دورانی نوشت

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k v_k^2 = \frac{1}{2}\sum_{k=1}^{N} m_k \boldsymbol{v_k}.\boldsymbol{v_k}$$

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k (\boldsymbol{\omega} \times \boldsymbol{r_k}).(\boldsymbol{\omega} \times \boldsymbol{r_k})$$

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \boldsymbol{\omega}.[\boldsymbol{r_k} \times (\boldsymbol{\omega} \times \boldsymbol{r_k})]$$

$$T = \frac{1}{2}\boldsymbol{\omega}.\sum_{k=1}^{N} m_k\,[\boldsymbol{r_k} \times (\boldsymbol{\omega} \times \boldsymbol{r_k})]$$

$$T = \frac{1}{2}\boldsymbol{\omega}.\sum_{k=1}^{N} m_k \left[\boldsymbol{r_k} \times (\boldsymbol{\omega} \times \boldsymbol{r_k})\right]$$

$$\boldsymbol{L} = \sum_{k=1}^{N} m_k \boldsymbol{r_k} \times (\boldsymbol{\omega} \times \boldsymbol{r_k})$$

$$T = \frac{1}{2}\boldsymbol{\omega}.\boldsymbol{L}$$

$$\boldsymbol{L} = \boldsymbol{I}.\boldsymbol{\omega}$$

$$T = \frac{1}{2}\boldsymbol{\omega}.\boldsymbol{I}.\boldsymbol{\omega}$$

$$T = \frac{1}{2}\boldsymbol{\omega}.\boldsymbol{I}.\boldsymbol{\omega}$$

$$T = \frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}$$

$$T = \frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\begin{pmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{pmatrix}$$

$$T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z\right) + \frac{1}{2}\left(I_{yx}\omega_y\omega_x + I_{yy}\omega_y^2 + I_{yz}\omega_z\omega_y\right) + \frac{1}{2}\left(I_{zx}\omega_z\omega_x + I_{zy}\omega_z\omega_y + I_{zz}\omega_z^2\right)$$

$$T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z\right) + \frac{1}{2}\left(I_{yx}\omega_y\omega_x + I_{yy}\omega_y^2 + I_{zy}\omega_z\omega_y\right) + \frac{1}{2}\left(I_{zx}\omega_z\omega_x + I_{zy}\omega_z\omega_y + I_{zz}\omega_z^2\right)$$

$$T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) + \frac{1}{2}\left(2I_{xy}\omega_y\omega_x + 2I_{xz}\omega_x\omega_z + 2I_{yz}\omega_y\omega_z\right)$$

$$T = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) + \left(I_{xy}\omega_y\omega_x + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z\right)$$

$$T = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 + I_{xy}\omega_y\omega_x + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z$$

$$T = \frac{1}{2} I_{xx}\omega_x^2 + \frac{1}{2} I_{yy}\omega_y^2 + \frac{1}{2} I_{zz}\omega_z^2 + I_{xy}\omega_y\omega_x + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z$$

انرژی جنبشی دورانی در دستگاه محورهای اصلی x',y',z' که در آن تانسور گشتاور لختی قطری است

$$T = \frac{1}{2} I'_{xx}\omega_x^2 + \frac{1}{2} I'_{yy}\omega_y^2 + \frac{1}{2} I'_{zz}\omega_z^2$$

$$T = \frac{1}{2} I'_{x}\omega_x^2 + \frac{1}{2} I'_{y}\omega_y^2 + \frac{1}{2} I'_{z}\omega_z^2$$