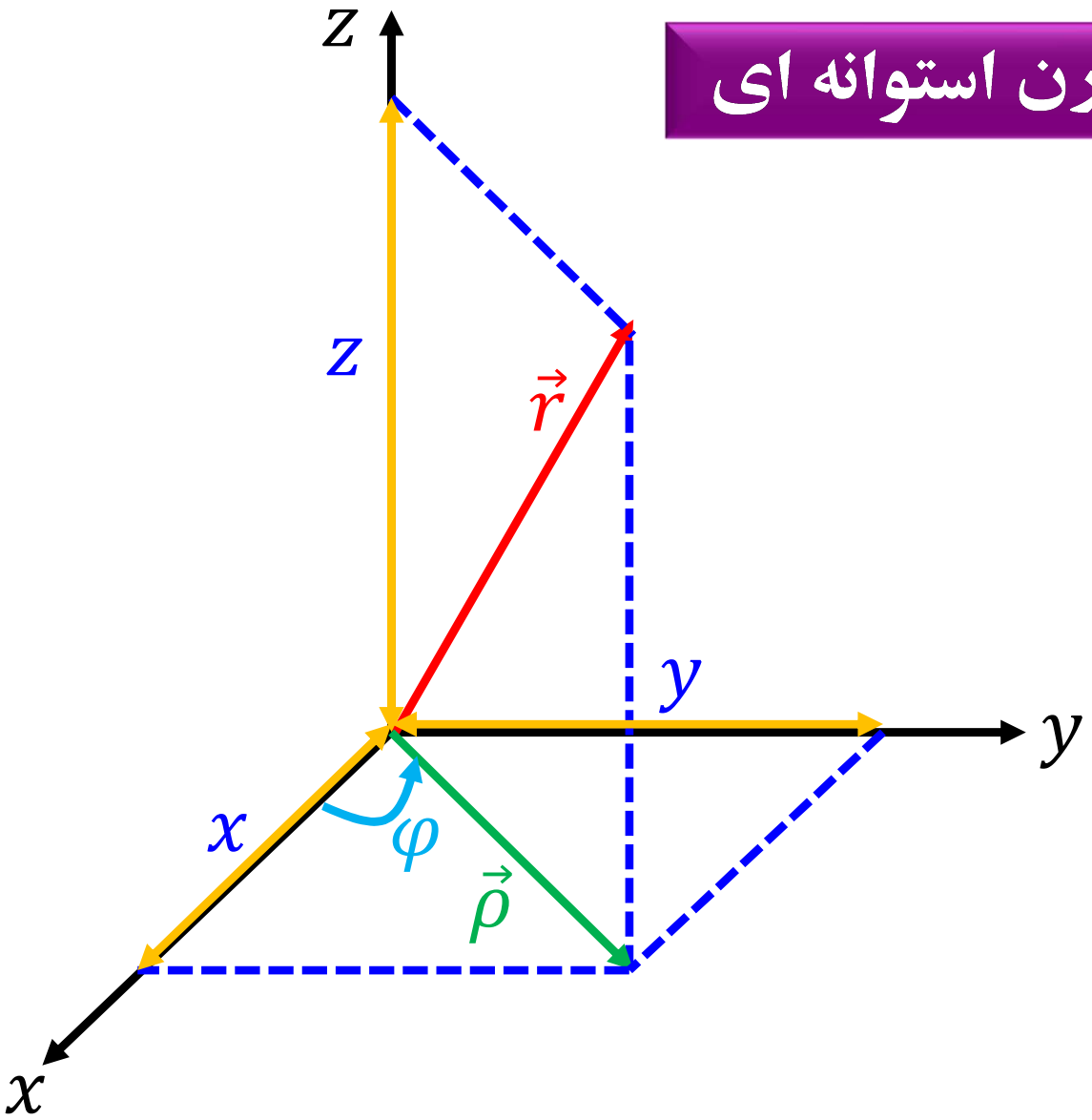
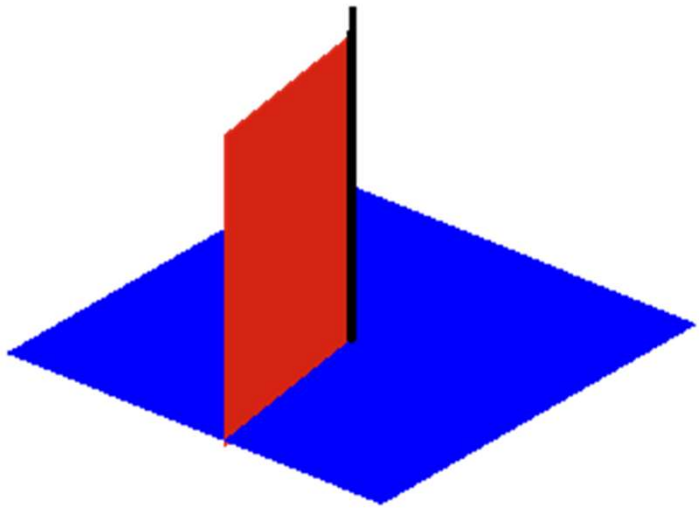
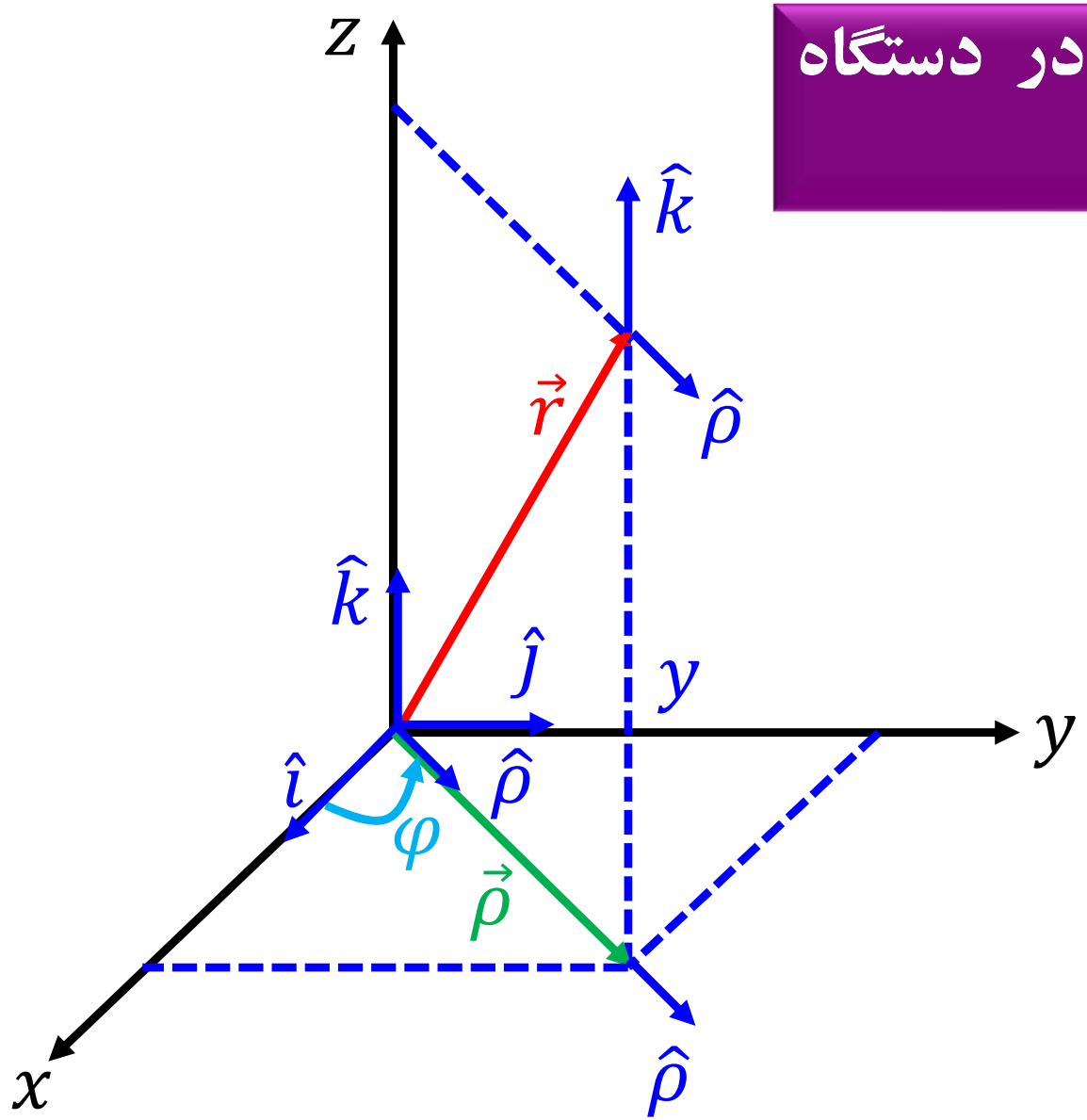
دستگاه مختصات 3 بعدی با تقارن استوانه ای





بردار مکان و بردارهای یکه در دستگاه استوانه ای



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \vec{\rho} + z\hat{k}$$

$$\vec{\rho} = x\hat{i} + y\hat{j}$$

$$\vec{r} = \rho\hat{\rho} + z\hat{k}$$

$$\hat{\rho} = \cos\varphi\hat{i} + \sin\varphi\hat{j}$$

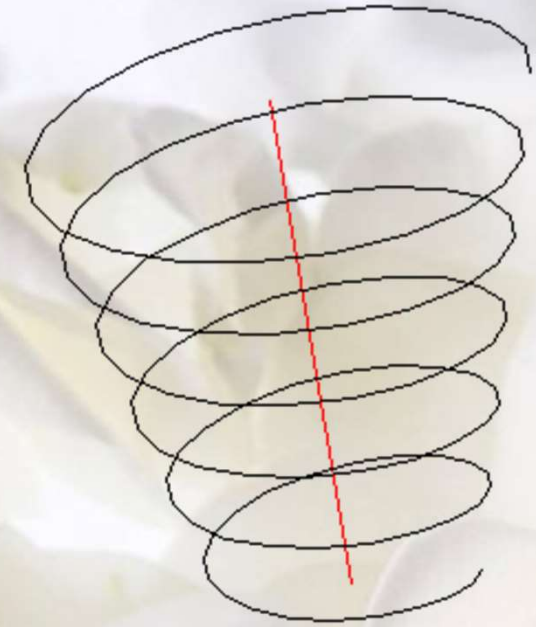
$$\hat{\varphi} = -\sin\varphi\hat{i} + \cos\varphi\hat{j}$$

$$\frac{d\hat{\rho}}{d\varphi} = -\sin\varphi\hat{i} + \cos\varphi\hat{j}$$

$$\frac{d\hat{\varphi}}{d\varphi} = -\cos\varphi\hat{i} - \sin\varphi\hat{j}$$

$$\frac{d\hat{\varphi}}{d\varphi} = -\hat{\rho}$$

$$\frac{d\hat{\rho}}{d\varphi} = \hat{\varphi}$$



بردار سرعت در دستگاه مختصات استوانه ای در حرکت روی یک هلیکس با شعاع فزآینده یا کاهنده

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k}$$

$$v_\rho = \dot{\rho}$$

$$v_\varphi = \rho\dot{\varphi}$$

$$v_z = \dot{z}$$

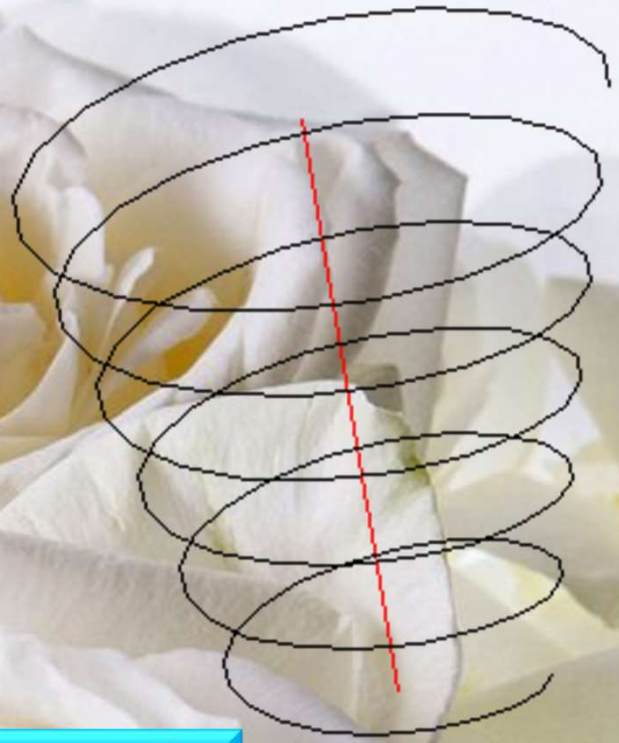
$$\vec{r} = \rho\hat{\rho} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = v_\rho\hat{\rho} + v_\varphi\hat{\varphi} + v_z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\rho\hat{\rho} + z\hat{k}\right)$$

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \dot{z}\hat{k}$$

بردار شتاب در دستگاه مختصات استوانه ای در حرکت روی یک هلیکس با شعاع فزآینده یا کاهنده



$$\vec{a} = \frac{d\vec{v}}{dt} = a_\rho \hat{\rho} + a_\varphi \hat{\varphi} + a_z \hat{k}$$

$$\vec{a} = \frac{d}{dt}\left(\dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k}\right)$$

$$\vec{a} = \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\varphi}\hat{\varphi} + \dot{\rho}\dot{\varphi}\hat{\varphi} + \rho\ddot{\varphi}\hat{\varphi} - \rho\dot{\varphi}\dot{\varphi}\hat{\rho} + \ddot{z}\hat{k}$$

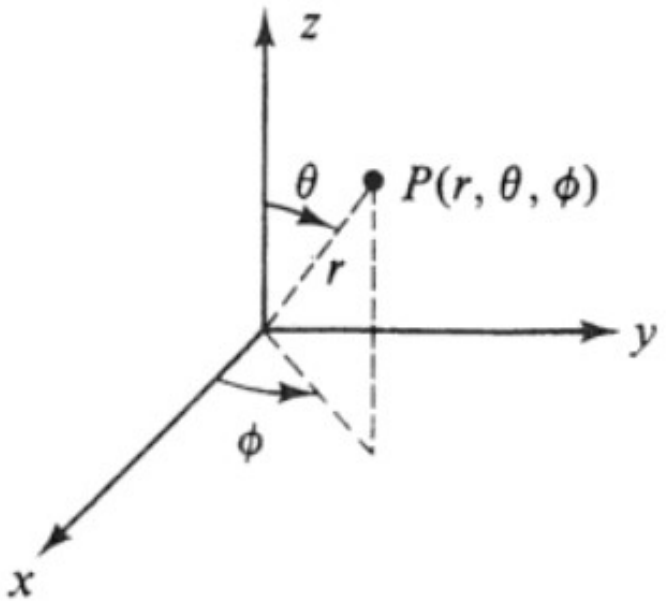
$$\vec{a} = \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\varphi}\hat{\varphi} + \rho\ddot{\varphi}\hat{\varphi} - \rho\dot{\varphi}\hat{\rho} - \rho\dot{\varphi}^2\hat{\rho} + \ddot{z}\hat{k}$$

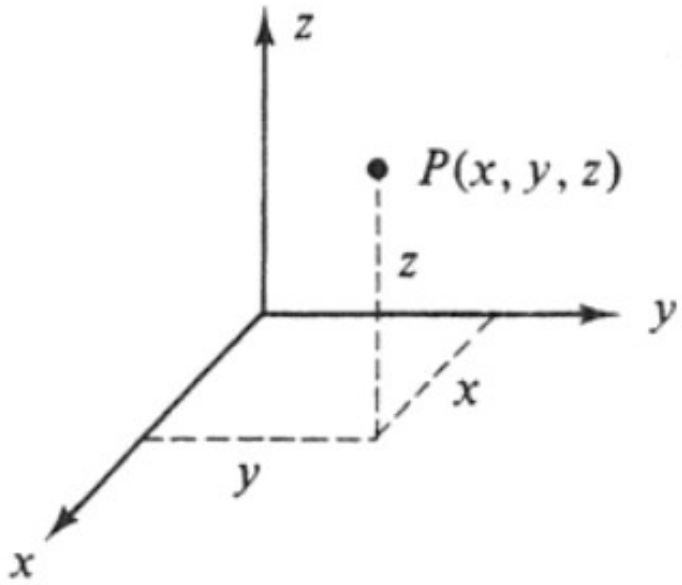
$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{k}$$

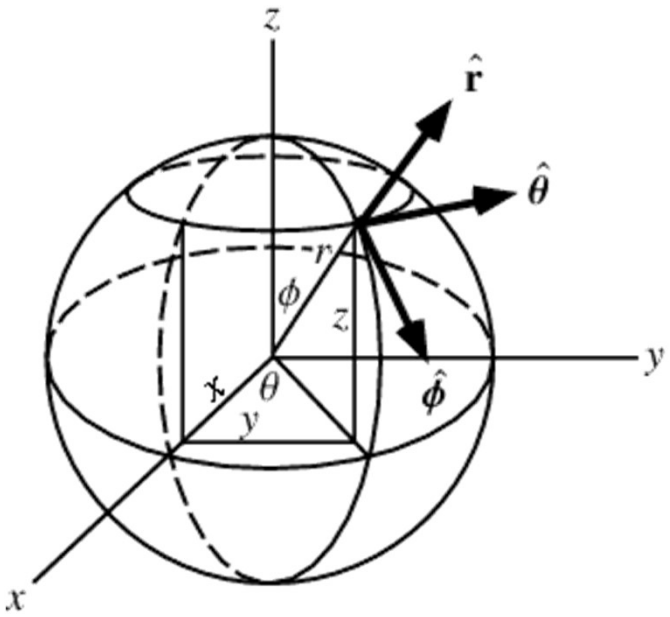
$$\vec{a} = a_\rho \hat{\rho} + a_\varphi \hat{\varphi} + a_z \hat{k} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{k}$$

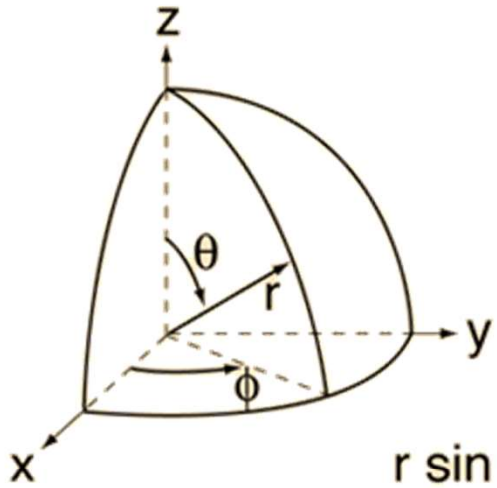
$$a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2 \qquad a_\varphi = \rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} \qquad a_z = \ddot{z}$$

دستگاه مختصات 3 بعدی با تقارن کروی



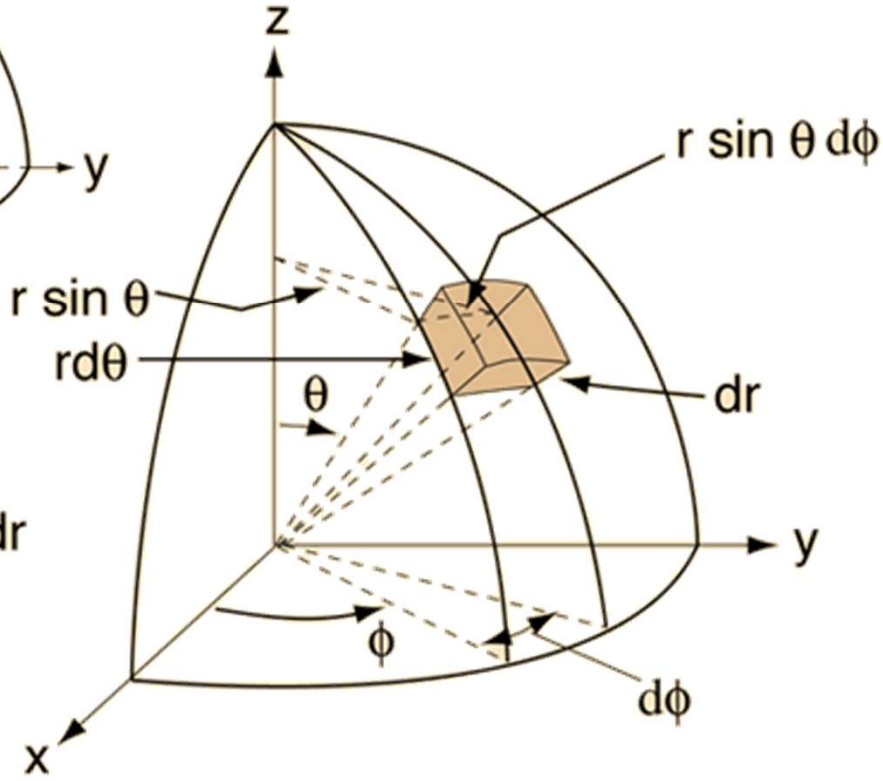




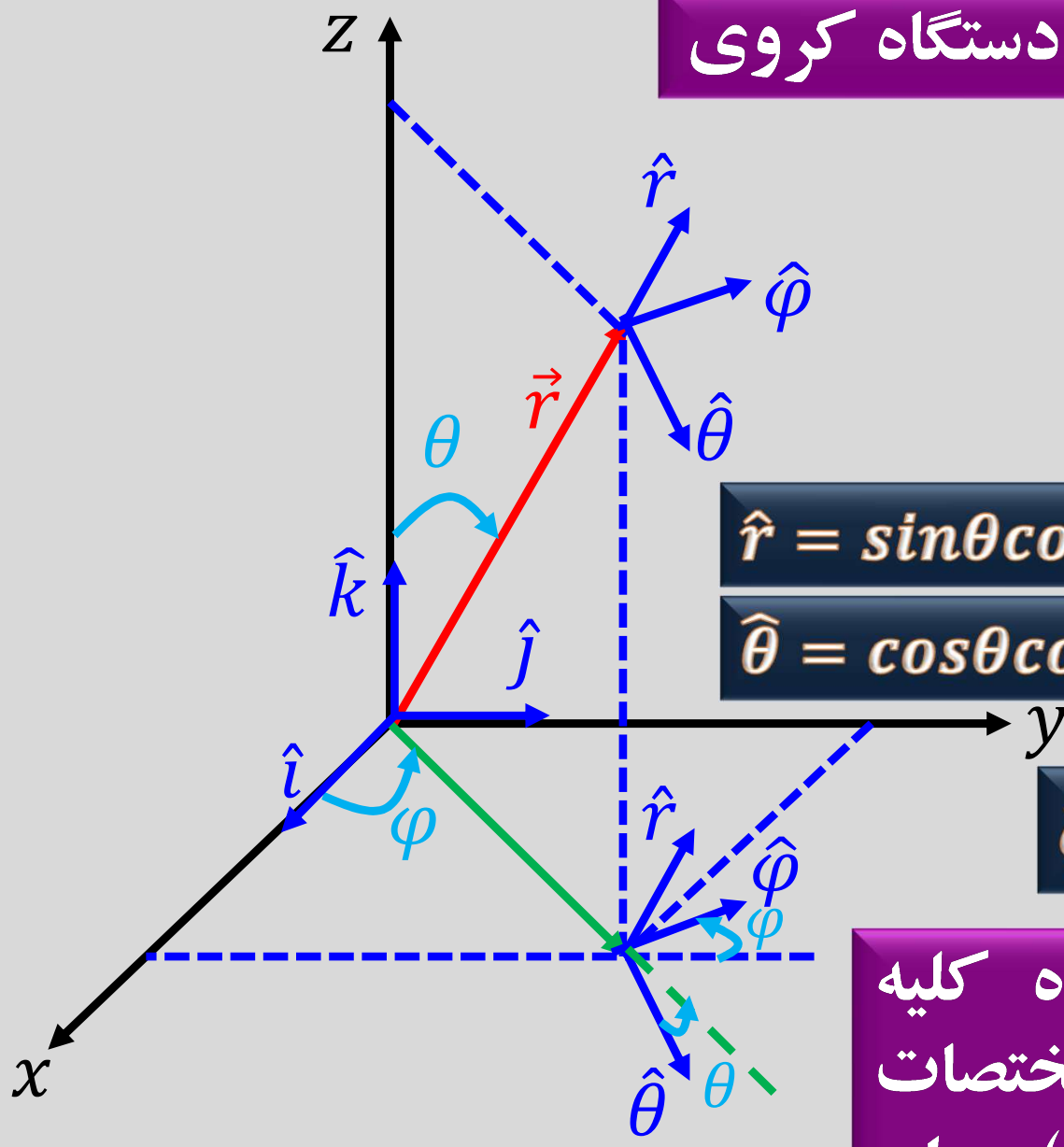


Volume element

$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$



بردار مکان و بردارهای یکه در دستگاه کروی



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = r\hat{r}$$

$$\hat{r} = sin\theta cos\varphi\hat{i} + sin\theta sin\varphi\hat{j} + cos\theta\hat{k}$$

$$\hat{\theta} = cos\theta cos\varphi\hat{i} + cos\theta sin\varphi\hat{j} - sin\theta\hat{k}$$

$$\hat{\varphi} = -sin\varphi\hat{i} + cos\varphi\hat{j}$$

تمرین: نشان دهید که اندازه کلیه بردارهای یکه (دستگاه های مختصات قطبی، استوانه ای و کروی) برابر واحد است

تمرین: نشان دهید که روابط زیر بین بردارهای یکه دستگاه مختصات کروی وجود دارد:

$$\begin{cases} \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} & \frac{\partial \hat{r}}{\partial \varphi} = \hat{\varphi} sin\theta \\ \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} & \frac{\partial \hat{\theta}}{\partial \varphi} = \hat{\varphi} cos\theta \\ \frac{\partial \hat{\varphi}}{\partial \theta} = 0 & \frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{\rho} = -\hat{r} sin\theta - \hat{\theta} cos\theta \end{cases}$$

راهنمایی: از بردارهای یکه زیر مشتق گیری کنید:

$$\hat{r} = sin\theta cos\varphi \hat{\imath} + sin\theta sin\varphi \hat{\jmath} + cos\theta \hat{k}$$

$$\hat{\theta} = cos\theta cos\varphi \hat{\imath} + cos\theta sin\varphi \hat{\jmath} - sin\theta \hat{k}$$

$$\hat{\varphi} = -sin\varphi \hat{\imath} + cos\varphi \hat{\jmath}$$

بردار سرعت در دستگاه مختصات کروی

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{\mathrm{dt}} + \frac{d\hat{r}}{d\varphi}\frac{d\varphi}{\mathrm{dt}}$$

$$\frac{d\hat{r}}{dt} = \hat{\theta}\dot{\theta} + sin\theta\hat{\varphi}\dot{\varphi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\varphi}sin\theta\hat{\varphi}$$

$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi}$$

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v_\varphi = r\dot{\varphi}sin\theta$$

بردار شتاب در دستگاه مختصات کروی

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\varphi}sin\theta\hat{\varphi}\right)$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\varphi}sin\theta\hat{\varphi}$$
$$+\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} + r\dot{\theta}\dot{\varphi}cos\theta\hat{\varphi}$$
$$+\dot{r}\dot{\varphi}sin\theta\hat{\varphi} + r\ddot{\varphi}sin\theta\hat{\varphi} + r\dot{\varphi}\dot{\theta}cos\theta\hat{\varphi}$$
$$+ r\dot{\varphi}sin\theta\dot{\varphi}(-\hat{r}sin\theta - \hat{\varphi}cos\theta)$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)\hat{r}$$
$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)\hat{\theta}$$
$$+ \left(r\ddot{\varphi}sin\theta + 2\dot{r}\dot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta\right)\hat{\varphi}$$

$$\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_\varphi\hat{\varphi}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta\right)\hat{r}$$
$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 sin\theta cos\theta\right)\hat{\theta}$$
$$+\left(r\ddot{\varphi}sin\theta + 2\dot{r}\dot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta\right)\hat{\varphi}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 sin\theta cos\theta$$

$$a_\varphi = r\ddot{\varphi}sin\theta + 2\dot{r}\dot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{k}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta\right)\hat{r}$$
$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 sin\theta cos\theta\right)\hat{\theta}$$
$$+ (r\ddot{\varphi}sin\theta + 2\dot{r}\dot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta)\,\hat{\varphi}$$